Marek Golasiński
Luciano Stramaccia

Weak homotopy equivalences of mapping spaces and Vogt’s lemma

Cahiers de topologie et géométrie différentielle catégoriques, tome 49, no 1 (2008), p. 69-80

<http://www.numdam.org/item?id=CTGDC_2008__49_1_69_0>
WEAK HOMOTOPY EQUIVALENCES OF MAPPING SPACES AND VOGT'S LEMMA

by Marek GOLASIŃSKI* and Luciano STRAMACCIA

RESUME. Dans cet article les auteurs présentent une caractérisation des équivalences de forme (au sens de la 'shape theory') et de forme forte dans le cadre général d'une Top-catégorie C tensorisée et co-tensorisée. Le cas des équivalences de forme équivariante est aussi considéré.

ABSTRACT. In this paper we give characterizations of shape and strong shape equivalences in the general setting of a tensored and cotensored Top-category C. The case of equivariant shape equivalences is also considered.

Introduction

A strong shape equivalence [20] is a map inducing an isomorphism in the strong shape category $s\text{Sh}(\text{Top}, \text{ANR})$. It turns out [24] that $f : X \to Y$ is such a map if it gives (by composition) an equivalence

*The first author would like to thank INDAM for a financial support and the Dipartimento di Matematica e Informatica, Università di Perugia, for its hospitality during his staying on September 15 - October 15, 2006.
\[ f^*_Z : \mathbb{Gd}(Y, Z) \to \mathbb{Gd}(X, Z) \] between fundamental groupoids, for all \( Z \in \text{Ob ANR} \).

J. Dydak and S. Nowak [11] provide a geometric explanation of strong shape theory and give a fairly simple way of introducing the strong shape category formally. There, those methods are applied to present a list of equivalent conditions for a map \( f : X \to Y \) of \( k \)-spaces (compactly generated Hausdorff spaces) to be a strong shape equivalence.

In [24], among other things, it was pointed out that the notion of strong shape equivalence, with respect to a full subcategory of models \( K \subseteq \text{Top} \), is a very general one. While that of homotopy equivalence is a specialization to the case \( K = \text{Top} \). This is based on the well known Vogt’s Lemma [25], from which it follows that a continuous map \( f : X \to Y \) is a homotopy equivalence if and only if it induces equivalences \( f^*_Z \) of track groupoids for all \( Z \in \text{Ob Top} \).

In the first part of this note, we give a generalization of the results recalled above and characterizing strong shape equivalences in the realm of a tensored and cotensored \( \text{Top} \)-category \( \mathcal{C} \), with respect to a full subcategory of models closed under cotensoring with finite CW-complexes. The case of shape equivalences follows easily.

Equivariant shape theory was started in [2] (see also [22] for a finite group action) and still many problems concerning it and its strong version remain open, mostly depending on the nature of the base group, see e.g., [4], [5]. In the second part of the paper, we deal with shape \( G \)-equivalences and strong shape \( G \)-equivalences. Using some results from [15] and [21], we are able to present a list of equivalent conditions for a \( G \)-map to be a shape \( G \)-equivalence, provided \( G \) is a finite group. In the case \( G \) is a compact group and \( H \subseteq G \) is closed normal subgroup, we follow [2] and [3] to deal with \( G/H \)-expansions and shape \( G/H \)-equivalences as well.

We also conjecture that an \( \text{ANR}_G \)-expansion \( X \to X \) of a normal \( G \)-space \( X \) yields \( \text{ANR} \)-expansions \( X^H \to X^H \) of the fixed point subspaces \( X^H \) for any closed subgroup \( H \subseteq G \). We point out that by [21], this holds provided \( G \) is finite.
1 Shape equivalences.

Throughout this paper $\text{Top}$ denotes the category of all compactly generated topological spaces and continuous maps. Let $C$ be a tensored and cotensored $\text{Top}$-category in the sense of [10] (see also [18]). Hence, for any objects $X \in \text{Ob} \text{Top}$ and $C \in \text{Ob} C$ there are a tensor object $X \otimes C$ and a cotensor object $C^X$ in $C$.

