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NORA GANTER

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SMASH PRODUCT OF E(1)-LOCAL SPECTRA AT AN ODD PRIME

by Nora GANTER

ABSTRACT. Soit $(\mathcal{M}, \wedge)$ une catégorie de modèles monoïdale stable. Nous analysons l'interaction entre la structure monoïdale et les morphismes structurels du système de catégories triangulées de diagrammes $\mathrm{Ho}(\mathsf{M}^{\mathsf{C}})$ telles que définies dans [Fra96]. Comme application, nous prouvons qu'une équivalence de catégories definie dans [loc.cit.] envoie le Smash produit de spectres $\mathsf{E}(1)$ locaux sur un produit tensoriel dérivé de complexes de cochaînes.

Let $(\mathcal{M}, \wedge)$ be a stable monoidal model category. We analyze the interaction of the monoidal structure with the structure maps of the system of triangulated diagram categories $\mathrm{Ho}(\mathsf{M}^{\mathsf{C}})$ defined in [Fra96]. As an application, we prove that an equivalence of categories defined in [ibid.] maps the smash product of $\mathsf{E}(1)$-local spectra to a derived tensor product of cochain complexes.

1. Background and Introduction

1.1. Chromatic localizations and Franke's algebraic models. Let $\mathcal{S}$ denote the stable homotopy category, and let E be a generalized homology theory. Then the E_*-localization of $\mathcal{S}$ is a triangulated category $\mathcal{S}_{\mathsf{E}}$ together with an exact functor

$$\gamma_{\mathsf{E}}\colon \mathcal{S} \longrightarrow \mathcal{S}_{\mathsf{E}},$$

which is universal with the property that it sends E_*-isomorphisms to isomorphisms. It is constructed as Verdier quotient by the thick subcategory $\mathcal{C}_{\mathsf{E}}$ of E-acyclic objects of $\mathcal{S}$, or, alternatively, using Bousfield localization in any of the model categories for $\mathcal{S}$.

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The term *chromatic localizations* refers to homology localization at the Johnson-Wilson homology theories

$$E(n)_*(-),$$

which are defined by Landweber exactness of their coefficient groups

$$E(n)_* = \mathbf{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}]$$

over the Brown-Peterson spectrum BP.

The spectrum $E(1)$ is also known as the Adams summand of the p-local K-theory spectrum: the latter decomposes as

$$K_{(p)} = \bigvee_{i=0}^{p-2} \Sigma^{2i} E(1),$$

so that the localizations at $E(1)$ and at $K_{(p)}$ agree.

Let $\mathcal{S}^{\text{fin}}_{(p)}$ denote the category of finite p-local spectra, and

$$\mathcal{C}^{\text{fin}}_{E(n)} := \mathcal{C}_{E(n)} \cap \mathcal{S}^{\text{fin}}_{(p)}.$$

Then the *thick subcategory theorem* [HS98], says that the Verdier quotients

$$\mathcal{S}^{\text{fin}}_{(p)}/\mathcal{C}^{\text{fin}}_{E(n)}$$

are all possible homology localizations of the category of finite p-local spectra (in the above sense). Note that this is not true for the whole stable homotopy category. Further, the question whether the canonical functor from $\mathcal{S}^{\text{fin}}_{(p)}/\mathcal{C}^{\text{fin}}_{E(n)}$ to $\mathcal{S}_{E(n)}$ is full and faithful is equivalent to the telescope conjecture for $E(n)$ [Kra05, 11.1(3),13.4(5)].

From a categorical, structure theoretic point of view, the thick subcategory theorem implies that it will not be possible to find a model for the p-local stable homotopy category which is as simple as Serre's model for the rational stable homotopy category. There is indeed a theorem by Schwede, asserting that $\mathcal{S}_{(p)}$ has no "exotic" model [Sch01b]. The localized categories $\mathcal{S}_{E(n)}$ promise to be simpler, and indeed there are purely algebraic descriptions of these categories due to Bousfield and Franke:

Bousfield [Bou85] gave a purely algebraic description of the objects of the $E(1)$-local stable homotopy category at an odd prime, but did not address homotopy classes of maps. He introduced an important tool, namely a construction for the Adams spectral sequence by injective resolutions,

$$E_2^{s,t} = \mathrm{Ext}^s_{E(1)_*E(1)}(E(1)_*X, E(1)_*Y[t]) \Longrightarrow [X, Y]_{E(1)\text{-local},t-s}.$$

Franke [Fra96] used Bousfield's work, in particular the computation of the injective dimension of the category of $E(1)_*E(1)$-comodules (it equals two) and this spectral sequence construction to show that $\mathcal{S}_{E(1)}$ is equivalent to the derived category of a certain type of cochain complexes of $E(1)_*E(1)$-comodules. He generalized the result to "higher chromatic primes", i.e. $E(n)$-local spectra and $E(n)_*E(n)$-comodules (for $n^2+n < 2p-2$).

