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WEAK HOMOTOPY EQUIVALENCES OF MAPPING SPACES AND VOGT'S LEMMA

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RESUME. Dans cet article les auteurs présentent une caractérisation des équivalences de forme (au sens de la 'shape theory') et de forme forte dans le cadre général d'une **Top**-catégorie **C** tensorisée et co-tensorisée. Le cas des équivalences de forme équivariante est aussi considéré.

ABSTRACT. In this paper we give characterizations of shape and strong shape equivalences in the general setting of a tensored and cotensored **Top**-category **C**. The case of equivariant shape equivalences is also considered.

Introduction

A strong shape equivalence [20] is a map inducing an isomorphism in the strong shape category $\mathbf{sSh}(\mathbf{Top}, \mathbf{ANR})$. It turns out [24] that $f : X \to Y$ is such a map if it gives (by composition) an equivalence

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$f_Z^* : \mathbf{Gd}(Y, Z) \to \mathbf{Gd}(X, Z)$ between fundamental groupoids, for all $Z \in \mathrm{Ob}\,\mathbf{ANR}$.

J. Dydak and S. Nowak [11] provide a geometric explanation of strong shape theory and give a fairly simple way of introducing the strong shape category formally. There, those methods are applied to present a list of equivalent conditions for a map $f : X \to Y$ of k-spaces (compactly generated Hausdorff spaces) to be a strong shape equivalence .

In [24], among other things, it was pointed out that the notion of strong shape equivalence, with respect to a full subcategory of models $\mathbf{K} \subseteq \mathbf{Top}$, is a very general one. while that of homotopy equivalence is a specialization to the case $\mathbf{K} = \mathbf{Top}$. This is based on the well known Vogt's Lemma [25], from which it follows that a continuous map $f : X \to Y$ is a homotopy equivalence if and only if it induces equivalences f_Z^* of track groupoids for all $Z \in \mathrm{Ob}\,\mathbf{Top}$.

In the first part of this note, we give a generalization of the results recalled above and characterizing strong shape equivalences in the realm of a tensored and cotensored **Top**-category **C**, with respect to a full subcategory of models closed under cotensoring with finite CW-complexes. The case of shape equivalences follows easily.

Equivariant shape theory was started in [2] (see also [22] for a finite group action) and still many problems concerning it and its strong version remain open, mostly depending on the nature of the base group, see e.g., [4], [5]. In the second part of the paper, we deal with shape G-equivalences and strong shape G-equivalences. Using some results from [15] and [21], we are able to present a list of equivalent conditions for a G-map to be a shape G-equivalence, provided G is a finite group. In the case G is a compact group and $H < G$ is closed normal subgroup, we follow [2] and [3] to deal with G/H-expansions and shape G/H-equivalences as well.

We also conjecture that an $\mathbf{ANR}_G$-expansion $X \to \underline{X}$ of a normal G-space X yields $\mathbf{ANR}$-expansions $X^H \to \underline{X}^H$ of the fixed point subspaces X^H for any closed subgroup $H < G$. We point out that by [21], this holds provided G is finite.

1 Shape equivalences.

Throughout this paper **Top** denotes the category of all compactly generated topological spaces and continuous maps. Let **C** be a tensored and cotensored **Top**-category in the sense of [10] (see also [18]). Hence, for any objects $X \in \operatorname{Ob}\mathbf{Top}$ and $C \in \operatorname{Ob}\mathbf{C}$ there are a tensor object $X \otimes C$ and a cotensor object C^X in **C**.

Tensoring with the unit interval I gives a cylinder $I \otimes C$, for every $C \in \operatorname{Ob}\mathbf{C}$. Consequently one obtains in **C** a *homotopy relation* between morphisms and, moreover, one gets the notions of *homotopy equivalence*, *Hurewicz cofibration* (*fibration*) and so on. We write $f \simeq_H g$ for a homotopy $H : I \otimes C \to C'$ joining the morphisms $f, g : C \to C'$ and $[C, C']$ for the set of all homotopy classes of morphisms from C to C' in **C**. $\pi(C, C')$ will denote the fundamental groupoid (called also the *track groupoid*, see e.g., [6],) of the topological space $\mathbf{C}(C, C')$ of morphisms from C to C'.

