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On synchronized relatively full embeddings and $Q$-universality


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ON SYNCHRONIZED RELATIVELY FULL EMBEDDINGS AND Q-UNIVERSALITY

To Jiří Adámek on his 60th birthday

by V. KOUBEK and J. SICHLER

Abstract

M. E. Adams et W. Dziobiak ont démontré que toute quasi-variété \( ff \) algébrique universelle de systèmes algébriques de signature finie est \( Q \)-universelle. Dans cet article on introduit la notion de plongement synchronisé relativement plein qu’on utilise ensuite afin de modifier leur résultat pour les quasi-variétés d’algèbres.

1 Introduction

We aim to show a new connection between two algebraic structures associated with quasivarieties of algebras. All needed definitions are given in the next section.

First, for any quasivariety \( Q \), the homomorphisms between its members form a concrete category. The richness of the categorical structure is reflected in the notion of algebraic universality studied in the monograph [18] by A. Pultr and V. Trnková.

When ordered by inclusion, the subquasivarieties of a given quasivariety \( Q \) form a lattice we denote \( Q\text{Lat}(Q) \). This is the second algebraic structure associated with \( Q \). Questions about the size of \( Q\text{Lat}(Q) \) or lattice identities satisfied in \( Q\text{Lat}(Q) \) motivated M. V. Sapir [19] to define and exhibit \( Q \)-universal quasivarieties, and W. Dziobiak [9, 10] to introduce what is now called an A-D family of objects of \( Q \). A survey of these notions and results concerning them is given in [2]. M. E. Adams

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and W. Dziobiak [3] linked the latter two properties by showing that every quasivariety $Q$ containing an A-D family is also $Q$-universal. The converse implication is still an open problem, originally stated by M. E. Adams and W. Dziobiak.

**Problem 1.1.** Is there a $Q$-universal quasivariety containing no A-D family?

In [4], M. E. Adams and W. Dziobiak proved the following remarkable and quite surprising result connecting the two algebraic structures associated with a quasivariety of algebraic systems.

**Theorem 1.2** [4]. Any finite-to-finites algebraically universal (ff-alg-universal) quasivariety of algebraic systems of finite similarity type contains an A-D family and hence it is $Q$-universal.

In [16], the present authors extended this result as follows.

**Theorem 1.3** [16]. Any almost ff-alg-universal quasivariety of algebraic systems of finite similarity type contains an A-D family and hence it is $Q$-universal.

Almost universality is a special case of relative universality, see Section 2. Here we aim to modify the latter result for quasivarieties of algebras. We assume that

$(\star)$ $Q$ is a quasivariety of finitary algebras and $V$ is a proper subvariety of $Q$ such that there exists a synchronized $I(V)$-relatively full embedding $F$ from the category of all undirected graphs into $Q$ such that $Ff$ is surjective for every graph quotient homomorphism $f$ and $FG$ is finite for every finite graph $G$.

**Theorem 1.4.** Any quasivariety $Q$ satisfying $(\star)$ contains an A-D family and hence it is $Q$-universal.

As already noted, all needed notions are reviewed in Section 2 below, and the proof of Theorem 1.4 is given in Section 3. It is based on the fact that any subquasivariety $R$ of a quasivariety $Q$ is an epireflective full subcategory of $Q$. In Section 3, it is also shown how Theorem 1.4 incorporates earlier results of [6, 7, 8].

2 Basic notions and their context

**Alg-universality.** A category $K$ is **alg-universal** if any category of algebras and all homomorphisms between them can be fully embedded into $K$. This is equivalent to the fact that there exists a full embedding from the category $GRA$ of all undirected graphs and all graph homomorphisms into $K$. Moreover, if $K$ is a concrete category
and there exists a full embedding \( F : \text{GRA} \rightarrow K \) such that the underlying set of \( FG \) for every finite graph is finite then we say that \( F \) preserves finiteness and that \( K \) is \( \text{ff-alg-universal} \). If \( K \) is a concrete category then any \( K \)-object \( A \) with a finite underlying set is called finite. Next we give several well-known properties of alg-universal categories. To do this, we say that a category \( K \) is a monoid universal if for every monoid \( M \) there exists a \( K \)-object \( A \) such that the endomorphism monoid of \( A \) is isomorphic to \( M \).

**Theorem 2.1** [18]. (a) Any concrete alg-universal category \( K \) is monoid universal; and if \( K \) is \( \text{ff-alg-universal} \), then for every finite monoid \( M \) there exists a finite \( K \)-object \( A \) such that the endomorphism monoid of \( A \) is isomorphic to \( M \).

(b) If \( K \) is alg-universal, then for a proper class \( I \) there exists a family \( \{ F_i : K \rightarrow K \mid i \in I \} \) of full embeddings such that \( F_i A \) is not isomorphic to \( F_j B \) for any \( K \)-objects \( A \) and \( B \) and for any distinct \( i, j \in I \). For any set \( I \) there exists a family \( \{ F_i : K \rightarrow K \mid i \in I \} \) of full embeddings such that there exists no \( K \)-morphism between \( F_i A \) and \( F_j B \) for any \( K \)-objects \( A \) and \( B \) and for any distinct \( i, j \in I \).

(c) If \( K \) is \( \text{ff-alg-universal} \) and \( I \) is a countable set, then there exists a family \( \{ F_i : K \rightarrow K \mid i \in I \} \) of full embeddings \( F_i \) preserving finiteness such that there exists no \( K \)-morphism between \( F_i A \) and \( F_j B \) for any \( K \)-objects \( A \) and \( B \) and any distinct \( i, j \in I \).

Theorem 2.1 provides a tool for proving that a given category \( K \) is not alg-universal. For example, if \( K \) is a concrete category such that for every set \( X \) there exists only a set of non-isomorphic \( K \)-objects with a given underlying set \( X \) and if there exists a cardinal \( \alpha \) such that every \( K \)-object whose underlying set has cardinality greater than \( \alpha \) has a non-identity endomorphism, then \( K \) is not alg-universal.

Hence for example the variety of lattices or the variety of monoids or the category of topological spaces and continuous mappings are not alg-universal because of the existence of constant morphisms. On the other hand, both the variety of semigroups [13] and the variety of \((0,1)\)-lattices ([11] or [12]) are alg-universal.

Thus we can say that monoids or lattices have sufficiently rich structure to be 'close' to being alg-universal while still permitting constant morphisms, although these categories are not alg-universal in the strict sense. This motivates a notion of almost alg-universality that ignores the constant morphisms. Next we define a more general concept expressing this idea.

Let \( K \) be a category. A class \( C \) of \( K \)-morphisms is an ideal if \( f \circ g \in C \) for \( K \)-morphisms \( f : a \rightarrow b, g : b \rightarrow c \) whenever \( f \in C \) or \( g \in C \). A faithful functor \( F : L \rightarrow K \) is called \( C \)-relatively full embedding if
(•) $Ff \notin C$ for any $\mathbb{L}$-morphism $f$;

(•) if $f : Fa \rightarrow Fb$ is a $\mathbb{K}$-morphism for $\mathbb{L}$-objects $a$ and $b$ then either $f \in C$ or $f = Fg$ for some $\mathbb{K}$-morphism $g : a \rightarrow b$.

