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COHOMOLOGY WITHOUT PROJECTIVES

by Dominique BOURN and Diana RODELO

Abstract

Une longue suite éxacte de cohomologie, sur le modèle de celle de Yoneda, est obtenue pour des catégories additives qui ne sont pas strictement abéliennes, sans projectifs et même sans object 0. Cela permet, entre autres, de faire entrer dans ce cadre les catégories des groupes topologiques et des groupes topologiques séparés et de jeter quelques lumières sur le parallelisme de traitement de la cohomologie des groupes et de la cohomologie des algèbres de Lie.

Introduction

One of the most illustrative cohomology results is that in any abelian (i.e. additive+exact) category $\mathbb{A}$, provided that there are enough projectives, any short exact sequence in $\mathbb{A}$:

$$0 \longrightarrow A \overset{k}{\rightarrowtail} B \overset{h}{\twoheadrightarrow} C \longrightarrow 0$$

and any object X produce a long exact sequence of abelian groups:

$$\begin{array}{ccccc}
0 \rightarrowtail Hom_{\mathbb{A}}(X,A) & \overset{k_*}{\rightarrowtail} & Hom_{\mathbb{A}}(X,B) & \overset{h_*}{\longrightarrow} & Hom_{\mathbb{A}}(X,C) \\
 & & \overset{\delta}{\swarrow} & & \\
Ext_{\mathbb{A}}(X,A) & \underset{k_*}{\longrightarrow} & Ext_{\mathbb{A}}(X,B) & \underset{h_*}{\longrightarrow} & Ext_{\mathbb{A}}(X,C) \\
\cdots & & \cdots & & \cdots \\
Ext^n_{\mathbb{A}}(X,A) & \overset{k_*}{\longrightarrow} & Ext^n_{\mathbb{A}}(X,B) & \overset{h_*}{\longrightarrow} & Ext^n_{\mathbb{A}}(X,C) \\
 & & \overset{\delta_n}{\swarrow} & & \\
Ext^{n+1}_{\mathbb{A}}(X,A) & \underset{k_*}{\longrightarrow} & Ext^{n+1}_{\mathbb{A}}(X,B) & \underset{h_*}{\longrightarrow} & Ext^{n+1}_{\mathbb{A}}(X,C) & \cdots
\end{array}$$

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see, for instance, the classical books [19] or [15] as well as a more recent approach [21]. The aim of this work is to show that this long exact sequence still holds even when the additive category A is not exact, has no projectives and even in the absence of an object 0 (see example 3 in Section 6).

An additive category without 0 is known by the notion of a naturally Mal'cev category and was introduced by P.T. Johnstone in [18], see also [6]. The principal examples of such categories dealing with homology theory are the full subcategory of the slice category Gp/C (objects: group homomorphisms with codomain the group C, morphisms: commutative triangles above C) whose objects are the homomorphisms with abelian kernel and, similarly, the full subcategory of the slice category R_{Lie}/A (objects: Lie-homomorphisms with codomain the Lie Algebra A over the ring R) whose objects are the Lie-homomorphisms with abelian kernel, i.e. equipped with trivial Lie brackets, see [2].

The "active" part of a cohomology theory, namely the Baer sums, though classically treated in the Barr exact context, is still available in the strictly weaker context of effective regularness, see [9].

The price to drop of projectives is to investigate the connected components of the monoidal categories $\mathbb{E}xt^n_{\mathbb{A}}(X, A)$ which produce the abelian groups $Ext^n_{\mathbb{A}}(X, A)$. It is shown that two objects X and X' are in the same connected component if and only if there are linked by a single pair of legs:

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr] & \\ X & & X' \end{tikzcd}$$

The absence of an object 0 also has its price. For any object X, the terminal map is no longer a split epimorphism and, in general, is not even a regular epimorphism (i.e. X may not have a *global support*), which is crucial for the development of our theory. Unfortunately, the subcategory of objects with global support is not closed under pulbacks, thus demanding an amount of work which results in an exposition a bit longer than initially expected.

Still in the absence of 0, we must determine which notion plays the role of a chain complex of length n. It appears that the notion of internal n-groupoids is suitable, as previously used in [4]. The tool organizing the whole machinery is the, subsequently defined, *direction* of an n-groupoid, see [7], [8] and [20].

The direct profit of this approach is mainly two-folded : providing some new light on the known classical parallelism in the treatment of the cohomology of groups and of the cohomology of Lie algebras; and extending the cohomology methods from ordinary abelian groups to topological and Hausdorff abelian groups, see Section 6.

1 General setting

1.1 Naturally Mal'cev categories

A ternary operation $p : X \times X \times X \to X$ is Mal'cev when it satisfies $p(x,y,y) = x$ and $p(x,x,y) = y$. A category $\mathbb{C}$ is *naturally Mal'cev* [18] when it is finitely complete and admits, for any object X, an internal natural Mal'cev operation:

$$p_X : X \times X \times X \longrightarrow X.$$

It was shown in [6] that this is equivalent to saying that, for every object X, the following upward left hand side pullback (with the simplicial notations) is actually a pushout:

$$\begin{tikzcd} & & X \\ X \times X \arrow[rru, "p_1"] \arrow[r, "s_1"] \arrow[d, shift right, "p_0"'] & X \times X \times X \arrow[ru, dotted, "p_X"] \arrow[d, shift right, "p_0"'] & \\ X \arrow[u, shift right, "s_0"'] \arrow[r, "s_0"'] & X \times X, \arrow[u, shift right, "s_0"'] \arrow[ruu, "p_0"'] & \end{tikzcd}$$

the natural Mal'cev operation being given by the induced dotted factorization. In a naturally Mal'cev category, any pointed object $e : 1 \to X$ is endowed with a canonical structure of internal abelian group:

$$X \times X \simeq X \times 1 \times X \xrightarrow{1_X \times e \times 1_X} X \times X \times \xrightarrow{\;p_X\;} X.$$

Examples 1.1. Naturally Mal'cev categories.

1) Any finitely complete additive category $\mathbb{A}$ is naturally Mal'cev. Thanks to the previous remark the converse is true: a pointed naturally Mal'cev category is necessarily additive. This is why a naturally Mal'cev category may be thought as an additive category without 0.

2) When $\mathbb{C}$ is naturally Mal'cev, any slice category $\mathbb{C}/X$ (objects: morphisms with codomain X, and morphisms: commutative triangles above X) is itself naturally Mal'cev, see 2.4.13 in [2]. Consequently any category $Pt_X\mathbb{C}$ (objects: split epimorphisms above X, and morphisms: commutative triangles between them), being pointed and naturally Mal'cev, is additive. This last point is actually another characteristic condition for naturally Mal'cev categories, see Theorem 7 of [6].

3) In particular, any slice category $\mathbb{A}/X$ of an additive category $\mathbb{A}$ is an example of a non pointed naturally Mal'cev category.

4) The two major examples of naturally Mal'cev categories we have in mind are the following ones: given any group C, the full subcategory $Mal(Gp/C)$ of the slice category Gp/C (objects: group homomorphisms with codomain C) whose objects are the homomorphisms with abelian kernel is naturally Mal'cev; and, given any R-Lie algebra A, the full subcategory $Mal(R_{Lie}/A)$ of the slice category R_{Lie}/A (objects: Lie-homomorphisms with codomain A) whose objects are the Lie-homomorphisms with abelian kernel (i.e. equipped with trivial Lie brackets).

5) As above, given a topological (resp. Hausdorff) group C, the full subcategory of the slice category $GpTop/C$ (resp. $GpHaus/C$) whose objects are the continuous homomorphisms with abelian kernel is naturally Mal'cev.

6) The three previous examples are particular cases of the following general situation: given any protomodular category $\mathbb{C}$, the full subcategory of abelian objects in $\mathbb{C}$ is naturally Mal'cev, see Corollary 2.7.6 and Proposition 3.1.19 in [2].

7) With any finitely complete category $\mathbb{E}$, we can associate the naturally Mal'cev category $\mathbb{C} = AutM\mathbb{E}$ of autonomous Mal'cev operations in $\mathbb{E}$. The objects are pairs (X, p) of an object X endowed with an internal Mal'cev operation p which is itself a morphism of Mal'cev operations, i.e. which, in set theoretical terms, satisfies:

$$p(p(x,x',x''),p(y,y',y''),p(z,z',z''))$$

$$= p(p(x,y,z),p(x',y',z'),p(x'',y'',z'')).$$

Remark then that we have $Ab\mathbb{E} = Ab\mathbb{C}$, where $Ab\mathbb{E}$ is the category of internal abelian groups in $\mathbb{E}$. Another characterization of a naturally Mal'cev category is then $\mathbb{C} = AutM\mathbb{C}$.

Any naturally Mal'cev category is Mal'cev [6], meaning that any reflexive relation is an equivalence relation [11], [12]. Moreover, in the naturally Mal'cev context, any pair of equivalence relations R and T on the same object X admits a canonical *connector* (in the sense of [10], where this situation is denoted by $[R,T] = 0$), namely a morphism:

$$p : R \times_X S \to X,\ (xRySz) \mapsto p(x,y,z)$$

which, internally speaking, satisfies the identities $p(x,y,y) = x$ and $p(y,y,z) = z$. In a way, Examples 1.1 1), 2) and 3) emphasized the fact that a naturally Mal'cev category is an additive category without an object 0; this commutation of any pair of equivalence relations makes this point even clearer. More importantly, this connector produces a double equivalence relation whose underlying diagram is the following:

$$\begin{tikzcd}
R \times_X T \arrow[r, shift left, "p_2"] \arrow[r, shift right, dashed, "(r_0 \cdot p_0, p)"'] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, dashed, "(p, t_1 \cdot p_2)"] & T \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] \\
R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_0"'] & X,
\end{tikzcd} \qquad (1)$$

where any commutative square is a pullback. In set theoretical terms, this means that with any $xRyTz$ we can associate a rectangle:

$$\begin{tikzcd} x \arrow[r, dotted, dash, "T"] \arrow[d, dash, "R"'] & p(x,y,z) \arrow[d, dotted, dash, "R"] \\ y \arrow[r, dash, "T"'] & z. \end{tikzcd}$$

1.2 Regular naturally Mal'cev categories

A finitely complete category $\mathbb{C}$ is regular [1] when the regular epimorphisms are stable under pullbacks and the effective equivalence relations (i.e. the kernel relations of some map) have a quotient. We repeatedly use objects with *global support*, i.e. such that the terminal map $X \to 1$ is a regular epimorphism. We shall need the following observation:

Proposition 1.2. *Let $\mathbb{C}$ be a regular naturally Mal'cev category. In any pullback of (downward) split epimorphisms with horizontal regular epimorphisms, the upward square is a pushout:*

$$\begin{tikzcd} X \arrow[r, two heads, "x"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\ Y \arrow[r, two heads, "y"'] \arrow[u, shift right, "s"'] & Y'. \arrow[u, shift right, "s'"'] \end{tikzcd}$$

Accordingly, the change of base functor $y^* : Pt_{Y'}\mathbb{C} \to Pt_Y\mathbb{C}$ is fully faithful.

Proof. Let us complete the diagram with the kernel equivalence relations of the horizontal maps:

$$\begin{tikzcd} R[x] \arrow[r, shift left=2, "x_1"] \arrow[r, shift right=2, "x_0"'] \arrow[d, shift right, "R(f)"'] & X \arrow[l, "s_0" description] \arrow[r, two heads, "x"] \arrow[d, shift right, "f"'] \arrow[rr, bend left, "\phi"] & X' \arrow[d, shift right, "f'"'] & W \\ R[y] \arrow[r, shift left, "y_1"] \arrow[r, shift right, "y_0"'] \arrow[u, shift right, "R(s)"'] & Y \arrow[l] \arrow[r, two heads, "y"'] \arrow[u, shift right, "s"'] & Y'. \arrow[u, shift right, "s'"'] \arrow[ur, "\psi"'] & \end{tikzcd}$$

Let (ϕ, ψ) be a pair of maps such that $\phi \cdot s = \psi \cdot y$. Actually, the map ϕ coequalizes the pair (x_0, x_1), whence the wished factorization. This coequalization can be checked by composing it with the pair $(R(s), s_0)$, which is jointly strongly epic. This last assertion is a consequence of the fact that the left hand side square indexed by 0 is a pullback in $\mathbb{C}$ and, therefore, a product in the additive category $Pt_Y\mathbb{C}$, see Example 1.1 2). Accordingly, this square is a sum in $Pt_Y\mathbb{C}$, and the pair $(R(s), s_0)$ is jointly strongly epic. □

Corollary 1.3. *Suppose that X has a global support and that B is group object. Then $p_B : X \times B \twoheadrightarrow B$ is the cokernel of $X \times o_B : X \rightarrowtail X \times B$:*

$$\begin{tikzcd} X \times B \arrow[r, two heads, "p_B"] \arrow[d, shift right, "p_X"'] & B \arrow[d, shift right] \\ X \arrow[r, two heads] \arrow[u, shift right, "X \times o_B"'] & 1. \arrow[u, shift right, "o_B"'] \end{tikzcd}$$

Notice that pushouts in any category $Pt_Y\mathbb{C}$ produce pushouts in $\mathbb{C}$ since this holds for $\mathbb{C}/Y$. There is an important consequence:

Lemma 1.4. *Let $\mathbb{C}$ be a regular naturally Mal'cev category. Consider a regular epimorphism $f : X \twoheadrightarrow Y$ and an equivalence relation R on Y. Then the following upper pullback is also a pushout in $\mathbb{C}$:*

$$\begin{tikzcd} R[f] \arrow[r, two heads, "f \cdot f_0"] \arrow[d, "j"'] & Y \arrow[d, "s_0"] \\ f^{-1}R \arrow[r, two heads, "R(f)"] \arrow[d, "(r_0', r_1')"'] & R \arrow[d, "(r_0, r_1)"] \\ X \times X \arrow[r, two heads, "f \times f"'] & Y \times Y. \end{tikzcd}$$

Proof. Consider the following diagram where the downward right hand side square is, by definition, a pullback:

$$\begin{tikzcd} f^{-1}R \arrow[r, two heads, "\psi"'] \arrow[rr, bend left, "R(f)"] & R_0 \arrow[r, two heads, "\phi"'] \arrow[d, shift right, "\rho_0"'] & R \arrow[d, shift right, "r_0"'] \\ R[f] \arrow[u, "j"] \arrow[r, two heads, "f_0"'] & X \arrow[r, two heads, "f"'] \arrow[u, shift right, "\sigma_0"'] & Y. \arrow[u, shift right, "s_0"'] \end{tikzcd}$$

Then by the previous proposition the upward right hand side square is a pushout. On the other hand the upward left hand side square is a pushout in the regular additive category $Pt_X\mathbb{C}$ (more precisely

$$1_X \longrightarrow R[f] \xrightarrow{j} f^{-1}R \xrightarrow{\psi} R_0 \longrightarrow 1_X$$

is a short exact sequence in $Pt_X\mathbb{C}$). Accordingly it is also a pushout in $\mathbb{C}$. Consequently the whole upward rectangle is a pushout. $\square$

1.3 Effectively regular categories

Our two major examples, Example 1.1 4), are Barr exact categories, and consequently regular ones. Actually we shall need here a slightly richer notion than the one of regular category, see [9].