Given a category **C**, let $\{\mathbf{C}, \mathbf{Set}\}$ be the category of functors from **C** to the category of sets and the natural transformations between them. If $(\mathbf{C}, \mathbf{P})$ is a pair of categories with $E : \mathbf{P} \hookrightarrow \mathbf{C}$ the inclusion functor, then the usual Yoneda embedding $Y_{\mathbf{P}} : \mathbf{P} \to \{\mathbf{P}, \mathbf{Set}\}^{\mathrm{op}}$ has the Kan extension

$$\gamma_E : \mathbf{C} \longrightarrow \{\mathbf{P}, \mathbf{Set}\}^{\mathrm{op}},$$

defined by $\gamma_E(C) = \mathbf{C}(\mathbf{C}, \mathbf{E}(-)) : \mathbf{P} \to \mathbf{Set}$ for every $C \in \operatorname{Ob}\mathbf{C}$. Recall, see [9] or [14], that the *shape category* $\mathbf{Sh}(\mathbf{C}, \mathbf{P})$ of the pair $(\mathbf{C}, \mathbf{P})$ is the full image of $\gamma_E : \mathbf{C} \longrightarrow \{\mathbf{P}, \mathbf{Set}\}^{\mathrm{op}}$, as described by the commutative diagram

$$\begin{tikzcd}
\mathbf{C} \arrow[rr, "\gamma_E"] \arrow[dr, "Sh"'] & & \{\mathbf{P}, \mathbf{Set}\}^{\mathrm{op}} \\
& \mathbf{Sh}(\mathbf{C}, \mathbf{P}), \arrow[ur, "\tilde{\gamma}_E"'] &
\end{tikzcd}$$

where the shape functor Sh is the identity on objects and $\tilde{\gamma}_E$ is fully faithful. Hence, the objects of $\mathbf{Sh}(\mathbf{C},\mathbf{P})$ are those of $\mathbf{C}$ and the morphisms can be described by

$$\mathbf{Sh}(\mathbf{C},\mathbf{C}') = \mathrm{Nat}(\mathbf{C}(\mathbf{C}',\mathbf{E}(-)),\mathbf{C}(\mathbf{C},\mathbf{E}(-)),$$

where Nat means the class of natural transformations.

Let now $(\mathbf{C},\mathbf{P})$ be a pair of categories, where $\mathbf{C}$ is a tensored and cotensored **Top**-category. A morphism $f : C \to C'$ in $\mathbf{C}$ is called:

(1) a *shape equivalence* for the pair $(\mathbf{C},\mathbf{P})$ if it fulfils the following properties:

(i) for each morphism $g : C \to P$, $P \in \mathrm{Ob}\,\mathbf{P}$, there is a morphism $h : C' \to P$ such that $h \circ f \simeq g$;

(ii) if $h_0, h_1 : C' \to P$, $P \in \mathrm{Ob}\,\mathbf{P}$, are such that $h_0 \circ f \simeq h_1 \circ f$ then $h_0 \simeq h_1$.

(2) $f : C \to C'$ is a *strong shape equivalence* the pair $(\mathbf{C},\mathbf{P})$ if, in addition to (i), the following strengthened form of (ii) holds:

(ii)* given $h_0, h_1 : C' \to P$, $P \in \mathrm{Ob}\,\mathbf{P}$, with $h_0 \circ f \simeq_F h_1 \circ f$ for a homotopy $F : I \otimes C \to P$, there is a homotopy $F' : I \otimes C \to P$ with $h_0 \simeq_{F'} h_1$ and such that $F' \circ (I \otimes f) \simeq F\,(\mathrm{rel}\ \{0,1\} \otimes C)$.

It is clear that any strong shape equivalence is a shape equivalence.

Those given above are the classical definitions for shape and strong shape equivalences of topological spaces, see e.g., [20], when $\mathbf{C} = \mathbf{Top}$ and $\mathbf{P} = \mathbf{ANR}$ is the full subcategory of absolute neighbourhood retracts. In [12] a map of spaces was defined to be a strong shape equivalence whenever the induced map $f^* : \mathbf{C}(Y,Z) \to \mathbf{C}(X,Z)$ is a weak homotopy equivalence, for all $Z \in \mathrm{Ob}\,\mathbf{ANR}$, and one of the main results there was to show that such a definition is equivalent to the classical one.