Tensoring with the unit interval $I$ gives a cylinder $I \otimes C$, for every $C \in \text{Ob} C$. Consequently one obtains in $C$ a homotopy relation between morphisms and, moreover, one gets the notions of homotopy equivalence, Hurewicz cofibration (fibration) and so on. We write $f \simeq_H g$ for a homotopy $H : I \otimes C \to C'$ joining the morphisms $f, g : C \to C'$ and $[C, C']$ for the set of all homotopy classes of morphisms from $C$ to $C'$ in $C$. $\pi(C, C')$ will denote the fundamental groupoid (called also the track groupoid, see e.g., [6],) of the topological space $C(C, C')$ of morphisms from $C$ to $C'$.

Given a category $C$, let $\{C, \text{Set}\}$ be the category of functors from $C$ to the category of sets and the natural transformations between them. If $(C, P)$ is a pair of categories with $E : P \hookrightarrow C$ the inclusion functor, then the usual Yoneda embedding $Y_P : P \to \{P, \text{Set}\}^{\text{op}}$ has the Kan extension

$$\gamma_E : C \to \{P, \text{Set}\}^{\text{op}},$$

defined by $\gamma_E(C) = C(C, E(-)) : P \to \text{Set}$ for every $C \in \text{Ob} C$. Recall, see [9] or [14], that the shape category $\text{Sh}(C, P)$ of the pair $(C, P)$ is the full image of $\gamma_E : C \to \{P, \text{Set}\}^{\text{op}}$, as described by the commutative diagram

$$\begin{array}{ccc}
C & \xrightarrow{\gamma_E} & \{P, \text{Set}\}^{\text{op}} \\
\downarrow{\text{Sh}} & & \downarrow{\tilde{\gamma}_E} \\
\text{Sh}(C, P), & & \\
\end{array}$$
where the shape functor $Sh$ is the identity on objects and $\tilde{\gamma}_E$ is fully faithful. Hence, the objects of $Sh(C, P)$ are those of $C$ and the morphisms can be described by

$$Sh(C, C') = Nat(C(C', E(-)), C(C, E(-)),$$

where $Nat$ means the class of natural transformations.

Let now $(C, P)$ be a pair of categories, where $C$ is a tensored and cotensored $\text{Top}$-category. A morphism $f : C \to C'$ in $C$ is called:

1. a \textit{shape equivalence} for the pair $(C, P)$ if it fulfils the following properties:
   (i) for each morphism $g : C \to P$, $P \in \text{ObP}$, there is a morphism $h : C \to P$ such that $h \circ f \simeq g$;
   (ii) if $h_0, h_1 : C' \to P$, $P \in \text{ObP}$, are such that $h_0 \circ f \simeq h_1 \circ f$ then $h_0 \simeq h_1$.

2. $f : C \to C'$ is a \textit{strong shape equivalence} the pair $(C, P)$ if, in addition to (i), the following strengthened form of (ii) holds:
   (ii)* given $h_0, h_1 : C' \to P$, $P \in \text{ObP}$, with $h_0 \circ f \simeq_F h_1 \circ f$ for a homotopy $F : I \otimes C \to P$, there is a homotopy $F' : I \otimes C \to P$ with $h_0 \simeq_F h_1$ and such that $F' \circ (I \otimes f) \simeq F(\text{rel } \{0,1\} \otimes C)$.

It is clear that any strong shape equivalence is a shape equivalence.

Those given above are the classical definitions for shape and strong shape equivalences of topological spaces, see e.g., [20], when $C = \text{Top}$ and $P = \text{ANR}$ is the full subcategory of absolute neighbourhood retracts. In [12] a map of spaces was defined to be a strong shape equivalence whenever the induced map $f^* : C(Y, Z) \to C(X, Z)$ is a weak homotopy equivalence, for all $Z \in \text{Ob ANR}$, and one of the main results there was to show that such a definition is equivalent to the classical one.