Thus $F$ is a full embedding exactly when it is $C$-relatively full embedding for $C = \emptyset$. Observe that, if $F : \mathbb{L} \rightarrow \mathbb{K}$ is a $C$-relatively full embedding for some ideal $C$ then $f$ is an $\mathbb{L}$-isomorphism if and only if $Ff$ is a $\mathbb{K}$-isomorphism. If there exists a $C$-relatively full embedding $F : \text{GRA} \rightarrow \mathbb{K}$ then we say that $\mathbb{K}$ is $C$-relatively alg-universal. If, moreover, $\mathbb{K}$ is concrete and $F$ preserves finiteness, then $\mathbb{K}$ is called $C$-relatively $ff$-alg-universal. Clearly, $\mathbb{K}$ is $C$-relatively alg-universal (or $C$-relatively $ff$-alg-universal) for $C = \emptyset$ just when $\mathbb{K}$ is alg-universal (or $ff$-alg-universal, respectively). If $\mathbb{K}$ is concrete category and $C$ is the ideal consisting of all $\mathbb{K}$-morphisms with constant underlying mapping then we say that $F : \mathbb{L} \rightarrow \mathbb{K}$ is almost full embedding instead of $C$-relatively full embedding and that $\mathbb{K}$ is almost alg-universal or almost $ff$-alg-universal instead of $C$-relatively alg-universal or $C$-relatively $ff$-alg-universal. The variety of lattices [20] and the variety of monoids [17] or [15] are almost alg-universal but not alg-universal. A second consequence of Theorem 2.1 is that a category $\mathbb{K}$ which is not monoid-universal is not alg-universal. This fact was exploited by M. E. Adams and W. Dziobiak in [5], where they proved that the variety of monadic Boolean algebras is not alg-universal, yet contains a proper class of non-isomorphic algebras whose endomorphism monoids consist of the identity map alone.

Theorem 2.1 naturally leads to the following question.

**Problem 2.2.** Is there a variety $\mathbb{V}$ of algebras which is monoid universal but not alg-universal?

We shall consider ideals of a special type. Let $O$ be a class of $\mathbb{K}$-objects. Then $I(O)$ denotes a class of all $\mathbb{K}$-morphisms $f : a \rightarrow b$ such that there exist $\mathbb{K}$-morphisms $g : a \rightarrow c$ and $h : c \rightarrow b$ with $c \in O$ and $f = h \circ g$. Clearly, $I(O)$ is an ideal of $\mathbb{K}$ called an object ideal of $O$. In what follows, we shall consider even more specific object ideals.

$Q$-**universality.** A class $Q$ of algebraic systems of a finitary type $\Delta$ is a quasivariety if it is closed under all products, all ultraproducts, all subsystems and all isomorphic images. For any class $\mathbb{K}$ of algebraic systems of type $\Delta$, there exists the least quasivariety $Q$ containing $\mathbb{K}$, which we shall denote $Q = \text{Qua}(\mathbb{K})$. Quasivarieties will be viewed as categories whose morphisms are all homomorphisms, that is, mappings preserving all operations and relations.
M. V. Sapir [19] defined a quasivariety $Q$ of finite type $\Delta$ as $Q$-universal if for every quasivariety $\mathcal{R}$ of finite type the lattice $QLat(\mathcal{R})$ is a homomorphic image of a sublattice of $QLat(Q)$.

Let $\mathcal{P}(\omega_0)$ be the set of all finite subsets of natural numbers and $\mathcal{P}(\omega) = \mathcal{P}(\omega_0) \setminus \{\emptyset\}$ the set of all finite non-empty subsets of natural numbers. W. Dziobiak [9, 10] studied families $\{S_A \mid A \in \mathcal{P}(\omega_0)\}$ of finite algebraic systems of a given type $\Delta$ we now call Adams-Dziobiak families (or A-D families) defined by these four conditions:

(p1) $S_\emptyset$ is the terminal algebraic system;

(p2) if $A = B \cup C$ for $A, B, C \in \mathcal{P}(\omega_0)$, then $S_A \in Qua(\{S_B, S_C\})$;

(p3) if $A \in \mathcal{P}(\omega)$ and $B \in \mathcal{P}(\omega_0)$ with $S_A \in Qua(\{S_B\})$, then $A = B$;

(p4) if $U, V \in Qua(\{S_A \mid A \in \mathcal{P}\})$ are finite algebraic systems for some finite $\mathcal{P} \subset \mathcal{P}(\omega)$ and if there exists an injective homomorphism $f : S_A \rightarrow U \times V$ for some $A \in \mathcal{P}(\omega)$, then there exists an injective homomorphism $g : S_A \rightarrow U$ or there exists an injective homomorphism $g : S_A \rightarrow V$ or there exist $B, C \in \mathcal{P}(\omega)$ and injective homomorphisms $g_B : S_B \rightarrow U$ and $g_C : S_C \rightarrow V$ with $A = B \cup C$.

We recall some known results.

**Theorem 2.3.** (a) If $Q$ is a $Q$-universal quasivariety then $QLat(Q)$ has cardinality $2^{\aleph_0}$ and the free lattice over a countable set can be embedded into $QLat(Q)$. Thus $QLat(Q)$ satisfies no non-trivial lattice identity [2].

(b) If a quasivariety $Q$ contains an A-D family, then the lattice of all ideals of the free lattice over a countable set can be embedded into $QLat(Q)$ [3].

Thus to prove that a quasivariety $Q$ of finite type is $Q$-universal, it suffices to prove that $Q$ has an A-D family. We shall study only quasivarieties $Q$ of algebras.

In Section 3 we give certain conditions sufficient for the existence of an A-D family in a quasivariety of algebras of finite type. For this we use factorization systems and epireflection.

**Factorization systems and epireflections.** For a category $\mathcal{K}$, let $\mathcal{E}$ be a class of $\mathcal{K}$-epimorphisms and let $\mathcal{M}$ be a class of $\mathcal{K}$-monomorphisms. We say that $(\mathcal{E}, \mathcal{M})$ is a factorization system of $\mathcal{K}$ if $\mathcal{E}$ and $\mathcal{M}$ are closed under composition, $f \in \mathcal{E} \cap \mathcal{M}$ if and only if $f$ is a $\mathcal{K}$-isomorphism, and for every $\mathcal{K}$-morphism $f : a \rightarrow b$ there
exist unique, up to a commuting isomorphism, $g : a \to c \in \mathcal{E}$ and $h : c \to b \in \mathcal{M}$ with $f = h \circ g$, see [1]. Any factorization system has the diagonalization property. We formulate it for categories with products. If $\mathbb{K}$ is a category with products and an $(\mathcal{E}, \mathcal{M})$-factorization system, then we write $\{f_i : a \to b_i \mid i \in I\} \in \mathcal{M}$ if the morphism $f : a \to \prod_{i \in I} b_i$ such that $f_i = \pi_i \circ f$ for all $i \in I$ where $\pi_i : \prod_{j \in I} b_j \to b_i$ is the $i$-th projection belongs to $\mathcal{M}$. Then the diagonalization property says: if $g_i \circ f = k_i \circ h$ for all $i \in I$ where $f : a \to b \in \mathcal{E}$, $\{g_i : b \to c_i \mid i \in I\}$ is a family of $\mathbb{K}$-morphisms, $h : a \to d$ is a $\mathbb{K}$-morphism and $\{k_i : d \to c_i \mid i \in I\} \in \mathcal{M}$ then there exists a $\mathbb{K}$-morphism $l : b \to d$ such that $h = l \circ f$ and $g_i = k_i \circ l$ for all $i \in I$. If $h \in \mathcal{E}$ then $l \in \mathcal{E}$, and if $\{g_i \mid i \in I\} \in \mathcal{M}$ then $l \in \mathcal{M}$.

We say that a family $\{f_i : A \to A_i \mid i \in I\}$ is separating if for distinct $a, b \in A$ there exists $i \in I$ with $f_i(a) \neq f_i(b)$. If $\mathbb{K}$ is a concrete category then a family $\{f_i : a \to b_i \mid i \in I\}$ of $\mathbb{K}$-morphisms is separating if the family of underlying mapping is separating. For concrete categories $\mathbb{K}$ and $\mathbb{L}$ we say that a functor $F : \mathbb{K} \to \mathbb{L}$ preserves separating families if $\{Ff_i : Fa \to Fb_i \mid i \in I\}$ is a separating family in $\mathbb{L}$ whenever $\{f_i : a \to b_i \mid i \in I\}$ is a separating family in $\mathbb{K}$.