Recall [19] that a strong homotopy equivalence is a quadruple (f,g,H,K) where $f : X \to Y$, $g : Y \to X$ are maps and $H : g \circ f \simeq 1_X$, $K : f \circ g \simeq 1_Y$ are homotopies with $f \circ H \simeq K \circ (f \times 1)$ *rel* $X \times \partial I$ and $H \circ (g \times 1) \simeq g \circ K$ *rel* $Y \times \partial I$. Vogt's Lemma (see [25] and also [8, Proposition (1.14)]) asserts that every homotopy equivalence can be made in a strong one. In particular, a map $f : X \to Y$ is a homotopy equivalence iff it induces, for every space $Z \in \mathrm{Ob}\,\mathbf{Top}$, an equivalence of track groupoids $f_Z^* : \pi(Y, Z) \to \pi(X, Z)$. In [23] it was pointed out that conditions (i) and (ii)* for a continuous map $f : C \to C'$ amount to the fact that f induces an equivalence $f_Z^* : \pi(Y, Z) \to \pi(X, Z)$, for all $Z \in \mathrm{Ob}\,\mathbf{ANR}$. Hence, the concept of strong shape equivalence is a relativization of that of homotopy equivalence.

Let us assume that $(\mathbf{C}, \mathbf{P})$ is a pair of categories, where $\mathbf{C}$ is a tensored and cotensored **Top**-category with the further property that every morphism $f : C \to C'$ in $\mathbf{C}$ has a factorization

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "i"'] & & C' \\ & C'' \arrow[ur, "q"'] & \end{tikzcd}$$

where i is a Hurewicz fibration and q a homotopy equivalence.

The following theorem contains a strong shape analogue of Vogt's lemma and furthermore, a categorical version of the main result of [12].

Theorem 1.1. *Let $(\mathbf{C}, \mathbf{P})$ be as above, with $\mathbf{P}$ closed under cotensors with finite CW-complexes, that is $P^Q \in \mathrm{Ob}\,\mathbf{P}$ for every finite CW-complex Q and $P \in \mathrm{Ob}\,\mathbf{P}$. Then, for a morphism $f : C \to C'$ in $\mathbf{C}$ the following are equivalent:*

(1) *$f : C \to C'$ in $\mathbf{C}$ is a strong shape equivalence;*

(2) *for any $P \in \mathrm{Ob}\,\mathbf{P}$, the induced continuous map $f^* : \mathbf{C}(C', P) \to \mathbf{C}(C, P)$ is a weak homotopy equivalence;*

(3) *for any CW-complex Q, the induced morphism $Q \otimes f : Q \otimes C \to Q \otimes C'$ is a shape equivalence;*

(4) *for all $P \in \mathrm{Ob}\,\mathbf{P}$, the induced functor $\bar{f}^* : \pi(C', P) \to \pi(C, P)$ of track groupoids is an equivalence of groupoids;*

(5) *for any $P \in \mathrm{Ob}\,\mathbf{P}$ and any $\alpha : C' \to P$ the induced homomorphism $\pi_1(f^*) : \pi_1(\mathbf{C}(C', P), \alpha) \to \pi_1(\mathbf{C}(C, P), f \circ \alpha)$ of fundamental groups is surjective.*

Proof. (2) $\Longleftrightarrow$ (3): Let $P \in \mathrm{Ob}\,\mathbf{P}$. Then, by the Whitehead theorem, see e.g., [26, (7.17) Theorem], the continuous map f^* :

$\mathbf{C}(C', P) \to \mathbf{C}(C, P)$ is a weak equivalence if and only if the induced map $[Q, \mathbf{C}(C', P)] \to [Q, \mathbf{C}(C, P)]$ is a bijection for every CW-complex Q. By adjointness, we derive that $[Q, \otimes C', P] \to [Q, \otimes C, P]$ is a bijection for all $P \in \mathrm{Ob}\,\mathbf{P}$. Consequently, $Q \otimes f : Q \otimes C \to Q \otimes C'$ is a shape equivalence for any CW-complex Q.

It is easy to see that (1) $\iff$ (4). By means of [24, Lemma 1.1 and Lemma 1.2], (4) $\iff$ (5) is a consequence of the so-called Whitehead Lemma [13, 17].