Definition 1.5. *A regular category* $\mathbb{C}$ *is said to be* effectively regular *when any equivalence relation* T *on an object* X *which is a subobject* $j : T \rightarrowtail R$ *of an effective equivalence relation* R *on* X *by an effective monomorphism in* $\mathbb{C}$ *(which means that* j *is the equalizer of some pair of maps in* $\mathbb{C}$*) is itself effective.*

Examples 1.6. Effectively regular categories.

1) The regular categories $GpTop$ and $GpHaus$ of topological and Hausdorff groups are effectively regular. Indeed, in any of these categories, an internal equivalence relation $R \rightarrowtail X \times X$ on a topological group X is effective if and only if the topology of R is induced by the product topology. Now, if $j : T \rightarrowtail R$ is a subobject among the equivalence relations on X, and j is effective, then the topology of T is induced by the topology on R, and consequently induced by the inclusion $T \rightarrowtail R \rightarrowtail X \times X$, thus effective.

2) Accordingly, given any topological (resp. Hausdorff) group C, the full subcategory of the slice category $GpTop/C$ (resp. $GpHaus/C$) whose objects are the continuous homomorphisms with codomain C and abelian kernel is an effectively regular naturally Mal'cev category.

3) The previous two examples also hold in the analogous situation concerning a topological or Hausdorff Lie-algebra A.

4) When $\mathbb{E}$ is an effectively regular category, the categories $Ab\mathbb{E}$ and $Gp\mathbb{E}$ of internal abelian groups (resp. internal groups) in $\mathbb{E}$ are effectively regular.

5) A regular finitely complete additive category $\mathbb{A}$ is effectively regular if and only if the kernel maps are stable for composition. Then $\mathbb{A}$ admits pushouts of kernel maps along any map which preserve kernel maps, and these pushouts are pullbacks, see [9].

The main interest of this notion is the following:

Proposition 1.7. *Suppose $\mathbb{C}$ is effectively regular. Let R be an equivalence relation on an object U which is fibrant above an effective equivalence relation $R[q]$ on V:*

$$\begin{tikzcd}
R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_0"'] \arrow[d, "\tilde{h}"'] & U \arrow[l] \arrow[d, "h"] & \\
R[q] \arrow[r, shift left=2, "q_1"] \arrow[r, shift right=2, "q_0"'] & V \arrow[l] \arrow[r, "q"] & W.
\end{tikzcd}$$

Then R is effective.

Proof. The fact that R is fibrant above $R[q]$ means that any of the squares above are pullbacks. Now consider $R[q \cdot h] = h^{-1}(R[q])$. Then there is a natural inclusion $j : R \rightarrowtail R[q \cdot h]$. Because R is fibrant, then j is split in $\mathbb{C}$, thus an effective monomorphism:

$$\begin{tikzcd}
R \arrow[rr, "r_1"] \arrow[dr, "j"] \arrow[dd, "\tilde{h}"'] & & U \arrow[dd, "h"] \\
& R[qh] \arrow[ul, dashed, bend left] \arrow[ur, "d_1"] \arrow[dl, "R(h)"'] & \\
R[q] \arrow[rr, "q_1"'] & & V.
\end{tikzcd}$$

Accordingly R is effective. $\square$

And more specifically the following:

Corollary 1.8. *Suppose that $\mathbb{C}$ is effectively regular and naturally Mal'cev. Let $g : X \to C$ be a map and T an equivalence relation* *on X. Then the following equivalence relation on T induced by the double relation associated with the connector p making $[R[g], T] = 0$ (see diagram* (1)*):*

$$\begin{tikzcd} R[g] \times_X T \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(g_0 \cdot p_0, p)"'] & T \end{tikzcd}$$

is effective.

Proof. This is a particular case of the previous proposition since the equivalence relation in question is fibrant above $R[g]$:

$$\begin{tikzcd} R[g] \times_X T \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(g_0 \cdot p_0, p)"'] \arrow[d, "p_0"'] & T \arrow[d, "t_0"] \\ R[g] \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_0"'] & X. \end{tikzcd}$$

□

2 Metakernels and direction

2.1 Metakernels and kernels

From now on we shall suppose that $\mathbb{C}$ is an effectively regular naturally Mal'cev category. Now let $f : X \to Y$ be any map in $\mathbb{C}$ and consider the following part of the diagram associated with the centrality (namely $[\nabla_X, R[f]] = 0$) of the equivalence relation $R[f]$:

$$\begin{tikzcd} X \times R[f] \arrow[r, shift left, "p_R"] \arrow[r, shift right, "(p_0 \cdot 1 \times f_0, p)"'] \arrow[d, shift right, "1 \times f_0"'] & R[f] \arrow[r, dashed, two heads, "\nu(f)"] \arrow[d, shift right, "f_0"'] & N[f] \arrow[d] \\ X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u, shift right] & X \arrow[r] \arrow[u, shift right] & 1. \end{tikzcd}$$

Then, according to Corollary 1.8, the upper horizontal equivalence relation is effective and admits a quotient map $\nu(f)$. Moreover, since any of the left hand side squares is a pullback, the Barr-Kock theorem in the regular category $\mathbb{C}$ implies that the right hand side square is a pullback, thus $R[f] \simeq X \times N[f]$.

Definition 2.1. *The object $N[f]$ defined in the diagram above (or more precisely the pair $(N[f], \nu(f))$) will be called the* metakernel *of the map f.*

The reason for this terminology is that, when $\mathbb{C}$ is pointed (and thus additive), the metakernel $N[f]$ is nothing but the kernel $K[f]$ of f (the terminal map $\tau_X : X \to 1$, being split, makes $N[f]$ the kernel of f_0, thus the kernel of f).

We shall study three important particular cases:

1. X HAS GLOBAL SUPPORT:

then $N[f] \in Ab\mathbb{C}$ and $\nu(f) = coker(s_0)$:

$$\begin{tikzcd}
X \times R[f] \arrow[r, shift left, "p_R"] \arrow[r, shift right, "(p_0\cdot 1\times f_0, p)"'] \arrow[d, shift right, "1\times f_0"'] & R[f] \arrow[r, dashed, two heads, "\nu(f)"] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "f_1"] & N[f] \arrow[d, shift right] \\
X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u, shift right, dashed, "(p, f_1\cdot p_R)"'] & X \arrow[r, two heads] \arrow[u, "s_0"] & 1 \arrow[u, shift right, "o_f"']
\end{tikzcd} \tag{2}$$

There is a global element $o_f : 1 \to N[f]$ induced by the subdiagonal s_0 which makes $N[f]$ an abelian group object in $\mathbb{C}$. Moreover, according to Proposition 1.2, the right hand side upward square is a pushout. So, $\nu(f)$ is the cokernel of the subdiagonal s_0. Note that, if f is a regular epimorphism, then X has a global support if and only Y has a global support.

2. $(f, s) : X \leftrightarrows Y$ IS A SPLIT EPI, Y HAS GLOBAL SUPPORT

Proposition 2.2. *Any split epimorphism $(f, s) : X \leftrightarrows Y$ having a codomain with global support is a pullback of its metakernel. In other words, we necessarily have $X \simeq Y \times N[f]$:*

$$\begin{tikzcd}
X \arrow[r, "\nu(f)\cdot s_1"] \arrow[d, shift right, "f"'] & N[f] \arrow[d, shift right] \\
Y \arrow[r, two heads] \arrow[u, shift right, "s"'] & 1. \arrow[u, shift right, "o_f"']
\end{tikzcd}$$

Accordingly, when Y has a global support, the change of base functor $Y^* : Pt_1\mathbb{C} \to Pt_Y\mathbb{C}$ becomes an equivalence of categories.

Proof. Consider the following rectangle made of two pullbacks:

$$\begin{tikzcd} X \arrow[r, "s_1"] \arrow[d, shift right, "f"'] & R[f] \arrow[r, "\nu(f)"] \arrow[d, shift right, "f_0"'] & N[f] \arrow[d, shift right] \\ Y \arrow[r, "s"'] \arrow[u, shift right, "s"'] & X \arrow[r, two heads] \arrow[u, shift right, "s_0"'] & 1. \arrow[u, shift right, "o_f"'] \end{tikzcd}$$

□

Remark 2.3. The splittings of $f : X \to Y$ are unique up to isomorphism when Y has global support. In fact, the global element o_f is produced by s_0 (diagram (2)) and is independent of the choice of the splitting s. By Proposition 2.2, any pair (s, s') of splittings of f produces a unique isomorphism $\phi : X \to X$ such that $f \cdot \phi = f$, $\phi \cdot s = s'$ and $N_Y(\phi) = 1_{N[f]}$.

We also have the following:

Corollary 2.4. *Suppose that $\mathbb{C}$ is effectively regular and naturally Mal'cev. Then, for any map $h : Z \to Y$ between objects with global support, the change of base functor $h^* : Pt_Y\mathbb{C} \to Pt_Z\mathbb{C}$ is an equivalence of categories.*

Proof. Consider the following commutative diagram where X^* and Y^* are equivalences of categories, according to the previous proposition:

$$\begin{tikzcd} Pt_Y\mathbb{C} \arrow[rr, "f^*"] & & Pt_X\mathbb{C} \\ & Pt_1\mathbb{C}. \arrow[ul, "Y^*"] \arrow[ur, "X^*"'] & \end{tikzcd}$$

□

3. f IS THE TERMINAL MAP $\tau_X : X \to 1$

Definition 2.5. *The metakernel of τ_X will be denoted $N[\tau_X] = d(X)$ and is called the* direction *of the object X*

$$\begin{tikzcd} X \times X \times X \arrow[r, shift left, "p_2"] \arrow[r, shift right, "(p_0,p)"'] \arrow[d, shift right, "p_0"'] & X \times X \arrow[r, dashed, two heads, "\nu_X"] \arrow[d, shift right, "p_0"'] & d(X) \arrow[d] \\ X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u, shift right] & X \arrow[r] \arrow[u, shift right] & 1. \end{tikzcd} \tag{3}$$

When X has a global support, we shall denote by $o_X : 1 \rightarrowtail d(X)$ its associated global element which makes $d(X)$ an abelian group object in $\mathbb{C}$. If X has a global element $x : 1 \to X$, there is a canonical isomorphism $X \to d(X)$ which exchange x and o_f, according to Proposition 2.2.

We shall denote by $Ab\mathbb{C} = Pt_1\mathbb{C}$ the category of abelian group objects (or equivalently of pointed objects) in $\mathbb{C}$, by $\mathbb{C}_\#$ the full subcategory of $\mathbb{C}$ of objects with global support. Notice that $\mathbb{C}_\#$ has products and is still naturally Mal'cev but it is no longer finitely complete (objects with global support are not stable under pullbacks in general). However it has kernel relations $R[f]$ of any map f and pullback along regular epimorphisms. So it is still effectively regular.

The previous definition provides a *direction functor* $d : \mathbb{C}_\# \to Ab\mathbb{C}$. On the other hand, Corollary 2.4 says that the category $\mathbb{C}_\#$ is essentially affine and thus protomodular, see [6].

Examples 2.6. Direction functors.

1) When $\mathbb{A}$ is additive, the direction functor is just $1_\mathbb{A} : \mathbb{A} \to \mathbb{A}$.

2) When $\mathbb{A}$ is additive, then $Ab(\mathbb{A}/Y)$ is equivalent to $\mathbb{A}$, for any object Y. The direction functor $d_Y : \mathbb{A}/Y \to \mathbb{A}$ is nothing but the kernel functor $K : \mathbb{A}/Y \to \mathbb{A}$.

3) In the slice category Gp/C of groups above C, an object $g : G \to C$ has a global support when it is surjective. The direction functor $d : Mal(Gp/C) \to Ab(Gp/C)$ associates with any extension g having an abelian kernel the projection $C \ltimes_\phi A \to C$ of the semidirect product given by the classical group action $\phi : C \to AutA$ associated with the extension g (in other words, the direction of g is nothing but its associated C-module).

4) A similar result holds true for extensions $g : B \to A$ of R-Lie algebras having an abelian kernel.

5) Suppose now that C is a topological group. An object $g : G \to C$ in the slice category $GpTop/C$ has a global support when it is a surjective continuous group homomorphism. The direction of such an extension g with abelian kernel A tells us that the semi-direct product $C \ltimes_\phi A$ is equipped with a topology which makes it a topological group and such that the projection $C \ltimes A \to C$ is continuous. It is classically known that this topology is the product topology, see [3] for instance. This projection is the direction of g. The same holds true for Hausdorff groups.