For a concrete category $\mathbb{K}$, let $\text{Inj}_\mathbb{K}$ consist of all $\mathbb{K}$-homomorphisms such that the underlying mapping is injective and $\text{Surj}_\mathbb{K}$ consist of all $\mathbb{K}$-morphisms such that the underlying mapping is surjective. Clearly, every morphism from $\text{Inj}_\mathbb{K}$ is a monomorphism of $\mathbb{K}$ and every morphism from $\text{Surj}_\mathbb{K}$ is an epimorphism of $\mathbb{K}$. If $(\text{Surj}_\mathbb{K}, \text{Inj}_\mathbb{K})$ is a factorization system of $\mathbb{K}$ then we say $\mathbb{K}$ has a concrete factorization system and $(\text{Surj}_\mathbb{K}, \text{Inj}_\mathbb{K})$ is a concrete factorization system of $\mathbb{K}$. Clearly, for every quasivariety $\mathbb{Q}$ of algebras ($\text{Surj}_\mathbb{Q}, \text{Inj}_\mathbb{Q}$) is a concrete factorization system of $\mathbb{Q}$ (because every bijective homomorphism is an isomorphism). Observe that a family $\{f_i : A \to B_i \mid i \in I\}$ of $\mathbb{Q}$-morphisms is separating if and only if it belongs to $\text{Inj}_\mathbb{Q}$, i.e. if the homomorphism $f : A \to \prod_{i \in I} B_i$ with $f_i = f \circ \pi_i$ has an injective underlying mapping where $\pi_i : \prod_{j \in I} B_j \to B_i$ is the $i$-th projection for all $i \in I$. Thus for a concrete category $\mathbb{K}$ we shall say that a family $\{f_i : A \to B_i \mid i \in I\}$ of $\mathbb{K}$-morphisms belong to $\text{Inj}_\mathbb{K}$ just when its corresponding family of underlying mappings is separating. A functor $F : \mathbb{Q} \to \mathbb{R}$ between quasivarieties $\mathbb{Q}$ and $\mathbb{R}$ preserves surjectivity if $F(\text{Surj}_\mathbb{Q}) \subseteq \text{Surj}_\mathbb{R}$.

If $\mathbb{Q}$ is a quasivariety of algebraic systems and $\mathbb{R}$ is a subquasivariety of $\mathbb{Q}$ (of the same type) then, by Theorem 10.1.2 from [14], $\mathbb{R}$ is an epireflective subcategory of $\mathbb{Q}$. This means that for every algebraic system $A \in \mathbb{Q}$ there exists a surjective homomorphism $\rho_A : A \to RA$ where $RA \in \mathbb{R}$ such that for every homomorphism $f : A \to C$ where $C \in \mathbb{R}$ there exists exactly one homomorphism $f^* : RA \to C$ with $f = f^* \circ \rho_A$. Since $\mathbb{R}$ is a full subcategory of $\mathbb{Q}$ then $\rho_A$ is the identity.
morphism exactly when $A \in R$. Then $R : Q \to R$ such that $Rf = (\rho_B \circ f)\ast$ for every homomorphism $f : A \to B$ in $Q$ is a functor which is a left adjoint to the inclusion functor from $R$ to $Q$. We say that $R$ is an epireflection. Observe that $R(Surj_Q) \subseteq Surj_R$.

A quasivariety $Q$ of algebras closed under homomorphic images is a variety. If $Q$ is a quasivariety of algebras and $V$ is a subvariety of $Q$ then a homomorphism $f : A \to B \in Q$ belongs to the ideal $T(V)$ if and only if $\text{Im}(f) \in V$.

3 Sufficient conditions for $Q$-universality

**Definition.** Let $Q$ be a quasivariety of finitary algebraic systems, let $V$ be a proper subvariety of $Q$ and let $R : Q \to V$ be the corresponding epireflection. For any object $A \in Q$, let $A$ denote the underlying set of $A$ and let $\rho_A : A \to RA$ denote the surjective $Q$-morphism from the epitransformation $\rho$. Let $F : K \to Q$ be a $T(V)$-relatively full embedding. Let $S \in V$ be an algebraic system with the underlying set $S$. We say that $F$ is $S$-synchronized and call $S$ its synchronizer if for every $K$-object $k$ there exists an injective mapping $p_k$ from $S$ to the underlying set of $RFk$ such that $\text{Im}(p_k)$ is an induced subobject of $RFk$ and $p_k$ is an isomorphism of $S$ onto the subobject of $RFk$ on the set $\text{Im}(\mu_k)$, and for every $K$-morphism $f : k_1 \to k_2$ we have

(s1) if $Ff$ is injective on $(\rho_{Fk_1})^{-1}(\text{Im}(\mu_{k_1}))$, then $Ff$ is injective;

(s2) $RFf \circ \mu_{k_1} = \mu_{k_2}$;

(s3) if $Ff \in Surj_Q$ and $A_i$ is the underlying set of $RFk_i$ for $i = 1, 2$, then every mapping $h : A_2 \to A_1$ such that $RFf \circ h = 1_{A_2}$ is a homomorphism from $RFk_2$ to $RFk_1$;

(s4) for every $K$-object $k$, if $s$ is an element of the underlying set of $RFk$ such that $s \notin \text{Im}(\mu_k)$ then $\rho_{Fk}^{-1}\{s\}$ is a singleton.

Next we interpret the condition (s3) for algebras.

**Proposition 3.1.** Let $Q$ be a quasivariety of algebras of a finitary similarity type $\Delta$, let $V$ be a proper subvariety of $Q$ and let $F : K \to Q$ be a functor. Then (s3) holds
exactly when

(\bullet) if \( F \) is surjective and for every \( s \in A_2 \) with \( |RFf^{-1}\{s\}| > 1 \), if \( \sigma_{RFf_2}(a_1, a_2, \ldots, a_n) = s \) for an \( n \)-ary operation \( \sigma \) and \( a_1, a_2, \ldots, a_n \in A_2 \), then \( s = a_{i_0} \) for some \( i_0 \in \{1, 2, \ldots, n\} \) and \( k(s) = \sigma_{RFk_1}(k(a_1), k(a_2), \ldots, k(a_n)) \) for every mapping \( k : \{a_1, a_2, \ldots, a_n\} \to A_1 \) such that \( RFf \circ k(a_i) = a_i \) for all \( i \in \{1, 2, \ldots, n\} \).

Proof. Assume (s3). Let \( s = \sigma_{RFk_2}(a_1, a_2, \ldots, a_n) \) for some \( \sigma \in \Delta \), let \( a_1, a_2, \ldots, a_n, s \in A_2 \) and \( |RFf^{-1}\{s\}| > 1 \). Let \( h : A_2 \to A_1 \) be a mapping such that \( RFf \circ h \) is the identity mapping. Then \( h(s) = \sigma_{RFk_1}(h(a_1), h(a_2), \ldots, h(a_n)) \).

If \( s \notin \{a_1, a_2, \ldots, a_n\} \) then there exists a mapping \( h' : A_2 \to A_1 \) with \( RFf \circ h = RFf \circ h', h(s) \neq h'(s) \) and \( h(t) = h'(t) \) for all \( t \in A_2 \setminus \{s\} \). Hence \( h'(s) \neq \sigma_{RFk_1}(h'(a_1), h'(a_2), \ldots, h'(a_n)) \) and this contradicts the fact that \( h' : RFk_2 \to RFk_1 \) is a homomorphism. Thus there exists \( i_0 \in \{1, 2, \ldots, n\} \) with \( a_{i_0} = s \). If \( k : \{a_1, a_2, \ldots, a_n\} \to A_1 \) is a mapping such that \( RFf \circ k(a_i) = a_i \) for every \( i \in \{1, 2, \ldots, n\} \) then there exists a mapping \( h : A_2 \to A_1 \) such that \( RFf \circ h \) is the identity mapping of \( A_2 \) and \( h(a_i) = k(a_i) \) for all \( i = \{1, 2, \ldots, n\} \). But \( h : RFk_2 \to RFk_1 \) is a homomorphism, by (s3), and hence \( k(s) = \sigma_{RFk_1}(k(a_1), k(a_2), \ldots, k(a_n)) \) because \( s = a_{i_0} \). Whence the condition (\bullet) holds.