(1) $\implies$ (2): For any $n \geq 0$, let $\mathbb{S}^n_+$ be the n-th sphere with an extra disjoint point. Since the cotensor $P^{\mathbb{S}^n_+} \in \mathrm{Ob}\,\mathbf{P}$, for every $P \in \mathrm{Ob}\,\mathbf{P}$, the induced map $[C', P^{\mathbb{S}^n_+}] \to [C, P^{\mathbb{S}^n_+}]$ is a bijection for every $P \in \mathrm{Ob}\,\mathbf{P}$. By means of adjointness, we derive that $[\mathbb{S}^n_+, \mathbf{P}(C', P)] \to [\mathbb{S}^n_+, \mathbf{P}(C, P)]$ is a bijection for every $P \in \mathrm{Ob}\,\mathbf{P}$. By (1) $\iff$ (5), for all $P \in \mathrm{Ob}\,\mathbf{P}$ and $\alpha : C' \to P$ the induced homomorphism $\pi_1(f^*) : \pi_1(\mathbf{C}(C', P), \alpha) \to \pi_1(\mathbf{C}(C, P), f \circ \alpha)$ of fundamental groups is an isomorphism. Thus, by virtue of [7], we deduce that the induced continuous map $f^* : \mathbf{C}(C', P) \to \mathbf{C}(C, P)$ is a weak homotopy equivalence for all $P \in \mathrm{Ob}\,\mathbf{P}$.

(2) $\implies$ (1): Given a morphism $f : C \to C'$, consider its factorization $f = q \circ i$, where $i : C \to C''$ is a Hurewicz cofibration and $q : C'' \to C'$ a homotopy equivalence. Since $q : C'' \to C'$ is a strong shape equivalence then it suffices to show that $i : C \to C''$ is a strong shape equivalence. The induced continuous map $f^* : \mathbf{C}(C', P) \to \mathbf{C}(C, P)$ is a weak homotopy equivalence, so is $i^* : \mathbf{C}(C'', P) \to \mathbf{C}(C, P)$ for all $P \in \mathrm{Ob}\,\mathbf{P}$. But $i^* : \mathbf{C}(C'', P) \to \mathbf{C}(C, P)$ is a Hurewicz fibration in **Top**, so it is a trivial Serre fibration in **Top**, for all $P \in \mathrm{Ob}\,\mathbf{P}$. Then, by the commutativity of the diagram,

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathbf{C}(C'', P) \arrow[d, "i^*"] \\ \star \arrow[r] & \mathbf{C}(C, P) \end{tikzcd}$$

for any morphism $\alpha : C \to P$ and $P \in \mathrm{Ob}\,\mathbf{P}$, there is a $\beta : C'' \to P$ such that $\alpha = \beta \circ i$.

Let now $\beta_1, \beta_2 : C'' \to P$ be morphisms in $\mathbf{P}$ and $F : I \otimes C'' \to P$ a homotopy joining $i \circ \beta_1$ and $i \circ \beta_2$. By adjointness we get the continuous induced maps $\tilde{F}_0 : \{0,1\} \to \mathbf{P}(C'', P)$ and $\tilde{F} : I \to \mathbf{P}(C'', P)$. Since $i^* : \mathbf{C}(C'', P) \to \mathbf{C}(C, P)$ is a trivial Serre fibration in **Top**, by the commutativity of the diagram,

$$\begin{tikzcd}
\{0,1\} \arrow[r, "\tilde{F}_0"] \arrow[d] & \mathbf{C}(C'', P) \arrow[d, "i^*"] \\
I \arrow[r, "\tilde{F}"] & \mathbf{C}(C, P)
\end{tikzcd}$$

there is a continuous map $\tilde{G} : I \to \mathbf{P}(C'', P)$ such that $i^* \circ \tilde{G} = \tilde{F}$ and the restriction $\tilde{G} \mid_{\{0,1\}} = \tilde{F}_0$. Consequently, again by adjointness, we derive a homotopy $G : I \otimes C'' \to P$ joining β_1, β_2 and such that $F = G \circ (I \otimes f)$ and the proof is complete.

□

2 Equivariant shape equivalences.

Let G be a topological group. Then, the category $\mathbf{Top}_G$ of all compactly generated topological G-spaces (and continuous G-maps) is a tensored and cotensored **Top**-category. For $X \in \mathrm{Ob}\,\mathbf{Top}$ and $C \in \mathrm{Ob}\,\mathbf{Top_G}$, the tensor $X \otimes C$ is given by the product $X \times C$ and the contensor C^X by the mapping space with the usual G-action on $X \times C$ and C^X. In particular, for $G = E$, the trivial group we get that the category **Top** is a tensored and cotensored **Top**-category.

Write $\mathcal{O}_G$ be the category of canonical orbits for a topological group G. Its objects are given by the cosets G/H (with the usual G-action) for any closed subgroup $H < G$; morphisms $G/H \to G/K$ are represented by G-maps.