2.2 Properties of the direction functor d

We shall suppose throughout this section that $\mathbb{C}$ is an effectively regular naturally Mal'cev category. The direction functor preserves the terminal object and the products since the regular epimorphisms are stable by products. Moreover:

Proposition 2.7. *Suppose that X has a global support and B is group object in $\mathbb{C}$. There is a bijection between the retractions ϕ of $X \times o_B : X \rightarrowtail X \times B$ and the group homomorphisms $h : B \to d(X)$. The maps $X \times o_B : X \rightarrowtail X \times B$ are cocartesian with respect to $d : \mathbb{C}_\# \to Ab\mathbb{C}$.*

Proof. The group homomorphism associated to ϕ is the unique h such that $d(\phi) = (1, h) : d(X) \times B \to d(X)$ since $d(\phi)$ is a retraction of $\iota_{d(X)} : d(X) \rightarrowtail d(X) \times B$. Conversely, the retraction associated with a group homomorphism h is determined by the unique factorization $(p_0, \phi) : X \times B \to X \times X$ such that $\nu_X \cdot (p_0, \phi) = h \cdot p_B$ which is induced by the pullback (3) defining $d(X)$. Now to see that $X \times 0_B$ is cocartesian, consider a map $f : X \to Y$ and a factorization $(d(f), h) : d(X) \times d(Y) \to d(Y)$ of $d(f)$ through $\iota_{d(X)} : d(X) \rightarrowtail d(X) \times B$. Then the factorization of f through $X \times o_B : X \rightarrowtail X \times B$ is nothing but:

$$X \times B \xrightarrow{f \times 1_B} Y \times B \xrightarrow{\phi} Y,$$

where ϕ is the map associated with $h : B \to d(Y)$. □

Recall also from [7] and [9] the following very powerful result, with a minor adaptation switching from Barr exact to effectively regular categories:

Proposition 2.8. *Consider the direction functor $d : \mathbb{C}_{\#} \to Ab\mathbb{C}$. Then:*

1) *d preserves regular epimorphisms;*

2) *any regular epimorphism in $\mathbb{C}_{\#}$ is cocartesian;*

3) *d reflects isomorphisms;*

4) *d preserves all existing pullbacks in $\mathbb{C}_{\#}$ and in particular any pullback along regular or split epimorphisms;*

5) *any regular epimorphism $\theta : d(X) \twoheadrightarrow B$ in $Ab\mathbb{C}$ produces a cocartesian map $\bar{\theta} : X \twoheadrightarrow Y$ above it;*

6) *the cocartesian maps obtained in* **(5)** *are stable for products.*

Proof. Point 1) is a consequence of the fact that, when $f : X \twoheadrightarrow Y$ is a regular epimorphism in $\mathbb{C}_{\#}$, it is a regular epimorphism in $\mathbb{C}$, thus $f \times f : X \times X \twoheadrightarrow Y \times Y$ is a regular epimorphism in $\mathbb{C}$. Point 3) is a consequence of the Barr-Kock Theorem valid in any regular category. The proof of point 4) is essentially the proof of Proposition 6 in [7].

Next we briefly sketch the main point 5). Let $\theta : d(X) \twoheadrightarrow B$ be a regular epimorphism in $Ab\mathbb{C}$. Consider the kernel $k : K \rightarrowtail d(X)$ of θ in $Ab\mathbb{C}$ and the following upper pullback in $\mathbb{C}$:

$$\begin{tikzcd}
R \arrow[r, two heads, "n"] \arrow[d, "j"] \arrow[dd, bend right=60, "r_0"'] & K \arrow[d, "k"] \\
X \times X \arrow[r, two heads, "\nu_X"'] \arrow[d, "p_0"] & d(X) \arrow[d] \\
X \arrow[r, two heads] & 1.
\end{tikzcd}$$

The map j is an effective monomorphism, as a pullback of an effective one, and produces a relation $(r_0, r_1) : R \rightrightarrows X$ which is reflexive since the splitting o_K of the terminal map of K produces a splitting $s_0 : X \to R$ of r_0. Consequently R is an equivalence relation which is effective. Moreover, this upper pullback is also a pushout; for this observe that in the following diagram:

$$\begin{tikzcd}
X \arrow[r, shift left, "s_0"] \arrow[d, "j"'] \arrow[rr, bend left=40, "s_0"] & R \arrow[l, shift left, "r_0"] \arrow[r, dashed, "j"] \arrow[d, "n"] & X \times X \arrow[d, "\nu_X"] \arrow[ll, bend right=30, "p_0"'] \\
1 \arrow[r, shift right, "o_K"'] \arrow[rr, bend right=40, "0_X"'] & K \arrow[l, shift right] \arrow[r, dashed, "k"] & d(X)
\end{tikzcd}$$

the two solid leftward diagrams are pullbacks, and that consequently the two associated solid rightward diagrams are pushouts ($\mathbb{C}_{\#}$ is essentially affine). Accordingly, the dashed one is a pushout. Denote by $\bar{\theta} : R \rightrightarrows X \twoheadrightarrow Y$ the quotient of R. It is clear that if X has a global support, then this is also the case of Y. We must show that $d(Y) = B$. Consider the following diagram:

$$\begin{tikzcd}
R \arrow[rr, two heads] \arrow[dr, two heads, "n"] \arrow[dd, "j"'] & & Y \arrow[dr, two heads] \arrow[dd, dotted, "s_0"] & \\
& K \arrow[rr, two heads] \arrow[dd, "k"] & & 1 \arrow[dd] \\
X \times X \arrow[rr, dotted, two heads, "\bar{\theta}\times\bar{\theta}"] \arrow[dr, two heads, "\nu_X"'] & & Y \times Y \arrow[dr, dotted, two heads, "\nu_Y"] & \\
& d(X) \arrow[rr, two heads, "\theta"'] & & B,
\end{tikzcd}$$

the back face is a pushout by Lemma 1.4 and induces a regular epimorphism, we call ν_Y. Since the left and front faces are pushouts, then the right face is also a pushout and consequently $d(Y) = B$. The proof of the cocartesian universality of $\bar{\theta}$ is straightforward.

Finally, point 2) is a consequence of 5) and point 6) is straightforward. □

Points 1) and 4) imply that the functor d is regular in the sense of [1]. Then the metakernel is a function of the direction:

Corollary 2.9. *The metakernel $N[f]$ of any map $f : X \to Y$ in $\mathbb{C}_{\#}$ is the kernel of the map $d(f)$ in $Ab\mathbb{C}$.*

Proof. Since $N[f]$ is in $Ab\mathbb{C}$, then $d(N[f]) \simeq N[f]$. The functor d, being exact, preserves the metakernels. So that $d(N[f]) = N[d(f)]$. Moreover $Ab\mathbb{C}$ is additive and $N[d(f)] = K[d(f)]$. □

2.3 Cofibrations above additive categories

In this subsection we focus on two consequences of Propositions 2.7 and 2.8:

1) the direction d is a cofibration;

2) any fibre of d is endowed with a symmetric monoidal structure.

Actually, this is a very general process. So, in the following $\mathbb{E}$ is a category with products and $\mathbb{A}$ is a finitely complete additive category.

Proposition 2.10. *Let $d : \mathbb{E} \to \mathbb{A}$ be a functor preserving products such that any split monomorphism and any split epimorphism in $\mathbb{A}$ admit cocartesian morphisms above them which are stable for products. Then d is a cofibration such that cocartesian maps are stable for products.*

Proof. This comes from the fact that any map $h : d(X) \to B$ in the additive category $\mathbb{A}$ can be written as $h = (h, 1_B) \cdot \iota_{d(X)}$, where $\iota_{d(X)} : d(X) \rightarrowtail d(X) \times B$ is a split monomorphism and $(h, 1_B) : d(X) \times B \to B$ is a split epimorphism. $\square$

We also have the following striking result [7]:

Proposition 2.11. *Suppose that $d : \mathbb{E} \to \mathbb{A}$ is a cofibration which preserves products and whose cocartesian maps are stable for products. Then any fibre $d^{-1}(A)$ of d is endowed with a symmetric monoidal structure.*

Proof. Let (X, Y) be any pair of objects in $d^{-1}(A)$. The tensor product $X \otimes Y$ is defined as the codomain of the cocartesian map $\mu_{X,Y} : X \times Y \to X \otimes Y$ above the map $+ : A \times A \to A$. The codomain of the cocartesian map $\mathbb{I}(o_A) : 1 \to \mathbb{I}(A)$ above $o_A : 0 \to A$ is a unit on the left for this tensor product since the map

$$X \xrightarrow{\mathbb{I}(o_A) \times 1_X} \mathbb{I}(A) \times X \xrightarrow{\mu_{\mathbb{I}(A),X}} \mathbb{I}(A) \otimes X$$

is cocartesian above 1_A (the global element o_A being a left unit for the internal law on A). The same property holds on the right. This tensor product is associative, since the two following cocartesian maps:

$$X \times Y \times Z \xrightarrow{1_X \times \mu_{Y,Z}} X \times (Y \otimes Z) \xrightarrow{\mu_{X,Y\otimes Z}} X \otimes (Y \otimes Z)$$

$$X \times Y \times Z \xrightarrow{\mu_{X,Y} \times 1_Z} (X \otimes Y) \times Z \xrightarrow{\mu_{X\otimes Y,Z}} (X \otimes Y) \otimes Z$$

are sent onto the same map, namely $A \times A \times A \to A$; $(a, b, c) \mapsto a + b + c$, thanks to the associativity of the internal law on A. The symmetry isomorphism $X \otimes Y \to Y \otimes X$ is induced by the twisting isomorphism $X \times Y \to Y \times X$ and the fact that the internal law on A is commutative. $\square$

We have the following precision:

Theorem 2.12. *Let us consider the assumptions of the previous proposition. Suppose, moreover, that $\mathbb{E}$ has kernel equivalence relations which are preserved by d and that any cocartesian map $\bar{\theta} : X \to Y$ in $\mathbb{E}$ admits a cocartesian subdiagonal $s_0 : X \rightarrowtail R[\bar{\theta}]$. Then, for any pair (X, Y) of objects in $d^{-1}(A)$, there is an object $[X, Y]$ in $d^{-1}(A)$ such that $X \otimes [X, Y]$ and Y are in the same connected component of $d^{-1}(A)$.*

Proof. Define $\nu_{X,Y} : X \times Y \to [X, Y]$ as the cocartesian map above the subtraction $\nu_A : A \times A \to A \,; (a, b) \mapsto b - a$. Then we have a natural isomorphism $[\mathbb{I}(A), X] \simeq X$, since $\nu_{\mathbb{I}(A),X} \cdot \mathbb{I}(o_A) \times 1_X$ is cocartesian above 1_A, thanks to the fact that $a - 0 = a$ for the internal law on A. Whence a cocartesian map $\zeta_X : \mathbb{I}(A) \times X \to X$ above ν_A. Next, we define η_X and $\bar{\nu}_X$ the cocartesian maps above $A \to 0$ with domain X and above $\nu_A : A \times A \to A$ with domain $R[\eta_X]$. Then consider the following diagram:

$$\begin{tikzcd}
R[\eta_X] \arrow[r, dashed, shift left=2, "p_0"] \arrow[r, dashed, shift right=2, "p_1"'] \arrow[d, "\bar{\nu}_X"'] & X \arrow[l, "s_0"' description] \arrow[d, "\eta_X"] \\
T(X) \arrow[r, dashed, shift right, "\pi_X"'] & \nabla X \arrow[l, shift right, "\sigma_X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times A \arrow[r, dashed, shift left=2, "p_1"] \arrow[r, dashed, shift right=2, "p_0"'] \arrow[d, "\nu_A"'] & A \arrow[l, "s_0"' description] \arrow[d] \\
A \arrow[r, dashed, shift right] & 0. \arrow[l, shift right, "o_A"']
\end{tikzcd}$$

The projections p_0 and p_1 produce the same dashed factorization π_X above the terminal map $A \to 0$. In the same way, the subdiagonal $s_0 : X \rightarrowtail R[\bar{\theta}]$ produces a factorization σ_X above o_A. Now, when s_0 is cocartesian, this also the case for σ_X, hence producing a factorization ξ in $d^{-1}(A)$:

$$\begin{tikzcd} T(X) \arrow[d, dotted, "\xi"'] & \nabla_X \arrow[l, "\sigma_X"'] \arrow[d] \\ \mathbb{I}(A) & 1. \arrow[l, "\mathbb{I}(o_A)"] \end{tikzcd}$$

Accordingly we have a map $\tilde{\nu}_X = \xi \cdot \bar{\nu}_X : R[\eta_X] \to \mathbb{I}(A)$ above ν_A. On the other hand we have a canonical inclusion $i : R[\eta_X] \rightarrowtail X \times X$ which lies in $d^{-1}(A \times A)$. Now consider the following diagram:

$$\begin{tikzcd} X \times X \times Y \arrow[r, "1_X \times \nu_{X,Y}"] & X \times [X,Y] \arrow[r, "\mu_{X,[X,Y]}"] & X \otimes [X,Y] \\ R[\eta_X] \times Y \arrow[u, "i \times 1_Y"] \arrow[rr, dashed] \arrow[dr, "\tilde{\nu}_X \times 1_Y"'] & & Z \arrow[u] \arrow[d] \\ & \mathbb{I}(A) \times Y \arrow[r, "\varsigma_Y"'] & Y, \end{tikzcd}$$

where the dashed arrow represents the cocartesian map above $p_A : A \times A \times A \to A$; $(a, b, c) \mapsto a - b + c$. The upper and lower composites, being both mapped to p_A, produce the two right hand side vertical maps in $d^{-1}(A)$. Accordingly $X \otimes [X, Y]$ and Y are in the same connected component of $d^{-1}(A)$. □

Remark. It is clear that the construction $\nabla : \mathbb{E} \to d^{-1}(0)$ is a left adjoint left inverse of the inclusion $d^{-1}(0) \rightarrowtail \mathbb{E}$.

Finally, the result we were aiming at:

Corollary 2.13. *When the assumptions of the two previous results are satisfied, the set $\pi_0(d^{-1}(A))$ of connected components of any fibre of the cofibration d is canonically endowed with an abelian group structure.*

Proof. The tensor product gives the binary operation, while the component of $[X, \mathbb{I}(A)]$ gives the inverse of the component of X. □

2.4 The monoidal closed groupoid $\mathbb{H}^1_{\mathbb{C}}(A)$

We shall denote by $\mathbb{H}^1_{\mathbb{C}}(A)$ the fibres $d^{-1}(A)$ of our originally defined direction functor $d : \mathbb{C}_\# \to Ab\mathbb{C}$. The fact that this functor d reflects the isomorphisms implies and that any map is cocartesian and that any fibre $\mathbb{H}^1_{\mathbb{C}}(A)$ is a groupoid. The stability of cocartesian maps under products is a consequence of the fact that, here, any map is cocartesian. This same fact makes any diagonal $s_0 : X \rightarrowtail X \times X$ and any terminal map cocartesian, so that $\nabla_X = 1$. As a consequence, the functor $X \otimes -$ is an equivalence of categories whose inverse equivalence is the functor $[X, -]$. In other words, the groupoid $\mathbb{H}^1_{\mathbb{C}}(A)$ has a symmetric closed monoidal structure.