1.2. **Systems of Triangulated Diagram Categories.** Franke's functors are only defined on the level of homotopy categories, and he actually proves that the corresponding models are not Quillen equivalent [Fra96, Rem.3.1.1]. This is the sense in which Schwede refers to them as "exotic" models. The equivalences do, however, preserve the triangulated structure as well as homotopy Kan extensions along maps of finite posets up to a certain length. In particular they preserve homotopy (co)limits over such diagrams. For the construction of Franke's functors it is essential to have functorial cones and homotopy Kan extensions. How can one never work on the level of model categories but still have a good handle on homotopy Kan extensions? The answer is to work with homotopy categories of strictly commuting diagrams of spectra; for each finite poset one category. This leads to the notion of a *system of triangulated diagram categories* (cf. Section 3.1).

1.3. **Introduction.** In this paper, we will describe how a monoidal structure interacts with the structure maps of a system of triangulated diagram categories. As an application, we will describe the interaction of Franke's functor with the smash product in the case $n = 1$. Denote Franke's functor by $\mathcal{R}ec$. We will prove:

Theorem 1.1. *There is a functor isomorphism*

$$\mathcal{R}ec(-) \wedge^L_{S_{E(1)}} \mathcal{R}ec(-) \cong \mathcal{R}ec(- \otimes^L_{E(1)_*} -).$$

Note that the category of $E(1)_*E(1)$-comodules does not have enough projectives, so that the existence of the derived tensor product on the right-hand side is a non-trivial statement. In spite of our theorem, $\mathcal{R}ec$ is not a monoidal functor:

Remark 1.2 (Schwede). Let $p = 3$, and denote the mod 3 Moore spectrum by $M(3)$. It has a unique multiplication, which is not associative. However,

$$\mathcal{R}ec^{-1}(M(3)) \simeq \cdots \to 0 \to E(1)_* \xrightarrow{\cdot 3} E(1)_* \to 0 \to \cdots$$

possesses an associative multiplication. Thus we cannot hope to find an associative functor isomorphism between $- \wedge^L -$ and the (derived) tensor product.

In order to make our work easily accessible for topologists, we stick to the terminology of stable homotopy theory. We use, however, only very general concepts, that could as well be formulated in the language developed in [Fra96].

1.4. **Plan.** In Section 3 we recall the parts of [Fra96] needed for our computations. We discuss systems of triangulated diagram categories, present the construction of the equivalence functor in a dual (but equivalent) version and state an easy generalization of the spectral sequence [Fra96, 1.4.35]. We do not assume familiarity with [Fra96], but some of our proofs will quote more of the results of that article. Section 4 is about the interaction of a monoidal structure on $\mathcal{M}$ with the system of triangulated diagram categories $\mathrm{Ho}(\mathcal{M}^C)$. It also contains some results about the computation of the edge morphisms of a homotopy Kan extension, which will be important tools for our computations in Section 7. In Section 5 we explain which model for $\mathcal{S}_{E(1)}$ we choose to work with. In Section 6 we show that the derived tensor product in Theorem 1.1 is well defined and reduce the proof of Theorem 1.1 to the case of flat complexes. Section 7 then contains the proof for flat complexes. Readers not interested in the appplication to $E(1)$-local spectra can skip Sections 3.2 – 3.3, 6 and 7.