For the converse, assume (\bullet) and let \( h : A_2 \to A_1 \) be a mapping such that \( RFf \circ h \) is the identity of \( A_2 \). Choose an \( n \)-ary operation \( \sigma \) of type \( \Delta \) and \( a_1, a_2, \ldots, a_n \in A_2 \). Write \( s = \sigma_{RFk_2}(a_1, a_2, \ldots, a_n) \). First we assume that \( |RFf^{-1}\{s\}| > 1 \). Then (\bullet) gives an \( i_0 \in \{1, 2, \ldots, n\} \) with \( s = a_{i_0} \) and \( h(s) = \sigma_{RFk_1}(h(a_1), h(a_2), \ldots, h(a_n)) \), as required. From \( F \in \text{Sur}_Q \) we infer that \( RFf \in \text{Sur}_Q \), and hence \( |RFf^{-1}\{s\}| = 1 \) is the only remaining case. If

\[
t = \sigma_{RFk_1}(h(a_1), h(a_2), \ldots, h(a_n))
\]

then

\[
RFf(t) = \sigma_{RFk_2}(RFf(h(a_1)), RFf(h(a_2)), \ldots, RFf(h(a_n)))
= \sigma_{RFk_2}(a_1, a_2, \ldots, a_n) = s
\]

and hence \( t = h(s) \). Thus \( h \) is a homomorphism, and the proof is complete. \( \square \)

Remark. Observe that if \( F : \mathbb{K} \to \mathbb{Q} \) is an almost full embedding then \( F \) is synchronized \( \mathbb{T}(\mathbb{T}) \)-relatively full embedding for the trivial variety \( \mathbb{T} \). Indeed, its synchronizer \( S \) is a singleton algebra and \( \mu_K \) is the identity automorphism of \( S \) for every
Clearly, the conditions (s1)-(s4) are satisfied. And $F : K \to Q$ is a full embedding exactly when $F$ is an almost full embedding and for every $K$-object $k$ there exists no $Q$-morphism from the terminal object of $Q$ into $Fk$.

Let $N_0$ be a poset viewed as a category whose objects are sets from the set $\mathcal{P}(\omega_0)$ of all finite subsets of $\omega$ and there exists an $N_0$-morphism from $A \in \mathcal{P}(\omega_0)$ into $B \in \mathcal{P}(\omega_0)$ if and only if $B \subseteq A$. Let $N$ be the full subcategory of $N_0$ whose objects belong to the set $\mathcal{P}(\omega) = \mathcal{P}(\omega_0) \setminus \{\emptyset\}$ of all non-void subsets of $\omega$. For $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$, let $\eta_{A,B}$ denote the unique $N$-morphism from $A$ to $B$.

**Theorem 3.2.** Let $Q$ be a quasivariety of finitary algebras and let $V$ be a subvariety of $Q$. If there exists a synchronized $I(V)$-relatively full embedding $F : N \to Q$ such that

1. $FA$ is a finite algebra for every $A \in \mathcal{P}(\omega)$;
2. $F\eta_{A,B} \in \text{Surj}_Q$ for every $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$ (then $RF\eta_{A,B}$ is a retract);
3. if $A = B \cup C$ for $A, B, C \in \mathcal{P}(\omega)$ then $\{F\eta_{A,B}, F\eta_{A,C}\}$ is a separating family.

Then $\{S_A \mid A \in \mathcal{P}(\omega_0)\}$ is an $A$-$D$ family where $S_0$ is a singleton algebra in $Q$ and $S_A = FA$ for all $A \in \mathcal{P}(\omega)$.

**Proof.** We need to prove (p1)-(p4). Clearly, (p1) is satisfied. To prove (p2), consider sets $A, B, C \in \mathcal{P}(\omega)$ with $A = B \cup C$. By (3), $\{F\eta_{A,B}, F\eta_{A,C}\}$ is a separating family and thus $FA$ is a subobject of $FB \times FC$. Hence we obtain $FA \in \text{Qua}\{FB, FC\}$ and the proof of (p2) is complete.

For every $A \in \mathcal{F}(\omega)$, let $\rho_A : FA \to RFA$ denote the epireflection homomorphism of $FA$ into $V$. Then $\rho_A \in \text{Surj}_Q$.

To prove (p3), let $A, B \in \mathcal{P}(\omega)$ be such that $FA \in \text{Qua}\{FB\}$. By the hypothesis, $FB$ is finite, so that the family of all homomorphisms from $FA$ to $FB$ is separating. Since $F$ is $I(V)$-relatively full embedding we infer that if $B \nsubseteq A$ then every homomorphism from $FA$ into $FB$ factorizes through $\rho_A$. Since $FA \notin V$ and $RFA \in V$, the mapping $\rho_A$ is not injective and thus $FA \notin \text{Qua}\{FB\}$ – a contradiction. Thus we can assume that $B \subseteq A$. If $f : FA \to FB$ is a homomorphism then either $h = F\eta_{A,B}$ or $h$ factorizes through $\rho_A$ because $F$ is $I(V)$-relatively full embedding. Since the family of all homomorphisms from $FA$ to $FB$ is separating, the pair $\{F\eta_{A,B}, \rho_A\}$ must be a separating family. We claim that this is impossible...
when $B \neq A$. Indeed, if $B \neq A$ then $F\eta_{A,B}$ is not injective; this is because from (2) it would follow that $F\eta_{A,B}$ is an isomorphism, contrary to the relative fulness of $F$. But then $F\eta_{A,B}$ is not injective on $(\rho_A)^{-1}(\text{Im}(\mu_A))$ by (s1) and hence, by (s2), for some $s \in S$ there are distinct $a, b \in \rho_A^{-1}\{s\}$ with $F\eta_{A,B}(a) = F\eta_{A,B}(b)$. Hence $\{F\eta_{A,B}, \rho_A\}$ is not a separating family, a contradiction. Thus $A = B$, and (p3) follows.

To prove (p4), let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a finite set and let $B, C \in \text{Qua}\{FX \mid X \in \mathcal{F}\}$ be finite algebras such that there exist $A \in \mathcal{P}(\omega)$ and an injective homomorphism $f : FA \to B \times C$. Hence there exist finite separating families $\{g_i : B \to FX_i \mid i \in I\}$ and $\{h_j : C \to FY_j \mid j \in J\}$ such that $X_i, Y_j \in \mathcal{P}(\omega)$ for all $i \in I$ and $j \in J$. Let $\pi_1 : B \times C \to B, \pi_2 : B \times C \to C$ be projections.

First we prove that we can assume that $\pi_1 \circ f, \pi_2 \circ f \in \text{Surj}_Q$. So assume that (p4) is satisfied if $\pi_1 \circ f, \pi_2 \circ f \in \text{Surj}_Q$. By the factorization property, there exist homomorphisms

$$f'_1 : FA \to B' \in \text{Surj}_Q, f''_1 : B' \to B \in \text{Inj}_Q,$$

$$f'_2 : FA \to C' \in \text{Surj}_Q, f''_2 : C' \to C \in \text{Inj}_Q$$

with $\pi_1 \circ f = f''_1 \circ f'_1$ and $\pi_2 \circ f = f''_2 \circ f'_2$. Since $f$ is injective we infer that $\{\pi_1 \circ f, \pi_2 \circ f\}$ is separating and hence $\{f'_1, f''_1\}$ is also separating. Thus there exists an injective homomorphism $f' : FA \to B' \times C'$ with $\pi'_1 \circ f' = f'_1$ and $\pi'_2 \circ f' = f'_2$ where $\pi'_1 : B' \times C' \to B'$ and $\pi'_2 : B' \times C' \to C'$ are projections. Then $\{g_i \circ f''_1 : B' \to FX_i \mid i \in I\}$ and $\{h_j \circ f''_2 : C' \to FY_j \mid j \in J\}$ are separating families and, by the assumption, the condition (p4) is satisfied for $f', B'$ and $C'$ because $\pi'_1 \circ f', \pi'_2 \circ f' \in \text{Surj}_Q$. Then (p4) is also satisfied for $f, B$ and $C$ because $f'_1 : B' \to B, f''_1 : C' \to C \in \text{Inj}_Q$. Thus with no loss of generality we can assume that $\pi_1 \circ f, \pi_2 \circ f \in \text{Surj}_Q$.