Recall [19] that a strong homotopy equivalence is a quadruple $(f, g, H, K)$ where $f : X \to Y$, $g : Y \to X$ are maps and $H : g \circ f \simeq 1_X$, $K : f \circ g \simeq 1_Y$ are homotopies with $f \circ H \simeq K \circ (f \times 1)$ rel $X \times \partial I$ and $H \circ (g \times 1) \simeq g \circ K$ rel $Y \times \partial I$. Vogt’s Lemma (see [25] and also [8, Proposition (1.14)]) asserts that every homotopy equivalence can be made in a strong one. In particular, a map $f : X \to Y$ is a homotopy equivalence iff it induces, for every space $Z \in \text{Ob Top}$, an equivalence
of track groupoids $f^*_Z : \pi(Y, Z) \to \pi(X, Z)$. In [23] it was pointed out that conditions (i) and (ii)* for a continuous map $f : C \to C'$ amount to the fact that $f$ induces an equivalence $f^*_Z : \pi(Y, Z) \to \pi(X, Z)$, for all $Z \in \text{Ob ANR}$. Hence, the concept of strong shape equivalence is a relativization of that of homotopy equivalence.

Let us assume that $(C, P)$ is a pair of categories, where $C$ is a tensored and cotensored Top-category with the further property that every morphism $f : C \to C'$ in $C$ has a factorization

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow{i} & & \downarrow{q} \\
C' & \xrightarrow{\pi} & C
\end{array}
\]

where $i$ is a Hurewicz fibration and $q$ a homotopy equivalence.

The following theorem contains a strong shape analogue of Vogt's lemma and furthermore, a categorical version of the main result of [12].

**Theorem 1.1.** Let $(C, P)$ be as above, with $P$ closed under cotensors with finite CW-complexes. That is $P^Q \in \text{Ob } P$ for every finite CW-complex $Q$ and $P \in \text{Ob } P$. Then, for a morphism $f : C \to C'$ in $C$ the following are equivalent:

1. $f : C \to C'$ in $C$ is a strong shape equivalence;
2. for any $P \in \text{Ob } P$, the induced continuous map $f^* : C(C', P) \to C(C, P)$ is a weak homotopy equivalence;
3. for any CW-complex $Q$, the induced morphism $Q \otimes f : Q \otimes C \to Q \otimes C'$ is a shape equivalence;
4. for all $P \in \text{Ob } P$, the induced functor $\tilde{f}^* : \pi(C', P) \to \pi(C, P)$ of track groupoids is an equivalence of groupoids;
5. for any $P \in \text{Ob } P$ and any $\alpha : C' \to P$ the induced homomorphism $\pi_1(f^*) : \pi_1(C(C', P), \alpha) \to \pi_1(C(C, P), f \circ \alpha)$ of fundamental groups is surjective.

**Proof.** (2) $\iff$ (3): Let $P \in \text{Ob } P$. Then, by the Whitehead theorem, see e.g., [26, (7.17) Theorem], the continuous map $f^* : \pi(Y, Z)$.
C(C', P) → C(C, P) is a weak equivalence if and only if the induced map [Q, C(C', P)] → [Q, C(C, P)] is a bijection for every CW-complex Q. By adjointness, we derive that [Q, ⊗C', P] → [Q, ⊗C, P] is a bijection for all P ∈ ObP. Consequently, Q ⊗ f : Q ⊗ C → Q ⊗ C' is a shape equivalence for any CW-complex Q.

It is easy to see that (1) ⇔ (4). By means of [24, Lemma 1.1 and Lemma 1.2], (4) ⇔ (5) is a consequence of the so-called Whitehead Lemma [13, 17].

(1) ⇒ (2): For any n ≥ 0, let Sn be the n-th sphere with an extra disjoint point. Since the cotensor PSn ∈ ObP, for every P ∈ ObP, the induced map [C', PSn] → [C, PSp] is a bijection for every P ∈ ObP. By means of adjointness, we derive that [Sn, P(C', P)] → [Sn, P(C, P)] is a bijection for every P ∈ ObP. By (1) ⇔ (5), for all P ∈ ObP and α : C' → P the induced homomorphism π1(f*) : π1(C(C', P), α) → π1(C(C, P), f ∘ α) of fundamental groups is an isomorphism. Thus, by virtue of [7], we deduce that the induced continuous map f* : C(C', P) → C(C, P) is a weak homotopy equivalence for all P ∈ ObP.