Definition 2.14. *We denote by $H^1_{\mathbb{C}}(A)$ the abelian group of connected components of the closed symmetric monoidal groupoid $\mathbb{H}^1_{\mathbb{C}}(A)$ and call it the* first cohomology group of $\mathbb{C}$ with coefficients in A.

Examples 2.15. The first cohomology group.

1) Let Y be an object of an effectively regular additive category $\mathbb{A}$. We observed in Example 2.6 2) that the direction functor for the category $\mathbb{A}/Y$ is nothing but the kernel functor $K : \mathbb{A}/Y \to \mathbb{A}$. Then the objects of fibre $\mathbb{H}^1_{\mathbb{C}}(A)$ are nothing but the short exact sequences in $\mathbb{A}$:

$$0 \longrightarrow A \overset{\alpha}{\rightarrowtail} G \overset{g}{\twoheadrightarrow} Y \longrightarrow 0.$$

The tensor product is given by the Baer sum and the group $H^1_{\mathbb{A}/Y}(A)$ is the classical Yoneda's $Ext_{\mathbb{A}}(Y, A)$.

2) Let C be a group, A an abelian group and $\phi : C \to Aut A$ a group action. This produces an abelian group $C \ltimes_\phi A \to C$ in Gp/C; we denote it by A_ϕ. Then $H^1_{Gp/C}(A_\phi)$ is nothing but the classical $Opext(C, A, \phi)$ of [19] equipped with the Baer sum. See also [9] for the extension of this result to any effectively regular pointed protomodular category $\mathbb{C}$.

3) Suppose now that C and A are topological (resp. Hausdorff) groups, and $\phi : C \to AutA$ is a group action such that the map $C \times A \to A$, associating $\phi_c(a)$ to (c, a), is continuous. We shall then say that the group action ϕ is continuous. This determines an internal abelian group $C \ltimes_\phi A \to C$ in $GpTop/C$ (resp. $GpHaus/C$), again see [3]; we denote it by A_ϕ. Then $H^1_{GpTop/C}(A_\phi)$ (resp. $H^1_{GpHaus/C}(A_\phi)$) is the group of $TOpext(C, A, \phi)$ of continuous extensions of A by C with operators ϕ.

4) It is shown in [7] that, in any Barr exact category $\mathbb{E}$ with products, given an internal group A and an object X, *there is a bijection between the simply transitive left actions of the group A on X and the associative Mal'cev operations on X with direction A.* Actually the same holds true when $\mathbb{E}$ is only effectively regular. On the other hand, when the group A is abelian, it was shown in [1] that the set $H^1_{\mathbb{E}}(A)$ of connected components of the groupoid $\underline{PLO}(A)$ of simply transitive left A-actions (also called A-torsors) in $\mathbb{E}$ is endowed with an abelian group structure which allows, for any short exact sequence of abelian groups, a Yoneda's six term long exact sequence, provided one defines $H^0_{\mathbb{E}}(A) = Hom_{\mathbb{E}}(1, A)$. So, in conclusion:

- considering the effectively regular and naturally Mal'cev category $\mathbb{C} = AutM\mathbb{E}$ of internal autonomous Mal'cev operations in $\mathbb{E}$, the direction functor $d : AutM\mathbb{E}_\# \to Ab\mathbb{E}$ gives an alternative way of describing $H^1_{\mathbb{E}}(A)$ as the set of connected components of $\mathbb{H}^1_{\mathbb{C}}(A)$ which makes $H^1_{\mathbb{E}}(A) = H^1_{\mathbb{C}}(A)$;
- any effective regular and naturally Mal'cev category $\mathbb{C}$ produces the same six term exact sequence, since in this case $\mathbb{C} = AutM\mathbb{C}$.

The previous six term exact sequence, valid when $\mathbb{E}$ is Barr exact, was completed at any level by Duskin and Glenn in [13] and [14] by means of simplicial objects in $\mathbb{E}$, and by the first author in [4] by means of internal n-groupoids in $\mathbb{E}$. One of the aims of this work is to show that the naturally Mal'cev category $AutM\mathbb{E}$ gives rise to a new homogeneous realization of this completed long exact sequence.

2.5 Amenable pullbacks

We know that the direction functor d preserves all existing pullbacks. Actually, the fact that it is a cofibration says more. Suppose we are given a pullback in $Ab\mathbb{C}$ on the right, and a pair of maps (f', y) above the pair (h', β) which admits the diamond on the left as a pullback in $\mathbb{C}_{\#}$:

$$\begin{tikzcd}
\bar{X} \arrow[dr, dotted, "\gamma"] \arrow[drr, dashed, "\bar{f}"] \arrow[ddr, dashed, "\bar{x}"'] & & \\
& X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\
& X' \arrow[r, "f'"'] & Y'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
A' \arrow[r, "h'"'] & B'.
\end{tikzcd}$$

Since d preserves pullbacks, then there is an isomorphism $g : d(\bar{X}) \to A$ and also a cocartesian isomorphism $\gamma : \bar{X} \to X$ above it. This allows us to complete the commutative solid square (which is certainly a pullback) strictly above the right hand side pullback. We call this solid square the *amenable pullback* above the pullback in $Ab\mathbb{C}$.

3 Groupoids and 1-dimensional direction

The prolongation of the six term exact sequence one step further uses the notion of internal groupoids.

3.1 The Lawvere condition

In [18] it is shown that a finitely complete category $\mathbb{C}$ is naturally Mal'cev if and only if it satisfies the:

Lawvere condition : *Any reflexive graph is canonically endowed with a unique groupoid structure.*

Of course, this condition makes any morphism of reflexive graphs (internally) functorial. Clearly this condition is an extension of the *Mal'cev condition* claiming that any reflexive relation is an equivalence relation [11].

Let us denote by $Grd\mathbb{C}$ the category of internal groupoids in $\mathbb{C}$, which is still naturally Mal'cev. We denote by $()_0 : Grd\mathbb{C} \to C$ the forgetful functor associating to any groupoid $\underline{X}_1$:

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "x_1"] \arrow[r, shift right=2, "x_0"'] & X_0 \arrow[l, "s_0"'] \end{tikzcd}$$

its "object of objects" X_0. It has a fully faithful right adjoint ∇_1 (the indiscrete groupoid functor) and a left adjoint Δ_1 (the discrete groupoid functor). All together, the functor $()_0$ is a fibration whose cartesian maps are the fully faithful internal functors, namely the internal functors $\underline{f}_1$ such that the following square is a pullback in $\mathbb{C}$:

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "{(x_0,x_1)}"'] & Y_1 \arrow[d, "{(y_0,y_1)}"] \\ X_0 \times X_0 \arrow[r, "f_0 \times f_0"'] & Y_0 \times Y_0. \end{tikzcd}$$

If the naturally Mal'cev $\mathbb{C}$ is also effectively regular, then $Grd\mathbb{C}$ is still effectively regular. It is clear that $(Grd\mathbb{C})_\# = Grd(\mathbb{C}_\#)$. Moreover, the direction functor d has a natural extension into a functor $\underline{d}_1 : Grd\mathbb{C}_\# \to GrdAb\mathbb{C}$. This functor $\underline{d}_1$ is precisely the direction functor of the effectively regular naturally Mal'cev category $Grd\mathbb{C}$, since we obviously have $AbGrd\mathbb{C} = GrdAb\mathbb{C}$.

Theorem 3.1. *The functor $\underline{d}_1 : Grd\mathbb{C}_\# \to GrdAb\mathbb{C}$ satisfies all the properties of the functor d listed in Propositions 2.7 and 2.8. Moreover, it preserves and reflects the $()_0$-cartesian maps and the $()_0$-invertible (i.e. whose image by $()_0$ is an isomorphism) maps.*

Proof. Only the last point has to be checked; it is straightforward since d, being left exact and reflecting the isomorphisms, also reflects all existing finite limits. □

Let us denote by $\underline{\eta}_1 : \underline{X}_1 \to \nabla_1 X_0$ the canonical projection with respect to the adjunction $(()_0, \nabla_1)$. In the effectively regular naturally Mal'cev category $Grd\mathbb{C}$ the metakernel $N_1[\underline{\eta}_1]$ of $\underline{\eta}_1$ produces an abelian group in $Grd\mathbb{C}$ when $\underline{X}_1$ has a global support, i.e. when $\underline{X}_1$ is in $Grd\mathbb{C}_\#$. We already know by Corollary 2.9 that $N_1[\underline{\eta}_1]$ is the kernel $\underline{A}_1$ of $\underline{d}_1(\underline{\eta}_1)$. Thus certainly $A_0 = 1$ and $A_1 = A$ is given by the following kernel sequence in $Ab\mathbb{C}$:

$$0 \longrightarrow A \rightarrowtail d(X_1) \xrightarrow{d(x_0,x_1)} d(X_0) \times d(X_0).$$

So that this metakernel is completely determined by this object A of $Ab\mathbb{C}$. Let us introduce the following:

Definition 3.2. *We call A the* 1-dimensional direction *of $\underline{X}_1$. This produces the* 1-dimensional direction functor $d_1 : Grd\mathbb{C}_\# \to Ab\mathbb{C}$.

According to the previous definition, we have $d_1\nabla_1 X = 1$ and, thus, $d_1(\underline{\eta}_1)$ is the terminal map $d_1(\underline{X}_1) \to 1$. Let us set also the following [4]:

Definition 3.3. *A groupoid $\underline{X}_1$ in $\mathbb{C}$ is said to be* aspherical *when X_0 has global support and $\underline{X}_1$ is connected, i.e. when the map $(x_0, x_1) : X_1 \to X_0 \times X_0$ is a regular epimorphism in $\mathbb{C}$ or, equivalently, when $\underline{\eta}_1 : \underline{X}_1 \to \nabla_1 X_0$ is a regular epimorphism in $Grd\mathbb{C}$.*

We shall denote by $Asp\mathbb{C}$ the full subcategory of $Grd\mathbb{C}_\#$ whose objects are the aspherical groupoids. Note that, the restriction of d_1 to $Asp\mathbb{C}$ can be factored as:

$$\begin{tikzcd} Asp\mathbb{C} \arrow[r, "\underline{d}_1"'] \arrow[rr, bend left, "d_1"] & AspAb\mathbb{C} \arrow[r, "d_1"'] & Ab\mathbb{C} \end{tikzcd}$$

and the properties of $\underline{d}_1$ are known from Section 2. In order to obtain the properties of $d_1 : Asp\mathbb{C} \to Ab\mathbb{C}$, we shall explore those of the simpler additive specification $d_1 : AspAb\mathbb{C} \to Ab\mathbb{C}$ next.

3.2 The additive specification: level 1

In this subsection we suppose that $\mathbb{A}$ is a finitely complete, effectively regular and additive category. Given any object A in $\mathbb{A}$, let us consider the following short exact sequence in $Grd\mathbb{A}$:

$$0 \longrightarrow \Delta_1 A \rightarrowtail \nabla_1 A \overset{\epsilon_1 A}{\twoheadrightarrow} K_1 A \longrightarrow 0.$$

Then $K_1 A$ is nothing but the following groupoid:

$$A \rightleftarrows 0.$$

The induced functor $K_1 : \mathbb{A} \to Grd\mathbb{A}$ determines a pullback:

$$\begin{tikzcd} \mathbb{A} \arrow[r, "K_1"] \arrow[d] & Grd\mathbb{A} \arrow[d, "()_0"] \\ 1 \arrow[r, "1"'] & \mathbb{A}, \end{tikzcd}$$

which makes the category $\mathbb{A}$ equivalent to the fibre of $()_0$ above the terminal object 1. The inverse equivalence is the restriction of d_1 to this fibre. According to the fact that, in any additive category, metakernels and kernels coincide, the 1-dimensional direction A of a groupoid $\underline{X}_1$ in $\mathbb{A}$ is characterized by the following kernel sequence in $Grd\mathbb{A}$:

$$0 \longrightarrow K_1 A \rightarrowtail \underline{X}_1 \xrightarrow{\eta_1} \nabla_1 X_0.$$

Lemma 3.4. *For the following pair of functors we have:*

$$\begin{tikzcd} & Grd\mathbb{A} \arrow[dl, "()_0"'] \arrow[dr, "d_1"] & \\ \mathbb{A} & & \mathbb{A} \end{tikzcd}$$

1) *both functors preserve the terminal object, products and pullbacks;*

2) *d_1 has a left exact section $K_1 : \mathbb{A} \to Grd\mathbb{A}$;*

3) *$()_0$ is a fibration and d_1 is a cofibration;*

4) *the subdiagonal of a cocartesian map is cocartesian;*

5) *the image of a $()_0$-cartesian map by d_1 is an isomorphism;*

6) *the image of a cocartesian map by $()_0$ is an isomorphism.*

Proof. The two first points are straightforward. The functor $()_0$: $Grd\mathbb{A} \to \mathbb{A}$ is a fibration thanks to the existence of pullbacks. The functor $d_1 : Grd\mathbb{A} \to \mathbb{A}$ is a cofibration since the category $Grd\mathbb{A}$ is effectively regular and additive, and consequently admits pushouts of kernel maps along any map (as previously mentioned in Example 1.6 6)). So, let A be the 1-dimensional direction of $\underline{X}_1$ and $h : A \to B$ any map in $\mathbb{A}$. Then the cocartesian map with domain $\underline{X}_1$ above h is given by the following diagram:

$$\begin{tikzcd} K_1A \arrow[r, tail] \arrow[d, "K_1h"'] & \underline{X}_1 \arrow[r, "\eta_1"] \arrow[d, "\underline{h}_1"'] & \nabla_1 X_0 \\ K_1B \arrow[r, tail] & \underline{Y}_1, \arrow[ur, "\underline{\xi}_1"'] & \end{tikzcd}$$

where the left hand side square is a pushout. Notice that we can choose $\underline{Y}_1$ with $Y_0 = X_0$. The lower horizontal monomorphism is necessarily the kernel of the factorization $\underline{\xi}_1$. The rest of the statement is straightforward. □