Then, the category $\mathcal{O}_G$-**Top** of all contravariant functors $\mathcal{O}_G \to$ **Top** is also a tensored and cotensored **Top**-category, where tensor and cotensor objects are defined componentwise.

Remark 2.1. (1) Given a compact topological group G, consider the full subcategory $\mathbf{P} = \mathbf{ANR}_G$ of $\mathbf{Top}_G$ of all G-absolute neighborhood retracts. By [1], we get $P^{\mathbb{S}^n_+} \in \mathrm{Ob}\,\mathbf{ANR}_G$ for all $n \geq 0$ provided $P \in \mathrm{Ob}\,\mathbf{ANR}_G$. Hence, Theorem 1.1 for the pair of categories $(\mathbf{Top}_G, \mathbf{ANR}_G)$ holds. In particular, for $G = E$, the trivial group we get the result shown in [24] also for the pair of categories $(\mathbf{Top}, \mathbf{ANR})$, where $\mathbf{ANR}$ is the the full subcategory of $\mathbf{Top}$ of all absolute neighborhood retracts.

(2) Let now G be any topological group and take the full subcategory $\mathcal{O}_G$-$\mathbf{ANR}$ of $\mathcal{O}_G$-$\mathbf{Top}$ of all contravariant functors $\mathcal{O}_G \to \mathbf{ANR}$. Again by [1], we get that $P^{\mathbb{S}^n_+} \in \mathrm{Ob}\,\mathcal{O}_G$-$\mathbf{ANR}$ for all $n \geq 0$ provided $P \in \mathrm{Ob}\,\mathcal{O}_G$-$\mathbf{ANR}$. Hence, Theorem 1.1 for the pair of categories $(\mathbf{Top}_G, \mathcal{O}_G$-$\mathbf{ANR})$ holds as well.

(3) Let pro-$\mathbf{Top}_G$ be the category of pro-objects over $\mathbf{Top}_\mathbf{G}$ and take its full subcategory pro-$\mathbf{ANR}_G$. Then, it is obvious that Theorem 1.1 is also valid for the pair of categories (pro-$\mathbf{Top}_G$, pro-$\mathbf{ANR}_G$).

Given a topological group G, a closed subgroup $H < G$ and a G-space X, write X^H for the fixed point subspace of X. A (strong) shape equivalence in the category $\mathbf{Top}_G$ is called a *(strong) shape G-equivalence.*

If X is a G-space, write X_H for restriction of X to a subgroup $H < G$. In case the group G is finite and X is a normal G-space then, by [21, Theorem 2], for any subgroup $H < G$, its Čech G-expansion $X \to \check{C}_G(X)$ yields an expansion $X^H \to \check{C}(X)^H$. Furthermore, by [21, Lemma 4.1], one can easily derive that the restricted Čech G-expansion $X_H \to \check{C}_G(X)_H$ is an H-expansion of the H-space X_H. Hence, by virtue of [15] and [21], we can state:

Proposition 2.2. *Let G be finite group and $f : X \to Y$ a G-map of normal G-spaces. Then, the following are equivalent:*

(1) *$f : X \to Y$ is a shape G-equivalence;*

(2) *for any subgroup $H < G$, the map $f^H : X^H \to Y^H$ is a shape equivalence;*

(3) *for any subgroup $H < G$, the restricted H-map $f_H : X_H \to Y_H$ is a shape H-equivalence;*

(4) *for any subgroup* $H < G$ *and* $k \geq 0$, *the induced maps* $\pi_k(\check{C}(X)^H) \to \pi_k(\check{C}(Y)^H)$ *of homotopy pro-groups is an isomorphism.*

Now, given a topological group G and a closed normal subgroup $H < G$, write X/H for the quotient G/H-space. In the light of [3], $P/H \in \operatorname{Ob} \mathbf{ANR}_G$ provided $P \in \operatorname{Ob} \mathbf{ANR}_G$. If $H < G$ is also a compact subgroup then we can easily show that $P/H \in \operatorname{Ob} \mathbf{ANR}_{G/H}$ as well.

Proposition 2.3. *Let* G *be a compact (Hausdorff) group and* $H < G$ *a closed normal subgroup.*

(1) *If* $X \to \underline{X}$ *is an* $\mathbf{ANR}_G$*-expansion of a* G*-space* X *then* $X/H \to \underline{X}/H$ *is an* $\mathbf{ANR}_{G/H}$*-expansion of the* G/H*-space* X/H.