(2) ⇒ (1): Given a morphism f : C → C', consider its factorization f = q ∘ i, where i : C → C'' is a Hurewicz cofibration and q : C'' → C' a homotopy equivalence. Since q : C'' → C' is a strong shape equivalence then it suffices to show that i : C → C'' is a strong shape equivalence. The induced continuous map f* : C(C', P) → C(C, P) is a weak homotopy equivalence, so is i* : C(C'', P) → C(C, P) for all P ∈ ObP. But i* : C(C'', P) → C(C, P) is a Hurewicz fibration in Top, so it is a trivial Serre fibration in Top, for all P ∈ ObP. Then, by the commutativity of the diagram,

for any morphism α : C → P and P ∈ ObP, there is a β : C'' → P such that α = β ∘ i.

Let now β1, β2 : C'' → P be morphisms in P and F : I ⊗ C'' → P a
homotopy joining $i \circ \beta_1$ and $i \circ \beta_2$. By adjointness we get the continuous induced maps $\tilde{F}_0 : \{0, 1\} \to \mathbf{P}(C'', P)$ and $\tilde{F} : I \to \mathbf{P}(C'', P)$. Since $i^* : \mathbf{C}(C'', P) \to \mathbf{C}(C, P)$ is a trivial Serre fibration in $\mathbf{Top}$, by the commutativity of the diagram,

\[
\begin{array}{ccc}
\{0, 1\} & \xrightarrow{\tilde{F}_0} & \mathbf{C}(C'', P) \\
\downarrow & & \downarrow i^* \\
I & \xrightarrow{\tilde{F}} & \mathbf{C}(C, P)
\end{array}
\]

there is a continuous map $\tilde{G} : I \to \mathbf{P}(C'', P)$ such that $i^* \circ \tilde{G} = \tilde{F}$ and the restriction $\tilde{G} |_{\{0, 1\}} = \tilde{F}_0$. Consequently, again by adjointness, we derive a homotopy $G : I \otimes C'' \to P$ joining $\beta_1, \beta_2$ and such that $F = G \circ (I \otimes f)$ and the proof is complete.

\[\square\]

2 Equivariant shape equivalences.

Let $G$ be a topological group. Then, the category $\mathbf{Top}_G$ of all compactly generated topological $G$-spaces (and continuous $G$-maps) is a tensored and cotensored $\mathbf{Top}$-category. For $X \in \text{Ob} \mathbf{Top}$ and $C \in \text{Ob} \mathbf{Top}_G$, the tensor $X \otimes C$ is given by the product $X \times C$ and the contensor $C^X$ by the mapping space with the usual $G$-action on $X \times C$ and $C^X$. In particular, for $G = E$, the trivial group we get that the category $\mathbf{Top}$ is a tensored and cotensored $\mathbf{Top}$-category.

Write $\mathcal{O}_G$ be the category of canonical orbits for a topological group $G$. Its objects are given by the cosets $G/H$ (with the usual $G$-action) for any closed subgroup $H < G$; morphisms $G/H \to G/K$ are represented by $G$-maps.

Then, the category $\mathcal{O}_G \mathbf{- Top}$ of all contravariant functors $\mathcal{O}_G \to \mathbf{Top}$ is also a tensored and cotensored $\mathbf{Top}$-category, where tensor and cotensor objects are defined componentwise.
Remark 2.1. (1) Given a compact topological group $G$, consider the full subcategory $P = \text{ANR}_G$ of $\text{Top}_G$ of all $G$-absolute neighborhood retracts. By [1], we get $P^{n^2} \in \text{Ob ANR}_G$ for all $n \geq 0$ provided $P \in \text{Ob ANR}_G$. Hence, Theorem 1.1 for the pair of categories $(\text{Top}_G, \text{ANR}_G)$ holds. In particular, for $G = E$, the trivial group we get the result shown in [24] also for the pair of categories $(\text{Top}, \text{ANR})$, where $\text{ANR}$ is the full subcategory of $\text{Top}$ of all absolute neighborhood retracts.