We have then the following stricter specification:

Theorem 3.5. *For the following pair of functors we have:*

$$\begin{tikzcd} & Asp\mathbb{A} \arrow[dl, "()_0"'] \arrow[dr, "d_1"] & \\ \mathbb{A} & & \mathbb{A} \end{tikzcd}$$

1) *both functors preserve the terminal object, products and all existing pullbacks in $Asp\mathbb{A}$;*

2) *$()_0$ is a fibration and d_1 is a cofibration which has K_1 as a section;*

3) *a map is cartesian with respect to $()_0$ if and only if its image by d_1 is an isomorphism;*

4) *a map is cocartesian with respect to d_1 if and only if its image by $()_0$ is an isomorphism;*

5) *the subdiagonal of a cocartesian map is cocartesian;*

As a consequence:

6) *the cocartesian maps are stable for products and pullbacks along split epimorphisms;*

7) *d_1 reflects pullbacks with a pair of parallel cocartesian maps;*

8) *the change of base functors with respect to d_1 reflect isomorphisms.*

Proof. Points 1), 2) and 5) were asserted in the previous lemma. Now consider any internal functor $\underline{h}_1 : \underline{X}_1 \to \underline{Y}_1$ above $h : A \to B$ and the associated 1-dimensional direction diagram:

$$\begin{tikzcd} K_1A \arrow[r, tail] \arrow[d, "K_1h"'] & \underline{X}_1 \arrow[r, two heads, "\eta_1"] \arrow[d, "\underline{h}_1"'] & \nabla_1 X_0 \arrow[d, "\nabla_1 h_0"] \\ K_1B \arrow[r, tail] & \underline{Y}_1 \arrow[r, two heads, "\xi_1"'] & \nabla_1 Y_0. \end{tikzcd}$$

Then point 3) is a consequence of a classical result about pullbacks of short exact sequences in regular additive categories: $\underline{h}_1$ is cartesian if and only if the right hand side square is a pullback, which is the case if and only if the left hand side vertical arrow is an isomorphism or, equivalently, h is an isomorphism. Point 4) is a consequence of the dual result about pushouts of short exact sequences in effectively regular additive categories: $\underline{h}_1 : \underline{X}_1 \to \underline{Y}_1$ is cocartesian if and only if the above left hand side square is a pushout, which is the case if and only if $\nabla_1 h_0$ is an isomorphism or, equivalently, h_0 is an isomorphism. Point 6) is implied by the previous characterization of the cocartesian maps in $Asp\mathbb{A}$. For point 7), consider any commutative rectangle in $Asp\mathbb{A}$:

$$\begin{tikzcd} \underline{X}_1 \arrow[r, dotted, "\underline{\phi}_1"'] \arrow[rr, bend left, "\underline{f}_1"] \arrow[d, "\underline{\xi}_1"'] & \underline{P}_1 \arrow[r] \arrow[dl] & \underline{Y}_1 \arrow[d, "\underline{\gamma}_1"] \\ \underline{X}'_1 \arrow[rr, "\underline{f}'_1"'] & & \underline{Y}'_1, \end{tikzcd}$$

whose horizontal arrows are cocartesian and whose image by d_1 is a pullback. So the horizontal maps are $()_0$-invertible. Then consider the pullback in $Grd\mathbb{A}$ of $(\underline{f}'_1, \underline{\gamma}_1)$ with domain $\underline{P}_1$. Since $()_0$-invertible maps are stable for pullbacks, then the factorization $\underline{\phi}_1$ is $()_0$-invertible, thus cocartesian. Accordingly, $\underline{P}_1$ is aspherical because $\underline{X}_1$ is and the pullback in question lies in $Asp\mathbb{A}$. This pullback is mapped by d_1

onto a pullback, implying that $d_1(\underline{\phi}_1)$ is an isomorphism, i.e. $\underline{\phi}_1$ is cartesian. Being also cocartesian, the map $\underline{\phi}_1$ is itself an isomorphism and, consequently, the rectangle is a pullback. Finally, for point 8), let us consider any commutative square in $Asp\mathbb{A}$:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

whose horizontal arrows are cocartesian above the same map in $\mathbb{A}$ and the vertical ones are inside fibres of d_1. Then the horizontal maps are made invertible by $()_0$ while the vertical maps are cartesian with respect to $()_0$. Accordingly, this is a pullback in $Asp\mathbb{A}$ and, if the right hand side vertical arrow is an isomorphism, this is also the case for the left hand side one. $\square$

3.3 Properties of the 1-dimensional direction functor d_1

We can now extend the previous result from any effectively regular additive category $\mathbb{A}$ to any effectively regular naturally Mal'cev category $\mathbb{C}$:

Theorem 3.6. *For the following pair of functors we have:*

$$\begin{tikzcd} & Asp\mathbb{C} \arrow[dl, "()_0"'] \arrow[dr, "d_1"] & \\ \mathbb{C}_\# & & Ab\mathbb{C} \end{tikzcd}$$

1) *both functors preserve the terminal object, products and all existing pullbacks in $Asp\mathbb{A}$;*

2) *$()_0$ is a fibration and d_1 is a cofibration which has K_1 as a section;*

3) *a map is cartesian with respect to $()_0$ if and only if its image by d_1 is an isomorphism;*

4) *a map is cocartesian with respect to d_1 if and only if its image by $()_0$ is an isomorphism;*

5) *the subdiagonal of a cocartesian map is cocartesian;*

6) *the cocartesian maps are stable for products and pullbacks along split epimorphisms;*

7) d_1 *reflects pullbacks with a pair of parallel cocartesian maps;*

8) *the change of base functors with respect to* d_1 *reflect isomorphisms.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
Asp\mathbb{C} \arrow[r, "\underline{d}_1"'] \arrow[rr, bend left, "d_1"] \arrow[d, "()_0"'] & AspAb\mathbb{C} \arrow[r, "d_1"'] \arrow[d, "()_0"] & Ab\mathbb{C} \\
\mathbb{C}_{\#} \arrow[r, "d"'] & Ab\mathbb{C}. &
\end{tikzcd}$$

The result follows from Theorem 3.6 and the fact that the functor $\underline{d}_1$ is a cofibration which preserves and reflects both $()_0$-cartesian and $()_0$-invertible maps (Theorem 3.1). $\square$

3.4 The monoidal category $\mathbb{H}^2_{\mathbb{C}}(A)$

We shall denote by $\mathbb{H}^2_{\mathbb{C}}(A)$ the fibres $d_1^{-1}(A)$ of the 1-dimensional direction functor $d_1 : Asp\mathbb{C} \to Ab\mathbb{C}$. By Proposition 2.11, the stability of cocartesian maps under products gives any of these fibres a symmetric tensor product. The map $1 \to K_1A$ is clearly $()_0$-invertible and consequently cocartesian, which makes K_1A the unit of this tensor product. Moreover the subdiagonal of the cocartesian maps are still cocartesian so, by Theorem 2.12, the set of connected components of this fibre has an abelian group structure.

Definition 3.7. *We denote by* $H^2_{\mathbb{C}}(A)$ *the abelian group of connected components of the symmetric monoidal category* $\mathbb{H}^2_{\mathbb{C}}(A)$ *and call it the* second cohomology group of $\mathbb{C}$ with coefficients in A.

Example 3.8. Let $\mathbb{A}$ be an effectively regular additive category. Then consider the equivalence of categories given by the Moore normalization functor at level 1, namely the functor $M_1 : Grd\mathbb{A} \to Ch_1\mathbb{A}$ (see

[5] for instance) which associates to any groupoid $\underline{X}_1$ the 1-chain complex (= map) given by the left hand side vertical arrow in the following pullback:

$$\begin{tikzcd} K[x_0] \arrow[r] \arrow[d, "\bar{x}_1"'] & X_1 \arrow[d, "(x_0,x_1)"] \\ X_0 \arrow[r, "(0,1_{X_0})"'] & X_0 \times X_0. \end{tikzcd}$$

This equivalence makes $Grd(\mathbb{A}/Y)$ equivalent to the category of chain complexes of length 2 in $\mathbb{A}$ with codomain Y:

$$U_1 \xrightarrow{d} U_0 \xrightarrow{d} Y.$$

The "direction" of this chain complex is given by the kernel $K[d]$ of $d : U_1 \to U_0$. Such a chain complex corresponds to an aspherical groupoid in $Grd(\mathbb{A}/Y)$ if and only if $d : U_0 \to Y$ is a regular epimorphism and the chain complex is exact at U_0. So that the Moore equivalence at level 1 makes our group $H^2_{\mathbb{A}/Y}(A)$ the same as Yoneda's classical $Ext^2_{\mathbb{A}}(Y, A)$.

3.5 Connected components of $\mathbb{H}^2_{\mathbb{C}}(A)$

It is clear that the fibres $\mathbb{H}^2_{\mathbb{C}}(A)$ are no longer groupoids and that the determination of its connected components is not straightforward. In any category $\mathbb{E}$ two objects X and Y are in the same connected component when there is a zig-zag:

$$\begin{tikzcd}[column sep=small] & V_1 \arrow[dl] \arrow[dr] & & V_2 \arrow[dl] \arrow[dr] & & \cdots & & V_n \arrow[dl] \arrow[dr] & \\ X & & U_1 & & U_2 & \cdots & U_{n-1} & & Y. \end{tikzcd}$$

The aim of this subsection is to show that, in the category $\mathbb{H}^2_{\mathbb{C}}(A)$, it is always possible to reduce the length of the zig-zag to 1:

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

When the category $\mathbb{E}$ admits pullbacks it is quite clear how to reduce the length. Unfortunately, this is neither the case of $\mathbb{C}_{\#}$ nor of $Grd\mathbb{C}_{\#}$.

But, for any finitely complete category $\mathbb{E}$, there is, in $Grd\mathbb{E}$, a natural construction which will allow us to get round this difficulty. Given a groupoid $\underline{X}_1$, there necessarily exists a $()_0$-cartesian diagram in $Grd\mathbb{E}$:

$$\begin{tikzcd} Com_1\underline{X}_1 \arrow[r, shift left=2, "\underline{\omega}_1"] \arrow[r, shift right=2, "\underline{\alpha}_1"'] & \underline{X}_1 \arrow[l, "\underline{\sigma}_1"'] \end{tikzcd}$$

above the following one in $\mathbb{E}$:

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "x_1"] \arrow[r, shift right=2, "x_0"'] & X_0. \arrow[l, "s_0"'] \end{tikzcd}$$

$Com_1\underline{X}_1$ is the groupoid of "commutative squares" of $\underline{X}_1$, which universally classifies the internal natural transformations with codomain $\underline{X}_1$, see [4]. When $\mathbb{C}$ is effectively regular and naturally Mal'cev, because the morphism $\underline{\alpha}_1$ is $()_0$-cartesian and split, the groupoid $Com_1\underline{X}_1$ is aspherical whenever $\underline{X}_1$ is. Furthermore, by using amenable pullbacks we can force the images by d_1 of all maps involved in the definition of $Com_1\underline{X}_1$ to be 1_A. Now suppose we are given two morphisms $(\underline{f}_1, \underline{g}_1)$ in the fibre $\mathbb{H}^2_{\mathbb{C}}(A)$:

$$\begin{tikzcd} & \underline{P}_1 \arrow[dl, "\underline{\psi}_1"'] \arrow[dr, "\underline{\phi}_1"] & \\ \underline{U}_1 \arrow[dr, "\underline{f}_1"'] & & \underline{V}_1 \arrow[dl, "\underline{g}_1"] \\ & \underline{X}_1. & \end{tikzcd}$$

We are going to complete it into a (non commutative) square in the same fibre thanks to the following pullback in $Grd\mathbb{C}$:

$$\begin{tikzcd} \underline{P}_1 \arrow[r, "\underline{h}_1"] \arrow[d, "(\underline{\psi}_1, \underline{\phi}_1)"'] & Com_1\underline{X}_1 \arrow[d, "(\underline{\omega}_1, \underline{\alpha}_1)"] \\ \underline{U}_1 \times \underline{V}_1 \arrow[r, "\underline{f}_1 \times \underline{g}_1"'] & \underline{X}_1 \times \underline{X}_1. \end{tikzcd}$$

The map $\underline{h}_1$ is $()_0$-cartesian because so is $\underline{f}_1 \times \underline{g}_1$. Since it is $()_0$-cartesian and $Com_1\underline{X}_1$ is connected, the groupoid $\underline{P}_1$ is connected.

Because $\underline{X}_1$ is aspherical, the map (x_0, x_1) is a regular epimorphism in $\mathbb{C}$, so this is also the case of $(\psi_0, \phi_0) : P_0 \to U_0 \times V_0$ and, since both U_0 and V_0 have global support, P_0 also has a global support. Consequently, $\underline{P}_1$ is aspherical and the above pullback actually lies in $Asp\mathbb{C}$. By switching to the amenable pullback above:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "s_0"'] & A \arrow[d, "s_0"] \\ A \times A \arrow[r, "1_{A\times A}"'] & A \times A, \end{tikzcd}$$

we have $d_1(\underline{\psi}_1) = 1_A$ and $d_1(\underline{\phi}_1) = 1_A$.

3.6 Connected component of 0

The paradigmatic component is naturally the one of 0.