Let us define $I' = \{i \in I \mid g_i \circ \pi_1 \circ f = F\eta_{A,X_i}\}$ and $J' = \{j \in J \mid g_j \circ \pi_2 \circ f = F\eta_{A,Y_j}\}$. Then $X_i \subseteq A$ and $Y_j \subseteq A$ for all $i \in I$ and $j \in J$. Observe that $g_i \circ \pi_1 \circ f$ and $g_j \circ \pi_2 \circ f$ factorize through $\rho_A$ for all $i \in I \setminus I'$ and $j \in J \setminus J'$ because $F$ is $\mathcal{I}(\mathcal{V})$-relatively full embedding. Hence $I' \neq \emptyset$ or $J' \neq \emptyset$.

Set $U = \bigcup_{i \in I'} X_i$ and $V = \bigcup_{j \in J'} Y_j$. Then $U \cup V \subseteq A$ and $g_i \circ \pi_1 \circ f$ factorizes through $F(\eta_{A,U})$ for all $i \in I'$ and $g_j \circ \pi_2 \circ f$ factorizes through $F(\eta_{A,V})$ for all $j \in J'$. Since $\{g_i \circ \pi_1 \circ f \mid i \in I\} \cup \{g_j \circ \pi_2 \circ f \mid j \in J\} \in \text{Inj}_Q$ we infer, by (p3), that if $J' = \emptyset$ then $U = A$, if $I' = \emptyset$ then $V = A$, if $I' \neq \emptyset \neq J'$ then $A = U \cup V$.

Assume that $I' \neq \emptyset$. Since $\pi_1 \circ f \in \text{Surj}_Q$, $\{F\eta_{U,X_i} \mid i \in I'\} \in \text{Inj}_Q$ by (3) and $g_i \circ \pi_1 \circ f = F\eta_{U,X_i} \circ F\eta_{A,U}$ for all $i \in I$, by the diagonalization property there exists a homomorphism $\psi : B \to FU$ with $\psi \circ \pi_1 \circ f = F\eta_{A,U}$ and $F\eta_{U,X_i} \circ \psi = g_i$. - 298 -
for all \( i \in I' \). From \( F\eta_{A,U} \in \text{Surj}_Q \) it follows that \( \psi \in \text{Surj}_Q \).

Since \( \{ g_i \mid i \in I \} \) is a separating family, for distinct \( u, v \in FA \) we have \( \pi_1 \circ f(u) \neq \pi_1 \circ f(v) \) if and only if there exists \( i \in I \) with \( g_i \circ \pi_1 \circ f(u) \neq g_i \circ \pi_1 \circ f(v) \). If \( i \in I' \) then \( g_i \circ \pi_1 \circ f = F\eta_{A,X_i} = F\eta_{U,X_i} \circ F\eta_{A,U} \). Thus if \( F\eta_{A,U}(u) \neq F\eta_{A,U}(v) \) for \( u, v \in FA \) then \( \pi_1 \circ f(u) \neq \pi_1 \circ f(v) \). If \( i \in I \setminus I' \) then \( g_i \circ \pi_1 \circ f = h \circ \rho_A \) for some homomorphism \( h \) and thus \( \pi_1 \circ f(u) \neq \pi_1 \circ f(v) \) implies that \( \rho_A(u) \neq \rho_A(v) \) or \( F\eta_{A,U}(u) \neq F\eta_{A,U}(v) \) because \( \{ F\eta_{U,X_i} \mid i \in I' \} \) is a separating family.

Let \( S \) be a synchronizer of \( F \). Consider \( t \in \rho_A^{-1}(\text{Im}(\mu_A)) \) and \( u \in FA \setminus \rho_A^{-1}(\text{Im}(\mu_A)) \). Then \( \rho_A(t) = \mu_A(s) \) for some \( s \in S \). By (s2), \( \mu_U \circ \psi \circ \pi_1 \circ f(t) = \mu_U \circ F\eta_{A,U}(t) = \mu_U(s) \) and \( \mu_U \circ \psi \circ \pi_1 \circ f(u) = \mu_U \circ F\eta_{A,U}(u) \notin \text{Im}(\mu_U) \). Hence \( \psi^{-1}(\rho_U^{-1}(\text{Im}(\mu_U))) = \pi_1 \circ f(\rho_A^{-1}(\text{Im}(\mu_A))) \). If we combine this fact with the foregoing argument we conclude that for \( u, v \in \rho_A^{-1}(\text{Im}(\mu_A)) \) we have \( \pi_1 \circ f(u) = \pi_1 \circ f(v) \) if and only if \( F\eta_{A,U}(u) = F\eta_{A,U}(v) \). From (s2) it follows that \( (RF\eta_{A,U})^{-1}(\mu_U(s)) = \{ \mu_A(s) \} \) for all \( s \in S \). Thus \( (R\psi)^{-1}(\mu_U(s)) = \{ \rho_A \circ f(\mu_A(s)) \} \) for every \( s \in S \) because \( \psi \circ \pi_1 \circ f = F\eta_{A,U} \). Since \( F\eta_{A,U} \) is surjective, by (s3), every mapping \( \nu' \) from the underlying set of \( RFU \) into the underlying set of \( RF^A \) such that \( RF\eta_{A,U} \circ \nu' \) is the identity mapping is a homomorphism from \( RFU \) into \( RF^A \). From \( \phi \circ \pi_1 \circ f = F\eta_{A,U} \) we conclude \( R(\psi \circ \pi_1 \circ f) = RF\eta_{A,U} \).

For a homomorphism \( \nu' : RFU \rightarrow RF^A \) such that \( RF\eta_{A,U} \circ \nu' \) is the identity automorphism of \( RFU \) we set \( \nu = R(\pi_1 \circ f) \circ \nu' \) and hence \( \nu : RFU \rightarrow RB \) is a homomorphism such that \( R\psi \circ \nu \) is the identity homomorphism of \( RFU \). Since \( \nu' \) exists by (s3), we can assume that we have a homomorphism \( \nu : RFU \rightarrow RB \) such that \( R\psi \circ \nu \) is the identity homomorphism of \( RFU \).

For every \( i \in I \setminus I' \) there exists a homomorphism \( \tilde{g}_i : RFA \rightarrow FX_i \) with \( g_i \circ \pi_1 \circ f = \tilde{g}_i \circ \rho_A \). By the properties of factorization systems, there exist homomorphisms \( \sigma : RFA \rightarrow D \in \text{Surj}_Q \) and \( \sigma_i : D \rightarrow FX_i \) for \( i \in I \setminus I' \) such that \( g_i \circ \pi_1 \circ f = \sigma_i \circ \sigma \circ \rho_A \) for all \( i \in I \setminus I' \) and \( \{ \sigma_i \mid i \in I \setminus I' \} \in \text{Inj}_Q \). By the diagonalization property, there exists a homomorphism \( \phi' : B \rightarrow D \) such that \( \phi' \circ \pi_1 \circ f = \sigma \circ \rho_A \) and \( \sigma_i \circ \phi' = g_i \) for all \( i \in I \setminus I' \). From \( \rho_A, \sigma \in \text{Surj}_Q \) it follows that \( \phi' \in \text{Surj}_Q \).