(2) Let now $G$ be any topological group and take the full subcategory $\mathcal{O}_G-\text{ANR}$ of $\mathcal{O}_G-\text{Top}$ of all contravariant functors $\mathcal{O}_G \rightarrow \text{ANR}$. Again by [1], we get that $P^{n^2} \in \text{Ob \mathcal{O}_G-ANR}$ for all $n \geq 0$ provided $P \in \text{Ob \mathcal{O}_G-ANR}$. Hence, Theorem 1.1 for the pair of categories $(\text{Top}_G, \mathcal{O}_G-\text{ANR})$ holds as well.

(3) Let pro-$\text{Top}_G$ be the category of pro-objects over $\text{Top}_G$ and take its full subcategory pro-$\text{ANR}_G$. Then, it is obvious that Theorem 1.1 is also valid for the pair of categories (pro-$\text{Top}_G$, pro-$\text{ANR}_G$).

Given a topological group $G$, a closed subgroup $H < G$ and a $G$-space $X$, write $X^H$ for the fixed point subspace of $X$. A (strong) shape equivalence in the category $\text{Top}_G$ is called a (strong) shape $G$-equivalence.

If $X$ is a $G$-space, write $X_H$ for restriction of $X$ to a subgroup $H < G$. In case the group $G$ is finite and $X$ is a normal $G$-space then, by [21, Theorem 2], for any subgroup $H < G$, its Čech $G$-expansion $X \rightarrow \hat{C}_G(X)$ yields an expansion $X^H \rightarrow \hat{C}(X)^H$. Furthermore, by [21, Lemma 4.1], one can easily derive that the restricted Čech $G$-expansion $X_H \rightarrow \hat{C}_G(X)_H$ is an $H$-expansion of the $H$-space $X_H$. Hence, by virtue of [15] and [21], we can state:

**Proposition 2.2.** Let $G$ be finite group and $f : X \rightarrow Y$ a $G$-map of normal $G$-spaces. Then, the following are equivalent:

1. $f : X \rightarrow Y$ is a shape $G$-equivalence;
2. for any subgroup $H < G$, the map $f^H : X^H \rightarrow Y^H$ is a shape equivalence;
3. for any subgroup $H < G$, the restricted $H$-map $f_H : X_H \rightarrow Y_H$ is a shape $H$-equivalence;
(4) For any subgroup $H < G$ and $k \geq 0$, the induced maps $\pi_k(\tilde{C}(X)^H) \to \pi_k(\tilde{C}(Y)^H)$ of homotopy pro-groups is an isomorphism.

Now, given a topological group $G$ and a closed normal subgroup $H < G$, write $X/H$ for the quotient $G/H$-space. In the light of [3], $P/H \in \text{Ob ANR}_G$ provided $P \in \text{Ob ANR}_G$. If $H < G$ is also a compact subgroup then we can easily show that $P/H \in \text{Ob ANR}_{G/H}$ as well.

**Proposition 2.3.** Let $G$ be a compact (Hausdorff) group and $H < G$ a closed normal subgroup.

1. If $X \to X$ is an $\text{ANR}_G$-expansion of a $G$-space $X$ then $X/H \to X/H$ is an $\text{ANR}_{G/H}$-expansion of the $G/H$-space $X/H$.

2. If a $G$-map $f : X \to Y$ is a strong shape $G$-equivalence for the pair $(\text{Top}_G, \text{ANR}_G)$ then the induced $G/H$-map $f/H : X/H \to Y/H$ of quotients is also a strong shape $G/H$-equivalences for the pair $(\text{Top}_{G/H}, \text{ANR}_{G/H})$.

**Proof.** (1) Given an $\text{ANR}_G$-expansion $X \to X$ of a $G$-space $X$ and a closed normal subgroup $H < G$, we get by the above that $X/H$ is an inverse system in $\text{ANR}_{G/H}$. Then, one can easily check that $X/H \to X/H$ is an $\text{ANR}_{G/H}$-expansion of the $G/H$-space $X/H$ as well.