Proposition 3.9. *A groupoid $\underline{X}_1$ of $\mathbb{H}^2_{\mathbb{C}}(A)$ is in the component of 0 if and only if there is an object Z with global support and a map $\underline{\theta}_1 : \nabla_1 Z \to \underline{X}_1$. In other words, if and only if there is a map with codomain $\underline{X}_1$ coming from the fibre $\mathbb{H}^2_{\mathbb{C}}(1)$.*

Proof. Suppose we have such a map $\underline{\theta}_1 : \nabla_1 Z \to \underline{X}_1$. Consider the pullback in $Grd\mathbb{C}$ on the right hand side square of diagram (4), which produces a $()_0$-cartesian functor $\underline{\tau}_1$. The map $\underline{\theta}_1 : \nabla_1 Z \to \underline{X}_1$ induces the vertical splitting. The groupoid $\underline{Z}_1$ is aspherical, since $Z_0 = Z$ has a global support and the dotted splitting assures that $\underline{Z}_1$ is connected, thus this pullback lies in $Asp\mathbb{C}$. By switching to an amenable pullback above:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d] & A \arrow[d] \\ 1 \arrow[r, "1_1"'] & 1, \end{tikzcd}$$

we obtain the following diagram:

$$\begin{tikzcd} K_1(A) \arrow[d] & \underline{Z}_1 \arrow[l, dashed, "\underline{\psi}_1"'] \arrow[r, "\underline{\tau}_1"] \arrow[d, shift right, "\underline{\varsigma}_1"'] & \underline{X}_1 \arrow[d, "\underline{\eta}_1"] \\ 1 & \nabla_1 Z \arrow[l] \arrow[r, "\nabla_1\theta_0"'] \arrow[u, shift right, dotted] \arrow[ur, "\underline{\theta}_1"'] & \nabla_1 X_0. \end{tikzcd} \tag{4}$$

So, according to Proposition 2.2, we have $\underline{Z}_1 = \nabla_1 Z \times K_1(A)$ and a projection $\underline{\psi}_1 : \underline{Z}_1 \to K_1(A)$ which is nothing but $\nu_1(\underline{\zeta}_1).\underline{s}_1$. Accordingly, $d_1(\underline{\psi}_1) = \nu_A.s_1 = 1_A$. Consequently, we have $\underline{\tau}_1$ and $\underline{\psi}_1$ in the fibre $\mathbb{H}^2_{\mathbb{C}}(A)$. Conversely, suppose we have a pair of maps $\underline{\psi}_1 : \underline{Z}_1 \to K_1(A)$ and $\underline{\tau}_1 : \underline{Z}_1 \to X_1$ in the fibre $\mathbb{H}^2_{\mathbb{C}}(A)$. Then consider the following diagram where $\underline{\zeta}_1$ is the cocartesian map above ν_A which produces the isomorphism $[K_1(A), \underline{X}_1] \simeq \underline{X}_1$ of Theorem 2.12:

$$\begin{tikzcd} \underline{Z}_1 \arrow[r, "\underline{\eta}_1"] \arrow[d, "(\underline{\psi}_1, \underline{\tau}_1)"'] & \nabla_1 Z_0 \arrow[d, dotted, "\underline{\theta}_1"] \\ K_1(A) \times \underline{X}_1 \arrow[r, "\underline{\zeta}_1"'] & \underline{X}_1 \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "s_0"'] & 1 \arrow[d, "o_A"] \\ A \times A \arrow[r, "\nu_A"'] & A. \end{tikzcd}$$

Since $\underline{\eta}_1$, being $()_0$-invertible, is also cocartesian, there is a unique factorization $\underline{\theta}_1$ above o_A. And, $\underline{Z}_1$ being aspherical, implies that Z_0 has a global support. $\square$

4 n-groupoids and n-dimensional direction

Thanks to the Lawvere condition an internal n-groupoid in $\mathbb{C}$ is nothing but a reflexive n-globular object $\underline{X}_n$, i.e. a diagram:

$$\underline{X}_n : X_n \underset{x_0}{\overset{x_1}{\underset{\longleftarrow s_0}{\rightrightarrows}}} X_{n-1} \underset{x_0}{\overset{x_1}{\underset{\longleftarrow s_0}{\rightrightarrows}}} X_{n-2} \cdots X_1 \underset{x_0}{\overset{x_1}{\underset{\longleftarrow s_0}{\rightrightarrows}}} X_0$$

satisfying, at each level, the condition of a reflexive graph and such that:

$$x_0 \cdot x_0 = x_0 \cdot x_1 \quad x_1 \cdot x_0 = x_1 \cdot x_1.$$

The n-functors are then the natural transformations between such objects. This determines a category n-$Grd\mathbb{C}$ which is still naturally Mal'cev. By canceling level n, we get a forgetful functor $()_{n-1}$:n-$Grd\mathbb{C} \to (n-1)$-$Grd\mathbb{C}$ which is a fibration. It has both a left and a right adjoint (the discrete and indiscrete groupoid functors), respectively denoted by Δ_n and ∇_n. They are defined by the initial and final objects in the fibre:

$$\Delta_n \underline{X}_{n-1} : X_{n-1} \underset{1}{\overset{1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}}_{s_0} X_{n-1} \underset{x_0}{\overset{x_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}}_{s_0} X_{n-2} \cdots X_1 \underset{x_0}{\overset{x_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}}_{s_0} X_0$$

and

$$\nabla_n \underline{X}_{n-1} : X_{n-1}^{\hat{=}} \underset{p_0}{\overset{p_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}}_{s_0} X_{n-1} \underset{x_0}{\overset{x_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}}_{s_0} X_{n-2} \cdots X_1 \underset{x_0}{\overset{x_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}}_{s_0} X_0,$$

where the object $X_{n-1}^{\hat{=}}$ of *parallel* $(n-1)$*-cells* is defined by induction: $X_0^{\hat{=}} = X_0 \times X_0$ and $X_{n-1}^{\hat{=}}$ given by the following kernel pair in $\mathbb{C}$:

$$X_{n-1}^{\hat{=}} \underset{p_0}{\overset{p_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}}_{s_0} X_{n-1} \xrightarrow{(x_0,x_1)} X_{n-2}^{\hat{=}}.$$

Actually an n-groupoid is an internal groupoid inside a fibre of the fibration $()_{n-2}$:

$$\int \underline{X}_n \underset{\underline{x}_n^0}{\overset{\underline{x}_n^1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}} \underline{X}_{n-1},$$

where $\int \underline{X}_n$ is the integral $(n-1)$-groupoid of the n-cells of the n-groupoid $\underline{X}_n$:

$$\int \underline{X}_n : X_n \underset{x_0 \cdot x_0}{\overset{x_1 \cdot x_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}}_{s_0} X_{n-2} \cdots X_1 \underset{x_0}{\overset{x_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}}_{s_0} X_0.$$

We shall set $\int \nabla_{n+1}(\underline{X}_n) = \underline{X}_n^{\hat{=}} = R[\underline{\eta}_n]$, which produces the following kernel pair in n-$Grd\mathbb{C}$:

$$\underline{X}_n^{\hat{=}} \underset{\underline{p}_n^0}{\overset{\underline{p}_n^1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}}_{\underline{s}_n^0} \underline{X}_n \xrightarrow{\underline{\eta}_n} \nabla_n \underline{X}_{n-1}.$$

4.1 The n-dimensional direction functor

When $\mathbb{C}$ is moreover effectively regular, this is still the case of n-*Grd*$\mathbb{C}$. It is clear that $(n\text{-}Grd\mathbb{C})_\# = n\text{-}Grd(\mathbb{C}_\#)$. Moreover, the direction functor d extends naturally to a functor $\underline{d}_n : n\text{-}Grd\mathbb{C}_\# \to n\text{-}GrdAb\mathbb{C}$. This functor $\underline{d}_n$ is precisely the direction functor of the effectively regular naturally Mal'cev category n-*Grd*$\mathbb{C}$ since we have $Ab(n\text{-}Grd\mathbb{C}) = n\text{-}GrdAb\mathbb{C}$. As in level 1, we have:

Theorem 4.1. *The functor $\underline{d}_n : n\text{-}Grd\mathbb{C}_\# \to n\text{-}GrdAb\mathbb{C}$ satisfies all the properties of the functor d listed in Propositions* 2.7 *and* 2.8. *Moreover, it preserves and reflects the $()_{n-1}$-cartesian maps and the $()_{n-1}$-invertible maps.*

Given any n-groupoid $\underline{X}_n$ in n-*Grd*$\mathbb{C}_\#$, the metakernel $N_n[\underline{\eta}_n]$ of $\underline{\eta}_n$ produces an abelian group in n-*Grd*$\mathbb{C}$. We already know by Corollary 2.9 that $N_n[\underline{\eta}_n]$ is the kernel $\underline{A}_n$ of $\underline{d}_n(\underline{\eta}_n)$. Thus, certainly $\underline{A}_{n-1} = 1$ and the last level $A_n = A$ is given by the following kernel sequence in $Ab\mathbb{C}$:

$$0 \longrightarrow A \rightarrowtail d(X_n) \xrightarrow{d(x_0,x_1)} d(X_{n-1}^{\hat{=}}).$$

So that this metakernel is just determined by an object of $Ab\mathbb{C}$.

Definition 4.2. *We call A the n-*dimensional direction *of $\underline{X}_n$. This produces the n-*dimensional direction functor *$d_n : n\text{-}Grd\mathbb{C}_\# \to Ab\mathbb{C}$.*

According to the previous sequence, we have $d_n\nabla_n\underline{X}_{n-1} = 1$ and, consequently, $d_n(\underline{\eta}_n)$ is the terminal map $d_n(\underline{X}_n) \to 1$. We need now the following [4]:

Definition 4.3. *An n-groupoid $\underline{X}_n$ is called* aspherical *when X_0 has global support and, for each $1 \leq k \leq n$, the map $(x_0, x_1) : X_k \to X_{k-1}^{\hat{=}}$ is a regular epimorphism in $\mathbb{C}$, i.e. when $\underline{X}_{n-1}$ is aspherical and $\underline{X}_n$ is connected (which means that the projection $\underline{\eta}_n : \underline{X}_n \to \nabla_n\underline{X}_{n-1}$ is a regular epimorphism in n-Grd$\mathbb{C}$).*

We denote by n-*Asp*$\mathbb{C}$ the full subcategory of n-*Grd*$\mathbb{C}_\#$ whose objects are the aspherical n-groupoids. As in level 1, the restriction of d_n to n-$Asp\mathbb{C}$ can be factored as:

$$\begin{tikzcd} n\text{-}Asp\mathbb{C} \arrow[r, "\underline{d}_n"'] \arrow[rr, bend left, "d_n"] & n\text{-}AspAb\mathbb{C} \arrow[r, "d_n"'] & Ab\mathbb{C}. \end{tikzcd}$$

Next we explore the properties and the properties of the simpler additive specification $d_n : n\text{-}AspAb\mathbb{C} \to Ab\mathbb{C}$ to obtain the properties of the functor $d_n : n\text{-}Asp\mathbb{C} \to Ab\mathbb{C}$, since those of $\underline{d}_n$ are already known from Theorem 4.1.

Remark. Actually the notion of the n-dimensional direction of an n-groupoid $\underline{X}_n$ is valid in any finitely complete, effectively regular category $\mathbb{E}$, provided it is aspherical, see [20], and moreover abelian when $n = 1$, see [8].

4.2 The additive specification: level n

In this subsection we suppose that $\mathbb{A}$ is a finitely complete, effectively regular and additive category; then this is still the case for the category n-$Grd\mathbb{A}$. We are going to define a functor $K_n : \mathbb{A} \to n\text{-}Grd\mathbb{A}$ by induction. We already defined K_1 and suppose K_k is defined as far as level $n-1$. Given any object A in $\mathbb{A}$, let us consider the following short exact sequence in n-$Grd\mathbb{A}$:

$$0 \longrightarrow \Delta_n K_{n-1}A \rightarrowtail \nabla_n K_{n-1}A \xrightarrow{\epsilon_n A} K_n A \longrightarrow 0.$$

Then $K_n A$ is nothing but the following n-groupoid:

$$A \rightleftarrows 0 \rightleftarrows 0 \cdots 0 \rightleftarrows 0$$

The induced functor $K_n : \mathbb{A} \to n\text{-}Grd\mathbb{A}$ determines a pullback:

$$\begin{tikzcd} \mathbb{A} \arrow[r, "K_n"] \arrow[d] & n\text{-}Grd\mathbb{A} \arrow[d, "()_{n-1}"] \\ 1 \arrow[r] & (n-1)\text{-}Grd\mathbb{A}, \end{tikzcd}$$

which makes the category $\mathbb{A}$ equivalent to the fibre of $()_{n-1}$ above the terminal object 1. The inverse equivalence is the restriction of

d_n to this fibre. According to the fact that, in any additive category, metakernels and kernels coincide, the n-dimensional direction A of an n-groupoid $\underline{X}_n$ in $\mathbb{A}$ is characterized by the following kernel sequence in n-$Grd\mathbb{A}$:

$$0 \longrightarrow K_nA \rightarrowtail \underline{X}_n \xrightarrow{\underline{\eta}_n} \nabla_n X_{n-1}.$$

For exactly the same reasons as in level 1, we have:

Lemma 4.4. *For the following pair of functors we have:*

$$\begin{tikzcd} & n\text{-}Grd\mathbb{A} \arrow[dl, "()_{n-1}"'] \arrow[dr, "d_n"] & \\ (n-1)\text{-}Grd\mathbb{A} & & \mathbb{A} \end{tikzcd}$$

1) *both functors preserve the terminal object, products and pullbacks;*

2) *d_1 has a left exact section $K_n : Ab\mathbb{C} \to n$-$Grd\mathbb{A}$;*

3) *$()_{n-1}$ is a fibration and d_n is a cofibration;*

4) *the subdiagonal of a cocartesian map is cocartesian;*

5) *the image of a $()_{n-1}$-cartesian map by d_n is an isomorphism;*

6) *the image of a cocartesian map by $()_{n-1}$ is an isomorphism.*

Proof. The cocartesian map with domain $\underline{X}_n$ above h is given by the following diagram in n-$Grd\mathbb{A}$, where the left hand side square is pushout:

$$\begin{tikzcd} K_nA \arrow[r, tail] \arrow[d, "K_nh"'] & \underline{X}_n \arrow[r, "\underline{\eta}_n"] \arrow[d, "\underline{h}_n"'] & \nabla_n X_{n-1} \\ K_nB \arrow[r, tail] & \underline{Y}_n. \arrow[ur, dotted, "\underline{\xi}_n"'] & \end{tikzcd}$$

□

And also:

Theorem 4.5. *For the following pair of functors we have:*

$$\begin{tikzcd} & n\text{-}Asp\mathbb{A} \arrow[dl, "()_{n-1}"'] \arrow[dr, "d_n"] & \\ (n-1)\text{-}Asp\mathbb{A} & & \mathbb{A} \end{tikzcd}$$