From \( RFA \in V \) and \( \sigma : RFA \rightarrow D \in \text{Surj}_Q \) it follows that \( D \in V \) and if \( \rho_B : B \rightarrow RB \) is the epireflection morphism of \( B \) into \( V \), then there exists a homomorphism \( \phi : RB \rightarrow D \in \text{Surj}_Q \) with \( \phi' = \phi \circ \rho_B \). Then

\[
\sigma \circ \rho_A = \phi' \circ \pi_1 \circ f = \phi \circ \rho_B \circ \pi_1 \circ f = \phi \circ R(\pi_1 \circ f) \circ \rho_A
\]

and \( \sigma = \phi \circ R(\pi_1 \circ f) \) follows because \( \rho_A \in \text{Surj}_Q \). Since \( \{ g_i \mid i \in I \} \in \text{Inj}_Q \) we infer that the family \( \{ \psi, \rho_B \} \) is separating. Hence there exists a homomorphism
ω : B → FU × RB ∈ InjQ such that τ1 ◦ ω = ψ and τ2 ◦ ω = ρB where τ1 : FU × RB → FU and τ2 : FU × RB → RB are projections. Then τ1 ◦ ω ◦ π1 ◦ f = ψ ◦ π1 ◦ f = FηA,U and τ2 ◦ ω ◦ π1 ◦ f = ρB ◦ π1 ◦ f = R(π1 ◦ f) ◦ ρA. Hence for every b ∈ B and a ∈ FA with π1 ◦ f(a) = b we have ω(b) = (FηA,U(a), R(π1 ◦ f) ◦ ρA(a)). By the property of products, there exists a homomorphism λ : FU → FU × RB such that τ1 ◦ λ is the identity morphism of FU and τ2 ◦ λ = ν ◦ ρU, hence λ ∈ InjQ. Select u ∈ FU. If ρU(u) ∈ Im(μU) then, by (s2), ρA((FηA,U)−1(u)) = (RFηA,U)−1(ρU(u)) is a singleton and hence for every a ∈ FA with FηA,U(a) = u we have {R(π1 ◦ f) ◦ ρA(a)} = (Rψ)−1(ρU(u)) = {ν(ρU(u))}. Thus λ(u) = (FηA,U(a), R(π1 ◦ f) ◦ ρA(a)) ∈ Im(ω). If ρU(u) ∉ Im(μU), then there exists a ∈ FA such that ρB ◦ π1 ◦ f(a) = ν(ρU(u)) because ρB, π1 ◦ f ∈ SurjQ. Then

\[ Rψ ◦ ρB ◦ π1 ◦ f(a) = Rψ ◦ ν(ρU(u)) = ρU(u). \]

Since

\[ Rψ ◦ ρB ◦ π1 ◦ f = ρU ◦ ψ ◦ π1 ◦ f = ρU ◦ FηU,A \]

we conclude that ρU(u) = ρU(FηA,U(a)) and, by (s4), u = FηA,U(a). Thus λ(u) = (FηA,U(a), R(π1 ◦ f) ◦ ρA(a)) ∈ Im(ω) because R(π1 ◦ f) ◦ ρA = ρB ◦ π1 ◦ f. Thus Im(λ) ⊆ Im(ω), so that there exists an injective homomorphism from FU to B.

If J′ ≠ ∅ then the same proof gives the existence of an injective ν : FV → C, and (p4) follows. □

The technical statement below enables us to prove a generalized version of Theorem 3.2. We say that a surjective homomorphism f : A → B of algebraic systems of similarity type Δ is a quotient if for every relation r ∈ Δ we have that (b0, b1, . . . , bk) ∈ rB if and only if there exists (a0, a1, . . . , ak) ∈ rA with f(ai) = bi for all i = 0, 1, . . . , k. A quasivariety Q is closed under quotients if an algebraic system A ∈ Q whenever there exist an algebraic system B ∈ Q and a quotient f : B → A. Let QuotQ denote the class of all quotients of Q. It is well-known [1] that (QuotQ, InjQ) is a factorization system in Q, and that SurjQ = QuotQ if Q is a quasivariety of algebras. If Q is clear from the context, we write Quot instead of QuotQ.

**Proposition 3.3.** Let Q be a quasivariety of algebraic systems and let R be a proper subquasivariety of Q closed under quotients. If there exists an I(R)-relatively full embedding F : N → Q such that FA is finite for all A ∈ P(ω) and FηA,B ∈ QuotQ for all A, B ∈ P(ω) with B ⊆ A, then there exists an I(R)-relatively full embedding G : N → Q such that
KOUBEK & SICHLER - ON SYNCHRONIZED RELATIVELY FULL EMBEDDINGS;

(1) $GA$ is finite for all $A \in \mathcal{P}(\omega)$;

(2) if $A, B, C \in \mathcal{P}(\omega)$ satisfy $B \cup C \subseteq A$, then $\{G\eta_{A,B}, G\eta_{A,C}\}$ is a separating family if and only if $A = B \cup C$;

(3) $G\eta_{A,B} \in \text{Quot}_Q$ for all $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$.

Moreover, if $Q$ is a quasivariety of algebras and $F$ is synchronized then $G$ is synchronized.

The fairly technical proof of this Proposition can be found in the Appendix.

**Proof of Theorem 1.4 completed.** Let $\mathcal{G}RA$ denote the (concrete) category of all undirected graphs and compatible mappings. We recall that there exists a full embedding $\Phi$ of $\mathbb{N}$ into $\mathcal{G}RA$ such that $\Phi A$ is a finite graph of every $A \in \mathbb{N}$ and $\Phi\eta_{A,B} \in \text{Quot}_{\mathcal{G}RA}$ for every $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$, see [7]. Let $F : \mathcal{G}RA \to Q$ satisfy the hypothesis of Theorem 1.4. Then the composite $F \circ \Phi : \mathbb{N} \to Q$ satisfies the hypothesis of Proposition 3.3, and hence $Q$ contains an $A$-$D$ family, by Theorem 3.2. This concludes the proof of Theorem 1.4. □

**Remark.** The embeddings from $\mathcal{G}RA$ into the variety of semigroups generated by $M_2$ or $M_3$ or $M_3^d$ or $M_4$ or $M_4^d$ constructed in [6, 7, 8] are synchronized (here for a semigroup $S = (S, \cdot)$, its dual is defined as $S^d = (S, \circ)$ with $s \circ t = t \cdot s$ for all $s, t \in S$) and constitute special cases of Theorem 3.2. The semigroups $M_2, M_3$ and $M_4$ are defined in Table 1.

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Table 1: The semigroups $M_2$, $M_3$ and $M_4$

Finally, we show that for quasivarieties of algebras Theorem 1.4 generalizes Theorem 1.3 of [16]. So let $Q$ be a quasivariety of algebras and let $V$ be a proper subvariety of $Q$. We say that an epireflection $R : Q \to V$ is **constant on** a functor $F : \mathbb{N} \to Q$ if the composite $R \circ F$ is a constant functor. It is then clear that if
the epireflection $R$ is constant on an $\mathcal{I}(V)$-relatively full embedding $F$, then $F$ is synchronized. Thus we immediately obtain

**Corollary 3.4.** Let $Q$ be a quasivariety of algebras and let $V$ be a proper subvariety of $Q$. If $F : N \rightarrow Q$ is an $\mathcal{I}(V)$-relatively full embedding such that the epireflection of $Q$ into $V$ is constant on $F$, $F\eta_{A,B} \in \text{Sur}_Q$ for all $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$ and $FA$ is finite for all $A \in \mathcal{P}(\omega)$ then there exists an $A$-$D$ family in $Q$, and thus $Q$ is $Q$-universal.

Thus, in particular, the object ideal $\mathcal{I}(V)$ associated with such an $\mathcal{I}(V)$-relatively full embedding $F$ is *principal* in the sense that it is determined by a single object of $V$ and includes the case when the synchronizer is a singleton algebra, that is, the case of an almost full embedding.

**Appendix**

*Proof of Proposition 3.3.* Consider a functor $H : N_0 \rightarrow N$ defined by $H\emptyset = \{0\}$ and $HA = \{0\} \cup \{n + 1 \mid n \in A\}$ for all $A \in \mathcal{P}(\omega)$ and $H\eta_{A,B} = \eta_{HA,HB}$ for $A, B \in \mathcal{P}(\omega_0)$ with $B \subseteq A$. Then $H$ is a full embedding (since $A \subseteq B$ if and only if $HA \subseteq HB$ for $A, B \in \mathcal{P}(\omega_0)$, it is correctly defined). Thus the composite $F' = F \circ H : N_0 \rightarrow Q$ is an $\mathcal{I}(\mathbb{R})$-relatively full embedding such that $F'A$ is finite for all $A \in \mathcal{P}(\omega)$ and $F'\eta_{A,B} = F\eta_{HA,HB} \in \text{Quot}_Q$ for all $A, B \in \mathcal{P}(\omega_0)$ with $B \subseteq A$.