(2) Certainly, it follows directly from (1) that $f/H : X/H \to Y/H$ is a shape $G/H$-equivalence. Nevertheless, we present below a direct proof of (2).

(i) Any $G/H$-map $\alpha : X/H \to P$ for $P \in \text{Ob ANR}_{G/H}$ yields a $G$-map $\tilde{\alpha} : X \to P$. Hence, there is a $G$-map $\tilde{\beta} : Y \to P$ and a $G$-homotopy $f \circ \tilde{\beta} \simeq \tilde{\alpha}$. Those lead to a $G/H$-map $\beta : Y \to P$ and a $G/H$-homotopy $f/H \circ \beta \simeq \alpha$.

(ii) Let $\beta_0, \beta_1 : Y/H \to P$ be $G/H$-maps with $\beta_0 \circ f/H \simeq_F \beta_1 \circ f/H$ for a $G/H$-homotopy $F : I \times X/H \to P$ and $P \in \text{Ob ANR}_{G/H}$. Then, we get $G$-maps $\tilde{\beta}_0, \tilde{\beta}_1 : Y \to P$ and a $G$-homotopy $\tilde{F} : I \times X \to P$ with $\tilde{\beta}_0 \circ f \simeq_F \tilde{\beta}_1 \circ f$. Hence, there is a $G$-homotopy $\tilde{F}' : I \otimes X \to P$ with $\tilde{\beta}_0 \simeq_{\tilde{F}'} \tilde{\beta}_1$ and such that $\tilde{F}' \circ (I \times f) \simeq \tilde{F} (\text{rel } \{0,1\} \times X)$. Those yield a $G/H$-homotopy $F' : I \times X/H \to P$ with $\beta_0 \simeq_{F'} \beta_1$ and such that $F' \circ (I \times f/H) \simeq F (\text{rel } \{0,1\} \times X/H)$ and the proof is complete. □
When $G$ is a finite group then, in light of [22], any object of $\text{ANR}_G$ has the $G$-homotopy type of a $G$-CW complex and vice versa. By [21, Lemma 4.1], any open covering of a $G$-space $X$ admits an equivariant refinement. Consequently, by [21, Theorem 2], for any subgroup $H < G$ and a normal $G$-space $X$, the Čech $G$-expansion $X \to \check{C}_G(X)$ yields an expansion $X^H \to \check{C}(X)^H$.

Let $G$ be a compact (Hausdorff) group. Then, by [2], any $G$-space $X$ admits a $\text{ANR}_G$-expansion $X \to X$. This means that the full subcategory $[\text{ANG}_G]$ of $[\text{Top}_G]$ which consists of spaces having the $G$-homotopy type of $\text{ANR}_G$'s, is dense in $[\text{Top}_G]$. Furthermore, by [16] we get $X^H \in \text{Ob ANR}$ for any closed subgroup $H < G$ provided $X \in \text{Ob ANR}_G$. We close the paper with the following:

**Conjecture 2.4.** Let now $G$ be a compact (Hausdorff) group and $X$ a normal $G$-space. If $X \to X_\ast$ is an $\text{ANR}_G$-expansion of $X$ then $X^H \to X^H$ is an $\text{ANR}$-expansion of $X^H$, for any closed subgroup $H < G$.

For a finite group $G$, generalizations of some results on equivariant homotopy theory presented in [22] show that the equivariant shape theory can afford new problems involving $G$-spaces. Therefore, an affirmative solution of the conjecture above should throw a new light on another path between $G$-spaces and their fixed point subspaces associated with closed subgroups $H < G$.

**References**


Faculty of Mathematics and Computer Science
Nicholas Copernicus University
87-100 Toruń, Chopina 12/18 - Poland
email: marek@mat.uni.torun.pl

Dipartimento di Matematica e Informatica
Università di Perugia
via Vanvitelli, 06123 Perugia - Italia
email: stra@dipmat.unipg.it

- 80 -