1) *both functors preserve the terminal object, products and all existing pullbacks in n-$Asp\mathbb{A}$;*

2) $()_{n-1}$ *is a fibration and* d_n *is a cofibration;*

3) *a map is cartesian with respect to* $()_{n-1}$ *if and only if its image by* d_n *is an isomorphism;*

4) *a map is cocartesian with respect to* d_n *if and only if its image by* $()_{n-1}$ *is an isomorphism;*

5) *the subdiagonal of a cocartesian map is cocartesian;*

6) *the cocartesian maps are stable for products and pullbacks along split epimorphisms;*

7) d_n *reflects pullbacks with a pair of parallel cocartesian maps;*

8) *the change of base functors with respect to* d_n *reflect isomorphisms.*

4.3 Properties of the n-dimensional direction functor d_n

We can now extend the previous result from any finitely complete effectively regular additive category $\mathbb{A}$ to any effectively regular naturally Mal'cev category $\mathbb{C}$:

Theorem 4.6. *For the following pair of functor we have:*

$$\begin{tikzcd} & n\text{-}Asp\mathbb{C} \arrow[dl, "()_{n-1}"'] \arrow[dr, "d_n"] & \\ (n-1)\text{-}Asp\mathbb{C} & & Ab\mathbb{C} \end{tikzcd}$$

1) *both functors preserve the terminal object, products and all existing pullbacks in n-$Asp\mathbb{A}$;*

2) $()_{n-1}$ *is a fibration and* d_n *is a cofibration;*

3) *a map is cartesian with respect to* $()_{n-1}$ *if and only if its image by* d_n *is an isomorphism;*

4) *a map is cocartesian with respect to d_n if and only if its image by $()_{n-1}$ is an isomorphism;*

5) *the subdiagonal of a cocartesian map is cocartesian;*

6) *the cocartesian maps are stable for products and pullbacks along split epimorphisms;*

7) *d_n reflects pullbacks with a pair of parallel cocartesian maps;*

8) *the change of base functors with respect to d_n reflect isomorphisms.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
n\text{-}Asp\mathbb{C} \arrow[r, "\underline{d}_n"'] \arrow[rr, bend left, "d_n"] \arrow[d, "()_{n-1}"'] & n\text{-}AspAb\mathbb{C} \arrow[r, "d_n"'] \arrow[d, "()_{n-1}"] & Ab\mathbb{C} \\
(n-1)\text{-}Asp\mathbb{C} \arrow[r, "\underline{d}_{n-1}"'] & (n-1)\text{-}Ab\mathbb{C}. &
\end{tikzcd}$$

The result follows from Theorem 4.6 and the fact that the functor $\underline{d}_n$ is a cofibration which preserves and reflects both $()_{n-1}$-cartesian and $()_{n-1}$-invertible maps (Theorem 4.1). $\square$

4.4 The comprehensive factorization

The fact that $d_n : n\text{-}Asp\mathbb{C} \to Ab\mathbb{C}$ is a cofibration has a useful interpretation at level $n+1$, namely in $(n+1)\text{-}Grd\mathbb{C}$. First, the diagram (5) defining $d_n(\underline{X}_n) = A$

$$\begin{tikzcd}
\underline{X}_n^{\hat{=}} \arrow[r, shift left=2, "\underline{p}_n^1"] \arrow[r, shift right=2, "\underline{p}_n^0"'] \arrow[d, "\underline{\nu}_n"'] & \underline{X}_n \arrow[l] \arrow[r, "\eta_n"] \arrow[d] & \nabla_n \underline{X}_{n-1} \\
K_n(A) \arrow[r, shift right] & 1. \arrow[l, shift right] &
\end{tikzcd} \tag{5}$$

means that there is an $(n+1)$-functor $\underline{\nu}_{n+1} : \nabla_{n+1}\underline{X}_n \to K_{n+1}(A)$ which is an internal discrete fibration. Moreover any $(n+1)$-functor $\underline{w}_{n+1} : \nabla_{n+1}\underline{X}_n \to K_{n+1}(B)$ is just given by a map $w_{n+1} : X_n^{\hat{=}} \to B$

in $\mathbb{C}$ such that $w_{n+1} \cdot s_0 = o_B \cdot \tau : X_n \to 1 \to B$. So there is a unique group homomorphism $h : A \to B$ such that $\int \underline{w}_{n+1} = K_n(h) \cdot \underline{\nu}_n$. Consequently, there is a cocartesian map $\underline{h}_n : \underline{X}_n \to \underline{Y}_n$ in n-$Asp\mathbb{C}$ above h, which implies that $d_n(\underline{Y}_n) = B$ and means that the following diagram commutes in $(n+1)$-$Grd\mathbb{C}$, the vertical arrows being internal discrete fibrations:

$$\begin{tikzcd} \nabla_{n+1}\underline{X}_n \arrow[r, "\nabla_{n+1}\underline{h}_n"] \arrow[d, "\underline{v}_{n+1}"'] \arrow[dr, "\underline{w}_{n+1}"] & \nabla_{n+1}\underline{Y}_n \arrow[d, "\underline{v}_{n+1}"] \\ K_{n+1}(A) \arrow[r, "K_{n+1}(h)"'] & K_{n+1}(B). \end{tikzcd}$$

We shall call this double decomposition of $\underline{w}_{n+1}$ its *comprehensive factorization.*

4.5 The monoidal category $\mathbb{H}^{n+1}_{\mathbb{C}}(A)$

We shall denote by $\mathbb{H}^{n+1}_{\mathbb{C}}(A)$ the fibres $d_n^{-1}(A)$ of the n-dimensional direction functor $d_n : n\text{-}Asp\mathbb{C} \to Ab\mathbb{C}$. Again, by Proposition 2.11, the stability of cocartesian maps under products gives any of these fibres a symmetric tensor product. The map $1 \to K_nA$ is clearly $()_{n-1}$-invertible and consequently cocartesian, which makes K_nA the unit of this tensor product. Moreover the subdiagonal of the cocartesian maps are still cocartesian so, by Theorem 2.12, the set of connected components of this fibre has an abelian group structure.

Definition 4.7. *We denote by $H^{n+1}_{\mathbb{C}}(A)$ the abelian group of connected components of the symmetric monoidal category $\mathbb{H}^{n+1}_{\mathbb{C}}(A)$ and call it the $n+1$-th* cohomology group of $\mathbb{C}$ with coefficients in A.

Example 4.8. Let $\mathbb{A}$ be an effectively regular additive category. Then consider the equivalence of categories given by the Moore normalization functor at level n, namely the functor $M_n : n\text{-}Grd\mathbb{A} \to Ch_n\mathbb{A}$, see [5]. This equivalence makes n-$Grd(\mathbb{A}/Y)$ equivalent to the category of chain complexes of length $n+1$ in $\mathbb{A}$ with codomain Y:

$$U_n \xrightarrow{d} U_{n-1} \cdots U_0 \xrightarrow{d} Y.$$

The “direction” of this chain complex is given by the kernel $K[d]$ of $d : U_n \to U_{n-1}$. Such a chain complex corresponds to an aspherical n-groupoid in n-$Grd(\mathbb{A}/Y)$ if and only if $d : U_0 \to Y$ is a regular epimorphism and the chain complex is exact at each level. So that the Moore equivalence at level n makes our group $H^{n+1}_{\mathbb{A}/Y}(A)$ the same as Yoneda's classical $Ext^{n+1}_{\mathbb{A}}(Y, A)$.

4.6 Connected components of $\mathbb{H}^{n+1}_{\mathbb{C}}(A)$

As in level 1, the category $\mathbb{H}^{n+1}_{\mathbb{C}}(A)$ has its connectedness length equal to 1. For that we shall quickly explicit the natural inductive construction which allows us to get round the difficulty involved by the fact that neither $\mathbb{C}_{\#}$ nor n-$Grd\mathbb{C}_{\#}$ admit pullbacks in general. When $\mathbb{E}$ is finitely complete, there is, in n-$Grd\mathbb{E}$, a natural construction which determines a $()_{n-1}$-cartesian diagram, see [4]:

$$\begin{tikzcd} Coh_n\underline{X}_n \arrow[r, shift left=2, "\underline{\omega}_n"] \arrow[r, shift right=2, "\underline{\alpha}_n"'] & \underline{X}_n \arrow[l, "\underline{\sigma}_n"'] \end{tikzcd}$$

above the following one in $(n-1)$-$Grd\mathbb{E}$:

$$\begin{tikzcd} & \int \underline{X}_n \arrow[dr, "\underline{x}^1_{n-1}"] \arrow[dr, dashed, shift right=2, "\underline{x}^0_{n-1}"'] & \\ Coh_{n-1}\underline{X}_n \arrow[ur, "\underline{\widetilde{\omega}}_{n-1}"] \arrow[dr, "\underline{\pi}^0_{n-1}"'] & & \underline{X}_{n-1} \\ & \underline{X}_{n-1}, \arrow[ur, "\underline{\alpha}_{n-1}"'] \arrow[ur, dashed, shift left=2, "\underline{\omega}_{n-1}"] & \end{tikzcd}$$

where the inner diamond is a pullback. The object $Coh_n\underline{X}_n$ classifies the higher order lax natural transformations with codomain $\underline{X}_n$. When $\mathbb{C} = \mathbb{A}$ is additive, the Moore equivalence $M_n : n\text{-}Grd\mathbb{A} \to Ch_n\mathbb{A}$ at level n exchanges $Coh_n\underline{X}_n$ with the universal classifier of the chain homotopies with codomain $M_n\underline{X}_n$, see [5]. Now, when $\mathbb{C}$ is effectively regular and naturally Mal'cev, because $\underline{\alpha}_n$ is $()_{n-1}$-cartesian and splits, the n-groupoid $Coh_n\underline{X}_n$ is aspherical whenever $\underline{X}_n$ is. Furthermore, by using amenable pullbacks, we can force the images by d_n of all maps involved in the definition of $Coh_n\underline{X}_n$ to be 1_A. Suppose we are given two morphisms $(\underline{f}_n, \underline{g}_n)$ in the fibre of $\mathbb{H}^{n+1}_{\mathbb{C}}(A)$ and consider the following pullback in n-$Grd\mathbb{C}$:

$$\begin{tikzcd} \underline{P}_n \arrow[r, "\underline{h}_n"] \arrow[d, "(\underline{\psi}_n, \underline{\phi}_n)"'] & Coh_n \underline{X}_n \arrow[d, "(\underline{\omega}_n, \underline{\alpha}_n)"] \\ \underline{U}_n \times \underline{V}_n \arrow[r, "\underline{f}_n \times \underline{g}_n"'] & \underline{X}_n \times \underline{X}_n. \end{tikzcd}$$

The map $\underline{h}_n$ is $()_{n-1}$-cartesian because so is $\underline{f}_n \times \underline{g}_n$. Since it is $()_{n-1}$-cartesian and $Coh_n \underline{X}_n$ is connected , then $\underline{P}_n$ is connected. The fact that $\underline{P}_n$ is aspherical is shown by induction, again see [4]. Then the above pullback lies in n-$Asp\mathbb{C}$ and we can switch to an amenable pullback above:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "s_0"'] & A \arrow[d, "s_0"] \\ A \times A \arrow[r, "1_{A \times A}"'] & A \times A. \end{tikzcd}$$

We have $d_n(\underline{\psi}_n) = 1_A$ and $d_n(\underline{\phi}_n) = 1_A$.

For exactly the same reasons as at level 1, the connected component of 0 is characterized by the following:

Proposition 4.9. *An n-groupoid $\underline{X}_n$ of $\mathbb{H}^{n+1}_{\mathbb{C}}(A)$ is in the component of 0 if and only if there is an aspherical $(n-1)$-groupoid $\underline{Z}_{n-1}$ and a map $\underline{\theta}_n : \nabla_n \underline{Z}_{n-1} \to \underline{X}_n$. In other words, if and only if there is a map with codomain $\underline{X}_n$ coming from the fibre $\mathbb{H}^{n+1}_{\mathbb{C}}(1)$.*

5 The long exact sequence

Our aim now is to show that any short exact sequence in $Ab\mathbb{C}$:

$$0 \longrightarrow A \overset{k}{\rightarrowtail} B \overset{h}{\twoheadrightarrow} C \longrightarrow 0 \qquad (6)$$

produces a long exact cohomology sequence of abelian groups:

$$\begin{tikzcd}
0 \arrow[r] & H^0_{\mathbb{C}}(A) \arrow[r, tail, "k_*"] & H^0_{\mathbb{C}}(B) \arrow[r, "h_*"] & H^0_{\mathbb{C}}(C) \arrow[dll, "\delta"'] \\
& H^1_{\mathbb{C}}(A) \arrow[r, "k_*"'] & H^1_{\mathbb{C}}(B) \arrow[r, "h_*"'] & H^1_{\mathbb{C}}(C) \\
& \cdots & \cdots & \cdots \\
& H^n_{\mathbb{C}}(A) \arrow[r, "k_*"] & H^n_{\mathbb{C}}(B) \arrow[r, "h_*"] & H^n_{\mathbb{C}}(C) \arrow[dll, "\delta_n"'] \\
& H^{n+1}_{\mathbb{C}}(A) \arrow[r, "k_*"'] & H^{n+1}_{\mathbb{C}}(B) \arrow[r, "h_*"'] & H^{n+1}_{\mathbb{C}}(C) & \cdots
\end{tikzcd}$$

5.1 The change of base functor

Let $h : B \to C$ be a map in $Ab\mathbb{C}$. Then the change of base functor $h_* : \mathbb{H}^{n+1}_{\mathbb{C}}(B) \to \mathbb{H}^{n+1}_{\mathbb{C}}(C)$ with respect to d_n is a strong monoidal functor since the following commutative square:

$$\begin{tikzcd}
B \times B \arrow[r, "+"] \arrow[d, "h \times h"'] & B \arrow[d, "h"] \\
C \times C \arrow[r, "+"'] & C
\end{tikzcd}$$

induces a natural isomorphism $h_*(\underline{X}_n \otimes \underline{Y}_n) \simeq h_*\underline{X}_n \otimes h_*\underline{Y}_n$, and consequently a group homomorphism $h_* : H^{n+1}_{\mathbb{C}}(B) \to H^{n+1}_{\mathbb{C}}(C)$. The following proposition characterizes the kernel of this group homomorphism when the map h is a regular epimorphism:

Proposition 5.1. *Given any short exact sequence* (6) *in $Ab\mathbb{C}$, the following sequence is exact in Ab:*

$$H^{n+1}_{\mathbb{C}}(A) \xrightarrow{k_*} H^{n+1}_{\mathbb{C}}(B) \xrightarrow{h_*} H^{n+1}_{\mathbb{C}}(C).$$

Proof. It is clear that this composition is trivial. Conversely, let $\underline{\theta}_n : \nabla_n \underline{Z}_{n-1} \to h_*\underline{X}_n$ be the map which asserts that $h_*\underline{X}_n = 0$. Then consider the following diagram where the left hand side square is a pullback in n-$Grd\mathbb{C}$:

$$\begin{tikzcd} & \underline{W}_n \arrow[dl, "\underline{k}_n"'] \arrow[r, two heads, "\underline{\rho}_n"] \arrow[d, "\underline{\chi}_n"] & \nabla_1 Z_{n-1} \arrow[d, "\underline{\theta}_n"] \\ k_*\underline{W}_n \arrow[r, "\underline{\psi}_n"'] & \underline{X}_n \arrow[r, two heads, "\underline{h}_n"'] & h_*\underline{X}_n \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "k"'] & 1 \arrow[d, "o_C"] \\ B \arrow[r, two heads, "h"'] & C. \end{tikzcd}$$

Since h is a regular epimorphism, so is the cocartesian $(()_{n-1}$-invertible) map $\underline{h}_n$, and consequently $\underline{\rho}_n$. Accordingly $\underline{W}_{n-1} = \underline{Z}_{n-1}$ and $\underline{W}_n$ is aspherical. Then switch to an amenable pullback above the right hand side square. Now the decomposition of $\underline{\chi}_n$ through the cocartesian map $\underline{k}_n$ above k gives a map $\underline{\psi}_n$ in the fibre of $\mathbb{H}^{n+1}(B)$ which makes $\underline{X}_n = k_*\underline{W}_n$ in the group $H^{n+1}_{\mathbb{C}}(B)$. □

5.2 The connecting homomorphism

We are going to define a connecting homomorphism $\delta_n : H^n_{\mathbb{C}}(C) \to H^{n+1}_{\mathbb{C}}(A)$. In Section 4.4 we noticed that the equality $d_{n-1}(\underline{W}_{n-1}) = C$ holds if and only if there is a discrete fibration $\underline{v}_n : \nabla_n \underline{W}_{n-1} \to K_n(C)$ in n-$Grd\mathbb{C}$. To define $\delta_n \underline{W}_{n-1}$ we begin with the following pullback in n-$Grd\mathbb{C}$:

$$\begin{tikzcd} \delta_n\underline{W}_{n-1} \arrow[r, two heads, "\underline{\eta}_n"] \arrow[d, "\underline{\chi}_n"'] & \nabla_n\underline{W}_{n-1} \arrow[d, "\underline{v}_n"] \\ K_n(B) \arrow[r, two heads, "K_n(h)"'] & K_n(C) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "k"'] & 1 \arrow[d, "o_C"] \\ B \arrow[r, two heads, "h"'] & C. \end{tikzcd}$$

Since $K_n(h)$ is $()_{n-1}$-invertible, so is the upper horizontal arrow, which is consequently $\underline{\eta}_n$. This is a regular epimorphism since $K_n(h)$ is so; accordingly $\delta_n\underline{W}_{n-1}$ is aspherical. So the pullback in question lies in n-$Asp\mathbb{C}$ and we can switch to an amenable pullback above the right hand side one. Hence, the n-direction of $\delta_n\underline{W}_{n-1}$ is A.

We are now going to investigate what the kernel of this connecting homomorphism is:

Proposition 5.2. *Given any short exact sequence* (6) *in $Ab\mathbb{C}$, the following sequence is exact in Ab:*

$$H^n_{\mathbb{C}}(B) \xrightarrow{h_*} H^n_{\mathbb{C}}(C) \xrightarrow{\delta_n} H^{n+1}_{\mathbb{C}}(A).$$

Proof. It is easy to see that this composition is trivial. Now, let $\underline{\theta}_n : \nabla_n \underline{Z}_{n-1} \to \delta_n \underline{W}_{n-1}$ be the map which asserts that $\delta_n \underline{W}_{n-1} = 0$. Using the pullback of the definition of $\delta_n \underline{W}_{n-1}$, the existence of $\underline{\theta}_n$ is equivalent to that of a pair $(\underline{\beta}_n, \underline{\theta}_{n-1})$ of maps making the following diamond commute:

$$\begin{tikzcd}
\nabla_n \underline{Z}_{n-1} \arrow[dd, "\underline{v}_n"'] \arrow[ddr, "\underline{\beta}_n"'] \arrow[dr, "\nabla_n \underline{g}_{n-1}"] \arrow[drr, "\nabla_n \underline{\theta}_{n-1}"] & & \\
& \nabla_n g_* \underline{Z}_{n-1} \arrow[d, "\underline{v}_n"] \arrow[r, dotted, "\nabla_n \underline{h}_{n-1}"'] & \nabla_n \underline{W}_{n-1} \arrow[d, "\underline{v}_n"] \\
K_n(\bar{B}) \arrow[r, "K_n(g)"'] & K_n(B) \arrow[r, "K_n(h)"'] & K_n(C).
\end{tikzcd}$$

Suppose that $d_{n-1}(\underline{Z}_{n-1}) = \bar{B}$. Consider the maps $g : \bar{B} \to B$ and $\underline{g}_{n-1}$ determined by the comprehensive factorization of β_n. Let $\underline{h}_{n-1}$ be the induced factorization of $\underline{\theta}_{n-1}$ above h. Then there is a factorization $h_*(g_*\underline{Z}_{n-1}) \to \underline{W}_{n-1}$ which makes $h_*(g_*\underline{Z}_{n-1}) = \underline{W}_{n-1}$ in the group $H^n_{\mathbb{C}}(C)$. □

5.3 The last step

Now we must investigate the kernel of $k_* : H^{n+1}_{\mathbb{C}}(A) \to H^{n+1}_{\mathbb{C}}(B)$.

Proposition 5.3. *Given any short exact sequence* (6) *in $Ab\mathbb{C}$, the following sequence is exact in Ab:*

$$H^n_{\mathbb{C}}(C) \xrightarrow{\delta_n} H^{n+1}_{\mathbb{C}}(A) \xrightarrow{k_*} H^{n+1}_{\mathbb{C}}(B).$$

Proof. It is clear that this composition is trivial. Conversely, let $(\underline{\psi}_n, \underline{\phi}_n)$ be the pair of maps in $\mathbb{H}^{n+1}_{\mathbb{C}}(B)$ assuring that $k_*\underline{X}_n$ is in the component of 0. Let us consider the following diagram where the upper left hand side square is a pullback in n-$Grd\mathbb{C}$:

$$
\begin{tikzcd}
\underline{X}_n \arrow[r, "\underline{k}_n"] & k_*\underline{X}_n & & \\
\underline{T}_n \arrow[u, "\underline{\psi}'_n"] \arrow[r, "\underline{\kappa}_n"] \arrow[ddd, "\underline{\eta}_n"'] & \underline{W}_n \arrow[u, "\underline{\psi}_n"'] \arrow[r, "\underline{\phi}_n"] \arrow[dr, dotted, "\underline{\zeta}_n"'] & & K_n(B) \arrow[ddd, "K_n(h)"] \\
& & \delta_n\underline{U}_{n-1} \arrow[ur, "\underline{\chi}_n"'] \arrow[ddd] & \\
& & & \\
\nabla_n\underline{W}_{n-1} \arrow[rrr, dotted, "\underline{\gamma}_n"] \arrow[drr, "\nabla_n\underline{\zeta}_{n-1}"'] & & & K_n(C) \\
& & \nabla_n\underline{U}_{n-1} \arrow[ur, "\underline{\upsilon}_n"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "k"] & B & & \\
A \arrow[u, "1_A"] \arrow[r, "k"] \arrow[dr, "1_A"] \arrow[dd] & B \arrow[u, "1_B"'] \arrow[rr, "1_B"] & & B \arrow[dd, "h"] \\
& & A \arrow[ur, "k"] \arrow[dd] & \\
1 \arrow[rrr, "o_C"] \arrow[drr, "1_1"'] & & & C. \\
& & 1 \arrow[ur, "o_C"'] &
\end{tikzcd}
$$

The map $\underline{\kappa}_n$ is $()_{n-1}$-invertible, so that $\underline{T}_{n-1} = \underline{W}_{n-1}$ is aspherical. Moreover, $\underline{\psi}'_n$ is $()_{n-1}$-cartesian, so that $\underline{T}_n$ is connected since so is $\underline{X}_n$. Accordingly $\underline{T}_n$ is aspherical. Then switch to an amenable pullback above the upper right hand side one. Since $d_n(K_n(h) \cdot \underline{\phi}_n \cdot \underline{\kappa}_n) = h \cdot k$ and $\underline{\eta}_n$ is cocartesian above $A \to 1$, there is a unique factorization $\underline{\gamma}_n$ above $o_c : 1 \to C$ making the lower rectangle commutative. Since $k_n(h)$ is cocartesian above h, then this rectangle is a pullback by having parallel cocartesian maps. Let $\underline{U}_{n-1}$ the object in the fibre $\mathbb{H}^n_{\mathbb{C}}(C)$ determined by the comprehensive factorization of $\underline{\gamma}_n$. Now consider the pullback, which lies in n-$Asp\mathbb{C}$, defining $\delta_n\underline{U}_{n-1}$ formed by the right hand side diamond. Then there is a factorization $\underline{\zeta}_n$ which certainly lies in $\mathbb{H}^{n+1}_{\mathbb{C}}(A)$, k being a monomorphism. This is also the case for $\underline{\psi}'_n$. Accordingly $\delta_n\underline{U}_{n-1}$ is in the component of $\underline{X}_n$, and $\underline{X}_n = \delta_n\underline{U}_{n-1}$ in the group $H^{n+1}_{\mathbb{C}}(A)$. $\square$

6 Applications

1. We recalled and drew, in the introduction, the diagram of the long exact sequence we were interested in. We already noticed that, thanks to the Moore normalization functor M_n, the classical Yoneda's group $Ext^n_{\mathbb{A}}(Y, A)$ for an abelian category $\mathbb{A}$ is nothing but our $H^n_{\mathbb{A}/Y}(A)$ (Examples 2.15 1), 3.8 and 4.8). Our investigation on the connected length of $H^n_{\mathbb{A}/Y}(A)$ shows the classical result still holds without any assumption on projectives.

2. The well known *Tierney equation* says that *abelian=additive+Barr exact.* We obtained the same result about the long exact sequence in the strictly weaker context of effectively regular additive categories. This is the case, for instance, of the category $AbTop$ and $AbHaus$ of topological and Hausdorff abelian groups. Thanks to the Moore normalization functor associated with the additive category $AbTop$, given two topological abelian groups A and Y, then $H^n_{AbTop/Y}(A)$ is nothing but the group $Ext^n_{AbTop}(Y, A)$ of continuous exact sequences of topological abelian groups:

$$0 \longrightarrow A \longrightarrow B_{n-1} \longrightarrow B_{n-2} \longrightarrow \cdots \longrightarrow B_0 \longrightarrow Y \longrightarrow 0.$$

The same description holds for $AbHaus$.

3. Suppose $\mathbb{C}$ is any arbitrary effectively regular naturally Mal'cev category and Y an object with global support. Then, according to Corollary 2.4, $Ab(\mathbb{C}/Y) = Ab\mathbb{C}$. Consequently, for any short exact sequence in $Ab\mathbb{C}$, there is still a Yoneda's Ext long exact sequence by setting $Ext^n_{\mathbb{C}}(Y, A) = H^n_{\mathbb{C}/Y}(A)$.

4. It is well known that, by a Moore normalization process, the category *GrdGp* of internal groupoids in the category *Gp* of groups is equivalent to the category X-Mod of crossed modules. Moreover, the category $Mal(Gp/C)$ of group homomorphisms $H \to C$ with abelian kernel is naturally Mal'cev, so an internal groupoid in this category is just a reflexive graph. This means that, in this context, our cohomology groups will have a simplified description we shall detail in a further article.

5. By the Example 1.1 4), the category $Mal(R_{Lie}/A)$, having Lie-homomorphisms with abelian kernel (i.e. equipped with trivial Lie brackets) as objects, is naturally Mal'cev and Barr exact. Consequently, our theory also applies to R-Lie algebras.

6. The categories $GpTop$ and $GpHaus$ of topological and Hausdorff groups are protomodular, and thus Mal'cev. Consequently the categories $Mal(GpTop/C)$ and $Mal(GpHaus/C)$ of continous extensions with abelian kernels are naturally Mal'cev. And our results apply here.

We shall give the details about the description of these cohomology groups in a further article.

7. When $\mathbb{E}$ is a finitely complete and effectively regular category, it is possible to extend, using our approach, the classical six term exact sequence of [1]:

$$\begin{tikzcd} 0 \arrow[r] & H^0_{\mathbb{E}}(A) \arrow[r, tail, "k_*"] & H^0_{\mathbb{E}}(B) \arrow[r, "h_*"] & H^0_{\mathbb{E}}(C) \arrow[dll, "\delta"'] \\ & H^1_{\mathbb{E}}(A) \arrow[r, "k_*"'] & H^1_{\mathbb{E}}(B) \arrow[r, "h_*"'] & H^1_{\mathbb{E}}(C) \end{tikzcd}$$

to any level, since the classical definition of $H^1_{\mathbb{E}}(A)$ in terms of principal group actions coincides with our own in terms of autonomous Mal'cev operations with direction A, see [7], namely since $H^0_{\mathbb{E}}(A) = H^0_{\mathbb{C}}(A)$ and $H^1_{\mathbb{E}}(A) = H^1_{\mathbb{C}}(A)$, where $\mathbb{C} = AutM\mathbb{E}$.

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