(2) *If a* G*-map* $f : X \to Y$ *is a strong shape* G*-equivalence for the pair* $(\mathbf{Top}_G, \mathbf{ANR}_G)$ *then the induced* G/H*-map* $f/H : X/H \to Y/H$ *of quotients is also a strong shape* G/H*-equivalences for the pair* $(\mathbf{Top}_{G/H}, \mathbf{ANR}_{G/H})$.

Proof. (1) Given an $\mathbf{ANR}_G$-expansion $X \to \underline{X}$ of a G-space X and a closed normal subgroup $H < G$, we get by the above that $\underline{X}/H$ is an inverse system in $\mathbf{ANR}_{G/H}$. Then, one can easily check that $X/H \to \underline{X}/H$ is an $\mathbf{ANR}_{G/H}$-expansion of the G/H-space X/H as well.

(2) Certainly, it follows directly from (1) that $f/H : X/H \to Y/H$ is a shape G/H-equivalence. Nevertheless, we present below a direct proof of (2).

(i) Any G/H-map $\alpha : X/H \to P$ for $P \in \operatorname{Ob} \mathbf{ANR}_{G/H}$ yields a G-map $\bar{\alpha} : X \to P$. Hence, there is a G-map $\bar{\beta} : Y \to P$ and a G-homotopy $f \circ \bar{\beta} \simeq \bar{\alpha}$. Those lead to a G/H-map $\beta : Y \to P$ and a G/H-homotopy $f/H \circ \beta \simeq \alpha$.

(ii) Let $\beta_0, \beta_1 : Y/H \to P$ be G/H-maps with $\beta_0 \circ f/H \simeq_F \beta_1 \circ f/H$ for a G/H-homotopy $F : I \times X/H \to P$ and $P \in \operatorname{Ob} \mathbf{ANR}_{G/H}$. Then, we get G-maps $\bar{\beta}_0, \bar{\beta}_1 : Y \to P$ and a G-homotopy $\bar{F} : I \times X \to P$ with $\bar{\beta}_0 \circ f \simeq_{\bar{F}} \bar{\beta}_1 \circ f$. Hence, there is a G-homotopy $\bar{F}' : I \otimes X \to P$ with $\bar{\beta}_0 \simeq_{\bar{F}'} \bar{\beta}_1$ and such that $\bar{F}' \circ (I \times f) \simeq \bar{F}\,(\mathrm{rel}\,\{0,1\} \times X)$. Those yield a G/H-homotopy $F' : I \times X/H \to P$ with $\beta_0 \simeq_{F'} \beta_1$ and such that $F' \circ (I \times f/H) \simeq F\,(\mathrm{rel}\,\{0,1\} \times X/H)$ and the proof is complete. $\square$

When G is a finite group then, in light of [22], any object of $\mathbf{ANR}_G$ has the G-homotopy type of a G-CW complex and vice versa. By [21, Lemma 4.1], any open covering of a G-space X admits an equivariant refinement. Consequently, by [21, Theorem 2], for any subgroup $H < G$ and a normal G-space X, the Čech G-expansion $X \to \check{C}_G(X)$ yields an expansion $X^H \to \check{C}(X)^H$.

Let G be a compact (Hausdorff) group. Then, by [2], any G-space X admits a $\mathbf{ANR}_G$-expansion $X \to \underline{X}$. This means that the full subcategory $[\mathbf{ANG}_G]$ of $[\mathbf{Top}_G]$ which consists of spaces having the G-homotopy type of $\mathbf{ANR}_G$'s, is dense in $[\mathbf{Top}_G]$. Furthermore, by [16] we get $X^H \in \operatorname{Ob}\mathbf{ANR}$ for any closed subgroup $H < G$ provided $X \in \operatorname{Ob}\mathbf{ANR}_G$. We close the paper with the following:

Conjecture 2.4. Let now G be a compact (Hausdorff) group and X a normal G-space. If $X \to \underline{X}$ is an $\mathbf{ANR}_G$-expansion of X then $X^H \to \underline{X}^H$ is an $\mathbf{ANR}$-expansion of X^H, for any closed subgroup $H < G$.

For a finite group G, generalizations of some results on equivariant homotopy theory presented in [22] show that the equivariant shape theory can afford new problems involving G-spaces. Therefore, an affirmative solution of the conjecture above should throw a new light on another path between G-spaces and their fixed point subspaces associated with closed subgroups $H < G$.

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