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2. Notations and conventions

By p we will always mean an odd prime. Posets are depicted as follows: vertices represent elements, and $x \leqslant y$ if and only if the vertex corresponding to x is linked to the vertex corresponding to y by an ascending path. The length of a poset C is defined to be the supremum of all k such that there exists a sequence $x_0 < x_1 < \cdots < x_k$ in C. All posets considered are finite and therefore of finite length. The cocycles of a cochain complex $C^\bullet$ are denoted $Z^\bullet$ and its coboundaries $B^\bullet$. For our purposes it does not matter whether we work with symmetric spectra [HSS00] or S-modules [EKMM97]. For a strict ring spectrum R, we denote the category of strict R-module spectra by $\mathcal{M}_R$ and its derived category by $\mathrm{Ho}(\mathcal{M}_R)$. If E is a spectrum, we denote the underlying category of $\mathcal{M}_S$, endowed with the model structure used for Bousfield localization at $E_*(-)$, by

$$\mathcal{M}_S[E^{-1}].$$

Its derived category, i.e. the localization of the stable homotopy category at $E_*(-)$ is denoted by

$$\mathcal{S}_E := \mathrm{Ho}(\mathcal{M}_S[E^{-1}]).$$

The identity functors

$$\mathcal{M}_S \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} \mathcal{M}_S[E^{-1}]$$

form a Quillen pair, and we write $(-)_E$ for the composition of its derived functors[1]

$$\mathcal{S} \xrightarrow{\mathrm{id}^L} \mathcal{S}_E \xrightarrow{\mathrm{id}^R} \mathcal{S}.$$

It is induced by fibrant replacement in $\mathcal{M}_S[E^{-1}]$ viewed as an endofunctor of $\mathcal{M}_S$. We will sometimes also write $(-)_E$ for the (strict) fibrant replacement in $\mathcal{M}_S[E^{-1}]$. In particular, S_E stands for the (strict) E-local

[1]Franke works with the localized categories, but topologists like to think of this functor as localization functor and refer to its image as "E-local spectra".

sphere, not as in [EKMM97] for the free E-module spectrum (if E is a ring spectrum).

Definition 2.1. A quasi periodic cochain complex of period N is a cochain complex $C^\bullet_*$ of $E(1)_*E(1)$-comodules C^n_* together with an isomorphism of cochain complexes of comodules

$$C^\bullet_{*-N} = C^\bullet_*[N],$$

where the right hand side stands for the complex $C^\bullet_*$ shifted to the left N times.

We denote the category of $E(1)_*E(1)$-comodules which are concentrated in degrees congruent to 0 modulo $2p-2$ by[2]

$$\mathrm{Comod}^0_{E(1)_*E(1)}.$$

We denote the category of period $2p-2$ quasi periodic cochain complexes in $\mathrm{Comod}^0_{E(1)_*E(1)}$ by

$$\mathcal{C}^{2p-2}(\mathrm{Comod}^0_{E(1)_*E(1)}),$$

and its derived category by[3]

$$\mathcal{D}^{2p-2}(\mathrm{Comod}^0_{E(1)_*E(1)}).$$

3. Franke's Algebraic Models

This section is a collection of the parts of [Fra96] needed for our constructions. We only consider the following special case of the main theorem of [Fra96]:

Theorem 3.1 (Bousfield, Franke). *The category of $E(1)$-local spectra is in a unique way equivalent to the derived category of period 1 quasi-periodic cochain complexes of $E(1)_*E(1)$-comodules.*

Here "unique" means "unique up to canonical natural isomorphism, given that the equivalence is also valid for diagram categories for diagrams of length $\leqslant 2$, preserves certain additional structure between these, and transforms $E(1)_*(-)$ into something naturally isomorphic to $H^*(-)$".

[2]In [Bou85] this category is called $\mathcal{B}(p)$, in [Fra96] it is $\tilde{\mathcal{B}}$.

[3]Compare to Franke's notations $\mathfrak{C}^{2p-2,[2p-2]}(\mathcal{B})$ and $\mathfrak{D}^{2p-2,[2p-2]}(\mathcal{B})$.

3.1. **Systems of Triangulated Diagram Categories.** Rather than recalling Franke's definition of a system of triangulated diagram categories, we explain the most important example. A model category $\mathcal{M}$ is called *stable*, if the suspension functor is invertible in the homotopy category $\mathrm{Ho}(\mathcal{M})$. This condition implies that $\mathrm{Ho}(\mathcal{M})$ is a triangulated category, where the triangles are the cofibre sequences [Hov99, 7.1]. We will be interested in the category of E(1)-local spectra . This is a triangulated category, and all of its standard models[4] are stable. Franke's paper never does any constructions on the level of model categories. However, the homotopy category alone is not rigid enough. For example the cone is not a functor in a triangulated category. Therefore, Franke also allows himself to work with homotopy categories of categories of diagrams in $\mathcal{M}$: Let C be a finite[5] poset. The category $\mathcal{M}^{\mathsf{C}}$ of C shaped diagrams in $\mathcal{M}$ has a model structure with vertex-wise weak equivalences and fibrations. The cofibrations are characterized as follows[6]: A morphism from A to B in $\mathcal{M}^{\mathsf{C}}$ is a Reedy cofibration if and only if for all $\mathsf{c} \in \mathsf{C}$ the morphism