Since $F'$ is an $\mathcal{I}(\mathbb{R})$-relatively full embedding, $F'A \notin \mathbb{R}$ for all $A \in \mathcal{P}(\omega_0)$. For $n \in \omega$, set $G\{n\} = F'\{n\}$. For $A \in \mathcal{P}(\omega)$, define $\Pi(A) = \prod_{a \in A} F'\{a\}$ and let $\pi_a : \Pi(A) \rightarrow F'\{a\}$ be the $a$-th projection for each $a \in A$. By the universal property of products, there exists a unique homomorphism $\tau_A : F'A \rightarrow \Pi(A)$ such that $F'\eta_{A,\{a\}} = \pi_a \circ \tau_A$ for every $a \in A$. Factorizing $\tau_A$ in $Q$ in the factorization system $(\text{Quot}_Q, \text{Inj}_Q)$, we obtain homomorphisms (unique up to an isomorphism) $\chi_A : F'A \rightarrow GA \in \text{Quot}_Q$ and $\mu_A : GA \rightarrow \Pi(A) \in \text{Inj}_Q$ such that $\tau_A = \mu_A \circ \chi_A$. Since the underlying set of $F'A$ is finite and since $\chi_A$ is a quotient, the underlying set of $GA$ is finite for all $A \in \mathcal{P}(\omega)$. This proves (1).

Consider $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$. By the universal property of products, there exists a unique homomorphism $\Pi(\eta_{A,B}) : \Pi(A) \rightarrow \Pi(B)$ such that $\pi_b = \kappa_b \circ \Pi(\eta_{A,B})$ for all $b \in B \subseteq A$, where $\kappa_b : \Pi(B) \rightarrow F'\{b\}$ is the $b$-th projection for $b \in B$. Then for every $b \in B$ we have

$$\kappa_b \circ \Pi(\eta_{A,B}) \circ \tau_A = \pi_b \circ \tau_A = F'\eta_{A,\{b\}} = F'\eta_{B,\{b\}} \circ F'\eta_{A,B}$$

$$\Rightarrow \kappa_b \circ \tau_B \circ F'\eta_{A,B}$$
because \( \kappa_b \circ \tau_B = F' \eta_B, \{b\} \), and hence

\[
\Pi(\eta_{A,B}) \circ \mu_A \circ \chi_A = \Pi(\eta_{A,B}) \circ \tau_A = \tau_B \circ F' \eta_{A,B} = \mu_B \circ \chi_B \circ F' \eta_{A,B}
\]

because the family \( \{\kappa_b \mid b \in B\} \) of projections is separating.

By the diagonalization property, there exists a homomorphism \( G \eta_{A,B} : GA \rightarrow GB \) with \( G \eta_{A,B} \circ \chi_A = \chi_B \circ F' \eta_{A,B} \) and \( \Pi(\eta_{A,B}) \circ \mu_A = \mu_B \circ G \eta_{A,B} \) because \( \mu_B \in \text{Inj} \) and \( \chi_A \in \text{Quot} \). From \( \chi_B \circ F' \eta_{A,B} \in \text{Quot} \) it follows that \( \chi_B \circ F' \eta_{A,B} \in \text{Quot} \) and \( G \eta_{A,B} \in \text{Quot} \), and (3) is proved. Note the diagram below, commuting for every \( b \in B \subseteq A \).

\[
\begin{array}{ccc}
F' A & \xrightarrow{\chi_A} & GA \\
\downarrow \quad G \eta_{A,B} & & \downarrow \Pi(\eta_{A,B}) \\
F' B & \xrightarrow{\chi_B} & GB \\
\end{array}
\]

To prove that \( G \) is a functor, let \( A, B, C \in \mathcal{P}(\omega) \) satisfy \( C \subseteq B \subseteq A \). Then

\[
G \eta_{B,C} \circ G \eta_{A,B} \circ \chi_A = G \eta_{B,C} \circ \chi_B \circ F' \eta_{A,B} = \chi_C \circ F' \eta_{B,C} \circ F' \eta_{A,B} = \chi_C \circ \chi_B \circ \chi_A = G \eta_{A,C} \circ \chi_A
\]

and because \( \chi_A \in \text{Quot} \) we conclude that \( G \eta_{B,C} \circ G \eta_{A,B} = G \eta_{A,C} \). Since \( F' \eta_{A,A} \) is the identity homomorphism, from \( G \eta_{A,A} \circ \chi_A = \chi_A \circ F' \eta_{A,A} = \chi_A \in \text{Quot} \) it follows that \( G \eta_{A,A} \) is also the identity homomorphism. Altogether, \( G \) is a functor.

We turn to (2). Note that \( F' \eta_{A,\{a\}} = \pi_a \circ \tau_A = \pi_a \circ \mu_A \circ \chi_A \) and \( G \eta_{A,\{a\}} \circ \chi_A = F' \eta_{A,\{a\}} \) for every \( a \in A \) because \( \chi(\{a\}) \) is the identity morphism of \( F' \{a\} = G \{a\} \).

From \( \chi_A \in \text{Quot} \) we then obtain \( G \eta_{A,\{a\}} = \pi_a \circ \mu_A \) for every \( a \in A \). But then \( \{G \eta_{A,\{a\}} \mid a \in A\} \) is a separating family because \( \mu_A \in \text{Inj} \) and the family \( \{\pi_a \mid a \in A\} \) of projections is separating. Hence \( \{G \eta_{A,B}, G \eta_{A,C}\} \) is a separating family for any \( A, B, C \in \mathcal{P}(\omega) \) with \( A = B \cup C \). Conversely, assume that \( B \cup C \subseteq A \) and \( \{G \eta_{A,B}, G \eta_{A,C}\} \) is a separating family. Then \( \{G \eta_{A,\{a\}} \mid a \in B \cup C\} \) is clearly a separating family. Set \( A' = B \cup C \). Then \( G \eta_{A,A'} \in \text{Inj} \) and thus from the already proved (3) it follows that \( G \eta_{A,A'} \) is an isomorphism. Choose \( a \in A \setminus A' \).

Since \( G \eta_{A,\{a\}} \circ \chi_A = F' \eta_{A,\{a\}} \in \text{Quot} \), we have \( G \eta_{A,\{a\}} = \pi_a \circ \mu_A \in \text{Quot} \). But then \( \pi_a \circ \mu_A \circ (G \eta_{A,A'})^{-1} \circ \chi_{A'} : F' A' \rightarrow F' \{a\} \) is a quotient because \((G \eta_{A,A'})^{-1} \chi_{A'} \in \text{Quot} \). This is a contradiction because \( F' \) is an \( \mathcal{I}(\mathbb{R}) \)-relatively full embedding, \( F' \{a\} \) does not belong to \( \mathbb{R} \) and \( \{a\} \not\subseteq A' \). Hence (2) follows.