$$A_c \coprod_{\varinjlim_{c'<c} A_{c'}} \varinjlim_{c'<c} B_{c'} \longrightarrow B_c, \tag{1}$$

induced by the universal properties, is a cofibration in $\mathcal{M}$. Of course we could just as well define cofibrations and weak equivalences vertex-wise and thus force the fibrations to be the maps satisfying the dual condition to (1). Note that both model structures have the same weak equivalences, and therefore the same homotopy category. Franke considers (for fixed $\mathcal{M}$) the entire system of homotopy categories of diagram model categories. This allows him to use functorial homotopy Kan extensions, in particular homotopy (co)limits, cones etc.: Let $f\colon \mathsf{C} \to \mathsf{D}$ be a map of posets. The above model structure is made in such a way that pulling back diagrams along f preserves fibrations and trivial fibrations. Therefore the adjoint functors

$$\mathcal{M}^{\mathsf{C}} \underset{f^*}{\overset{\mathrm{LKan}_f}{\rightleftarrows}} \mathcal{M}^{\mathsf{D}}$$

[4]See Section 5.

[5]One could do with much weaker conditions on C, see [Hov99, 5.2.], [DHKS04] and the original source [Ree74].

[6]see e.g. [Fra96]. Here these cofibrations are called "diagram cofibrations"

form a Quillen pair[7], and LKan_f has a left derived functor. Similarly, using the other model structure, one defines right homotopy Kan extensions, right adjoint to f^*. The example of the cone functor [Fra96, 1.4.5] illustrates how homotopy Kan extensions are computed and why they are useful.

Notation 3.2. Write $\mathbb{I}$ for the poset $\{0 < 1\}$, and

$$\vee \subset \mathbb{I} \times \mathbb{I}$$

for the sub-poset with elements $\{(1,0),(0,0),(0,1)\}$.

Definition 3.3 (Franke). Let $f \in \mathrm{Ho}(\mathcal{M}^{\mathbb{I}})$. The cone of f is defined as homotopy colimit over the "cokernel diagram" of f:

$$\begin{array}{ccc} \star & & Y \\ & \diagdown \quad \diagup^{f} & \\ & X & \end{array}$$

More precisely, let

$$\overset{\star}{\vee} : \mathcal{M}^{\mathbb{I}} \longrightarrow \mathcal{M}^{\vee}$$

denote the functor that takes as input an element f of $\mathcal{M}^{\mathbb{I}}$ and returns the object of $\mathcal{M}^{\vee}$ that has f at the edge $((0,0) \leqslant (0,1))$ and the (strict) zero object at the vertex $(1,0)$. Then $\overset{\star}{\vee}$ preserves weak equivalences and thus induces a functor

$$\overset{\star}{\vee} : \mathrm{Ho}(\mathcal{M}^{\mathbb{I}}) \longrightarrow \mathrm{Ho}(\mathcal{M}^{\vee}).$$

We define the cone of f by

$$\mathrm{cone}(f) := \underset{\overrightarrow{\vee}}{\mathrm{Holim}}\ \overset{\star}{\vee}(f).$$

Remark 3.4. Note that

$$\overset{\star}{\vee} \cong \mathrm{HoRKan}_{\mathbb{I} \subset \vee}.$$

Remark 3.5 (Comparison with the classical cone definition). Let $f\colon X$ — be a cofibration between cofibrant objects. Let CX be a cone object of X, i.e. part of a factorization

$$X \rightarrowtail CX \overset{\sim}{\twoheadrightarrow} \star$$

of the map from X to the terminal object.

[7] For an introduction to Kan extensions, the reader is referred to [ML98, Ch. 10].

Consider the strict pushout diagram

(2)
$$\begin{array}{ccccc} & & CX & & \\ & \nearrow & & \searrow & \\ X & \xrightarrow{f} & Y & \longrightarrow & Y \coprod_{f} CX. \end{array}$$

Without the bottom right corner, this is a Reedy cofibrant replacement of $\overset{\star}{\vee}{}^{\circ}(\mathrm{f})$ in $\mathcal{M}^{\vee}$. Therefore the bottom right corner is $\mathrm{cone}(\mathrm{f})$.

The functor cone takes values in $\mathrm{Ho}(\mathcal{M})$. Sometimes that is not good enough. If, for instance, we want to define cofibre sequences, we have to take the cone of the "*cone inclusion map*", i.e. of the bottom right arrow in (2). We need a functor

$$\mathfrak{Cone}\colon \mathrm{Ho}(\mathcal{M}^{\mathfrak{z}}) \longrightarrow \mathrm{Ho}(\mathcal{M}^{\mathfrak{z}})$$

that returns the bottom right arrow of (2). This is given by [Fra96, 1.4.5]

(3)
$$\mathfrak{Cone}(-) := \Big(\operatorname*{HoLKan}_{\vee^{\circ}\subset(\mathfrak{z}\times\mathfrak{z})} \circ \overset{\star}{\vee}{}^{\circ}(-)\Big)_{(0,1)<(1,1)}.$$

3.2. **Construction of Franke's Functor.** Using the functor $\mathrm{E}(1)_*(-)$, Franke defines a "reconstruction" functor, $\mathcal{R}ec$, from the derived category of quasi periodic cochain complexes of $\mathrm{E}(1)_*\mathrm{E}(1)$-comodules into the category of $\mathrm{E}(1)$-local spectra. It turns out that $\mathcal{R}ec$ is an equivalence of categories. In order to motivate the way $\mathcal{R}ec$ is defined, we look at the simplest analogous situation: instead of the system of diagram categories corresponding to $\mathrm{E}(1)$-local spectra, we take the system corresponding to quasi periodic $\mathrm{E}(1)_*\mathrm{E}(1)$-comodules and pretend that we want to reconstruct the identity functor by using only information that can be obtained via $\mathrm{H}^*(-)$ (this now plays the role of $\mathrm{E}(1)_*(-)$). Note that the data of a period 1 quasi periodic cochain complex of $\mathrm{E}(1)_*\mathrm{E}(1)$-comodules are the same as the data of a period $2\mathrm{p}-2$ quasi periodic cochain complex of $\mathrm{E}(1)_*\mathrm{E}(1)$-comodules that are concentrated in degrees congruent to zero modulo $2\mathrm{p}-2$, i.e.

(4)
$$\mathcal{C}^1(\mathrm{Comod}_{\mathrm{E}(1)_*\mathrm{E}(1)}) \cong \mathcal{C}^{2\mathrm{p}-2}(\mathrm{Comod}^{0}_{\mathrm{E}(1)_*\mathrm{E}(1)}).$$

More precisely, any period one quasi periodic cochain complex $C_*^\bullet$ is uniquely determined by its zeroest differential

$$d^0 \colon C_*^0 \longrightarrow C_*^1 \cong C_{*-1}^0.$$

Then the corresponding $2p-2$ periodic cochain complex $D_*^\bullet$ is given by

$$(D_*^k, d^k) = \begin{cases} (C_{*-k}^0, d_{*-k}^0) & \text{if } * \equiv 0 \mod 2p-2 \\ 0 & \text{else.} \end{cases}$$

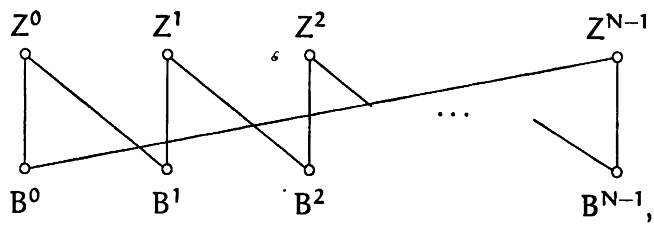
Now any quasi periodic cochain complex $C^\bullet$ of period N can be decomposed into N pieces of the form

(5)

$$\cdots \to 0 \to \underset{\substack{\vdots \\ n^{th}\text{ spot}}}{\hat{C^n}} \to B^{n+1} \to 0 \to \cdots \to 0 \to C^{n+N} \to B^{n+1+N} \to 0 \to \cdots$$

which can be glued back together along the inclusions of the B^n into C^n. More precisely, $C^\bullet$ is the colimit of the diagram of complexes