To prove that \( G \) is an \( \mathcal{I}(\mathbb{R}) \)-relatively full embedding consider \( A, B \in \mathcal{P}(\omega) \) with \( B \subseteq A \). Then \( \eta_{A,B} \) is a morphism of \( \mathbb{N} \) and we must prove that \( \text{Im}(G \eta_{A,B}) \not\subseteq \)
For every \( b \in B \), \( \eta_{B,(b)} \in \text{Quot} \) and \( G\{b\} \notin \mathbb{R} \). Since \( \mathbb{R} \) is closed under \( \text{Quot} \), we infer that \( GB \notin \mathbb{R} \) and because \( \eta_{A,B} \in \text{Quot} \) we conclude that \( \text{Im}(\eta_{A,B}) \notin \mathbb{R} \). Conversely, let \( f : GA \to GB \) for \( A, B \in \mathcal{P}(\omega) \) be a homomorphism such that \( \text{Im}(f) \notin \mathbb{R} \). To complete the proof it suffices to prove that \( B \subseteq A \) and \( f = \eta_{A,B} \). Let \( f' : A \to C \in \text{Quot} \) and \( f'' : C \to B \in \text{Inj} \) be homomorphisms with \( f = f'' \circ f' \) then \( C \) is isomorphic to \( \text{Im}(f) \). Since \( \{ \eta_{B,(b)} \mid b \in B \} \) is a separating family, we infer that \( \{ \eta_{B,(b)} \circ f'' \mid b \in B \} \) is a separating family and, by the universal property of products, the morphism \( h : C \to \prod_{b \in B} \text{Im}(\eta_{B,(b)} \circ f'') \in \text{Inj} \). Since \( \text{Im}(f) \notin \mathbb{R} \) we conclude that \( \prod_{b \in B} \text{Im}(\eta_{B,(b)} \circ f'') \notin \mathbb{R} \), and thus there exists \( b \in B \) such that \( \text{Im}(\eta_{B,(b)} \circ f'') = \text{Im}(\eta_{B,(b)} \circ f) \notin \mathbb{R} \). Thus \( \eta_{B,(b)} \circ f \notin \mathcal{I}(\mathbb{R}) \). Since \( \chi_A \in \text{Quot} \) we conclude that \( \eta_{B,(b)} \circ f \circ \chi_A : F'A \to F''\{b\} \notin \mathcal{I}(\mathbb{R}) \) and thus \( b \in A \) and \( \eta_{B,(b)} \circ f \circ \chi_A = F''\eta_{A,(b)} \) because \( F' \) is an \( \mathcal{I}(\mathbb{R}) \)-relatively full embedding. Then

\[
F'\eta_{(b),0} \circ \eta_{B,(b)} \circ f \circ \chi_A = F'\eta_{(b),0} \circ F'\eta_{A,(b)} = F'\eta_{A,0}.
\]

Since for every \( b' \in B \) we have

\[
F'\eta_{(b'),0} \circ \eta_{B,(b')} \circ \chi_B = F'\eta_{(b'),0} \circ F'\eta_{B,(b')}
= F'\eta_{B,(b')} \circ F'\eta_{B,(b')}
= F'\eta_{(b'),0} \circ G_{B,(b')} \circ \chi_B
\]

we infer that \( F'\eta_{(b'),0} \circ \eta_{B,(b')} = F'\eta_{(b'),0} \circ G_{B,(b')} \) for all \( b' \in B \) because \( \chi_B \in \text{Quot} \). From this it follows that

\[
F'\eta_{(b),0} = F'\eta_{(b),0} \circ G_{B,(b)} \circ f \circ \chi_A = F'\eta_{(b),0} \circ G_{B,(b')} \circ f \circ \chi_A
\]

for all \( b' \in B \). Since \( F'\eta_{A,0} \notin \mathcal{I}(\mathbb{R}) \) we conclude that \( G_{B,(b')} \circ f \circ \chi_A \notin \mathcal{I}(\mathbb{R}) \) for all \( b' \in B \) because \( F'\eta_{(b'),0} \in \text{Quot} \) and \( \mathbb{R} \) is closed under \( \text{Quot} \). Hence \( b' \in A \) and \( G_{B,(b')} \circ f \circ \chi_A = F'\eta_{A,(b')} \) for all \( b' \in B \) because \( F' \) is an \( \mathcal{I}(\mathbb{R}) \)-relatively full embedding. Thus \( B \subseteq A \) and

\[
G_{B,(b')} \circ \eta_{A,B} \circ \chi_A = G_{B,(b')} \circ \chi_B \circ F'\eta_{A,B}
= F'\eta_{A,(b')} \circ F'\eta_{A,B}
= F'\eta_{A,(b')} \circ G_{B,(b')} \circ f \circ \chi_A
\]

for all \( b' \in B \). By (2), \( \{ \eta_{B,(b')} \mid b' \in B \} \in \text{Inj} \) and thus \( \eta_{A,B} \circ \chi_A = f \circ \chi_A \). But \( \chi_A \in \text{Quot} \), and this completes the proof that \( f = \eta_{A,B} \).

It remains to prove that if \( Q \) is a quasivariety of algebras and \( F \) is synchronized then also \( G \) is synchronized. First observe that \( F' \) is also synchronized. For \( A \in \mathcal{I}(\mathbb{R}) \)
\(\mathcal{P}(\omega)\) let \(\rho_{F' A}\) and \(\rho_{GA}\) be the respective epireflection morphisms of \(F' A\) and \(GA\).

Let \(S\) be an algebra and for \(A \in \mathcal{P}(\omega)\) let \(\nu_A : S \rightarrow RF'A\) witness the fact that \(F'\) is synchronized. Since for every \(a \in A\) we have \(F'\eta_{A,\{a\}} = G\eta_{A,\{a\}} \circ \chi_A\) we conclude that \(RF'\eta_{A,\{a\}} = R(G\eta_{A,\{a\}} \circ \chi_A)\). Set \(\zeta_A = R\chi_A \circ \nu_A : S \rightarrow RGA,\) then the property that for every \(s \in S\) and \(A, B \in \mathcal{P}(\omega)\) with \(B \subseteq A\) we have \(RF'\eta_{A,B}(\nu_A(s)) = \nu_B(s)\) implies \(RG\eta_{A,B}(\zeta_A(s)) = \zeta_B(s)\) and the fact that \(\nu_A\) is injective for every \(A \in \mathcal{P}(\omega)\) and \(\chi_{\{a\}}\) is the identity mapping for every \(a \in \omega\) imply that \(\zeta_A\) is injective for all \(A \in \mathcal{P}(\omega)\). The validity of (s1) and (s2) for \(F'\) implies that \(G\) also satisfies (s1) and (s2). From the facts that \(F'\) satisfies (s4) and \(\zeta_{\{a\}} = \nu_{\{a\}}\) for all \(a \in \omega\) and \(\{G\eta_{A,\{a\}} \mid a \in A\}\) is a separating family for all \(A \in \mathcal{P}(\omega)\) it follows that (s4) holds for \(G\). Indeed, if \(u\) and \(v\) are distinct elements of \(RGA\) with \(\rho_{GA}(u), \rho_{GA}(v) \notin \text{Im}(\zeta_A)\) then there exists \(a \in A\) with \(F'\eta_{A,\{a\}}(u) \neq F'\eta_{A,\{a\}}(v)\) and hence \(\rho_{G\{a\}} \circ F'\eta_{A,\{a\}}(u) \neq \rho_{G\{a\}} \circ F'\eta_{A,\{a\}}(v)\).

Then \(\rho_{G\{a\}} \circ F'\eta_{A,\{a\}} = RF'\eta_{A,\{a\}} \circ \rho_{GA}\) implies that \(\rho_{GA}(u) \neq \rho_{GA}(v)\). If \(u\) and \(v\) are elements of \(RGA\) with \(\rho_{GA}(u) \notin \text{Im}(\zeta_A)\) and \(v \in \text{Im}(\zeta_A)\) then, by the same argument, we obtain that \(\rho_{GA}(u) \neq \rho_{GA}(v)\) and hence \(GA\) satisfies (s4). To prove (s3) consider \(A, B \in \mathcal{P}(\omega)\) with \(B \subseteq A\). Choose \(b \in B\). Since \(F'\) satisfies (s3), the condition (●) from Proposition 3.1 is satisfied for \(F'\eta_{A,B}\) and \(F'\eta_{B,b}\). Since \(\chi\) is a surjective natural transformation from \(F'\) onto \(G\) and since \(F'\{b\} = G\{b\}\) we conclude, by Proposition 3.1, that every mapping \(h\) from the underlying set of \(RGB\) into \(RGA\) such that \(RG\eta_{A,B} \circ h\) is the identity mapping is a homomorphism from \(RGA\) into \(RGB\). Thus \(G\) satisfies (s3) and whence \(G\) is synchronized. \(\square\)

References


- 305 -

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