(6)



where we abbreviate (5) by Z^n, and

$$\cdots \to 0 \to B^n \to 0 \to \cdots \to 0 \to B^{n+N} \to 0 \to \cdots$$

by B^n; the vertical edges are $B^n \hookrightarrow C^n$, and the diagonal edges are $B^{n+1} = B^{n+1}$ (recall our convention that arrows in posets always go upwards). Note that the colimit of the diagram (6) is equal to its homotopy colimit by (1). How can we read off $C^\bullet$ from the cohomology of such a diagram? Note first that

$$\cdots \to 0 \to C^n \to 0 \to \ldots$$

turns up as cone of the diagonal maps. The cone inclusion to the complex ΣB^{n+1} is d^n. (Note that B^{n+1} is concentrated in degrees congruent to $n+1$, and that Z^n and C^n are concentrated in degrees congruent to n.) Applying H^* to the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & C^n & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow d & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & B^{n+1} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array} \qquad \begin{array}{c} C^n \\ \downarrow \\ \Sigma B^{n+1}. \end{array}$$

does not destroy any information, and the same is true for the composite

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & B^{n+1} & \longrightarrow & 0 & \longrightarrow & \cdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & C^{n+1} & \longrightarrow & B^{n+2} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \| & & \downarrow & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & C^{n+1} & \longrightarrow & 0 & \longrightarrow & \cdots & & \end{array} \qquad \begin{array}{c} B^{n+1} \\ \downarrow \text{vertical edge} \\ Z^{n+1} \\ \downarrow \text{cone inclusion} \\ C^{n+1} \end{array}$$

In other words, it is possible to reconstruct $C^\bullet$ from (6) by applying H^*. We return to $E(1)$-local spectra.

Notation 3.6. We denote the underlying poset of (6) by C_N. It has vertices ζ_n and β_n, where $n \in \mathbb{Z}/_N$ and relations $\beta_n \leqslant \zeta_n$ and $\beta_{n+1} \leqslant \zeta_n$. In our case, $N = 2p-2$.

Notation 3.7. Following Franke, we write $\mathcal{L}$ for the full subcategory of objects A of $\mathrm{Ho}(\mathcal{M}^{C_N}_{S_{E(1)}})$ satisfying:

- $Z^n_* := E(1)_{*-n}(A_{\zeta_n})$ and $B^n_* := E(1)_{*-n}(A_{\beta_n})$ are concentrated in degrees $\equiv 0 \mod 2p-2$, and
- $E(1)_{*-n}(A_{\beta_n} \to A_{\zeta_n})\colon B^n_* \longrightarrow Z^n_*$ is injective.

Construction 3.8. Let A be an object of $\mathcal{L}$. We define

$$C^n_*(A) := E(1)_{*-n}\left(\mathrm{cone}(A_{\beta_{n+1}} \to A_{\zeta_n})\right).$$

If we apply $E(1)_{*-n}(-)$ to the exact triangle

$$A_{\beta_{n+1}} \to A_{\zeta_n} \to \mathrm{cone}\left(A_{\beta_{n+1}} \to A_{\zeta_n}\right) \to \Sigma A_{\beta_{n+1}} \to \Sigma A_{\zeta_n}$$

we obtain a (short) exact sequence

$$B^{n+1}_{*+1} \xrightarrow{0} Z^n_* \to C^n_* \to B^{n+1}_* \xrightarrow{0} Z^n_{*-1}.$$

It follows that C_*^n is also concentrated in degrees $\equiv 0 \mod 2p-2$. In order to define the differential, we apply $E(1)_{*-n}(-)$ to

$$\operatorname{cone}\left(A_{\beta_{n+1}} \to A_{\zeta_n}\right) \to \Sigma A_{\beta_{n+1}} \to \Sigma A_{\zeta_{n+1}} \to \Sigma \operatorname{cone}\left(A_{\beta_{n+2}} \to A_{\zeta_{n+1}}\right)$$

and obtain

$$C_*^n(A) \twoheadrightarrow B_*^{n+1} \rightarrowtail Z_*^{n+1} \rightarrowtail C_*^{n+1}(A), \qquad \underbrace{\qquad\qquad}_{d^n} \tag{7}$$

Franke proves that this defines an equivalence of categories

$$\begin{aligned} \mathcal{L} &\to \mathcal{C}^{2p-2}(\mathrm{Comod}^0_{E(1)_*E(1)}) \\ A &\mapsto C_*^\bullet(A). \end{aligned}$$

Notation 3.9. We call this equivalence Q.

Let Q^{-1} be a quasi-inverse of Q. Franke further shows that

$$\mathcal{R}ec := \underrightarrow{\mathrm{Holim}} \circ Q^{-1}\colon \mathcal{C}^{2p-2}(\mathrm{Comod}^0_{E(1)_*E(1)}) \longrightarrow \mathcal{S}_{E(1)} \tag{8}$$

factors through the derived category and induces an equivalence of categories

$$\mathcal{D}^1(E(1)_*E(1)\text{-comod}) \simeq \mathcal{D}^{2p-2}(\mathrm{Comod}^0_{E(1)_*E(1)}) \to \mathcal{S}_{E(1)},$$

the first equivalence being (4). We denote this equivalence also by $\mathcal{R}ec$.

Remark 3.10. Franke's construction uses cokernels and coimages, rathe than kernels and images. The reason for this is that we need to use Adams spectral sequences via injective resolutions – there are not enough projectives. The theorem is also valid for diagram categories up to a certain length of the diagrams. In order to improve the restrictions on this length, it is advantageous to work with cokernels and coimages. However, Franke also proves a uniqueness statement about $\mathcal{R}ec$, which implies that the dual construction presented here results in the same functor on $\mathcal{S}_{E(1)}$.

3.3. **A spectral sequence.** We now recall the construction of the spectral sequence [Fra96, 1.4.35]. We rephrase Franke's proof to get a generalization for homotopy Kan extensions: Let $\mathcal{M}$ be a stable model category, let $f\colon D \to C$ be a map of finite posets, and let $Y \in \mathrm{Ho}(\mathcal{M}^D)$. Further, let

$$F_*\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{A}$$

be a homological functor into a Grothendieck category of graded objects, such that

$$F_{t+1}(X) = F_t(\Sigma X).$$

Then there is a spectral sequence

$$E_2^{s,t} = \underset{f}{\mathrm{LKan}}_{-s} F_t(Y) \Longrightarrow F_{s+t}(\underset{f}{\mathrm{HoLKan}}\, Y). \tag{9}$$

The example relevant for us is

$$F_t(-) := E(1)_{-t}(-) \colon \mathrm{Ho}(\mathcal{M}_{S_{E(1)}}) \longrightarrow \mathrm{Comod}_{E(1)_*E(1)}.$$

The **Construction** of (9) is analogous to that of [Fra96, 1.4.35]: Let D be a poset. For $d \in D$, let $i_d \colon \{d\} \longrightarrow D$ denote the inclusion of the vertex d in D. For $Y \in \mathrm{Ho}(\mathcal{M}^D)$, define

$$\tilde{P}Y := \bigoplus_{d \in D} \underset{i_d}{\mathrm{HoLKan}}\, Y_d.$$

Then we have:

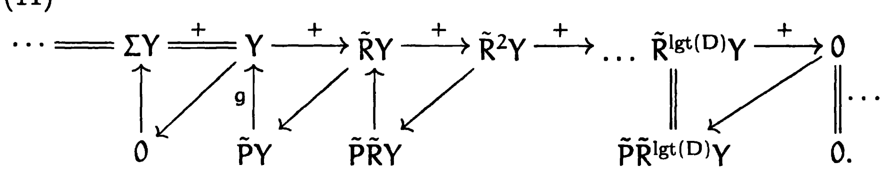
$$(\tilde{P}Y)_{d'} = \bigoplus_{\{d \in D \mid d \leqslant d'\}} Y_d. \tag{10}$$

We define a morphism $g \colon \tilde{P}Y \to Y$ by letting

$$g|_{\mathrm{HoLKan}_{i_d} Y_d} \colon \underset{i_d}{\mathrm{HoLKan}}\, Y_d \longrightarrow Y$$

be the counit of the adjunction. Taking the homotopy fibre $\tilde{R}Y$ of g and iterating the whole process with $\tilde{R}Y$ playing the role of Y, one obtains a resolution

(11)



Applying the homological functor $F_*(\mathrm{HoLKan}_f(-))$, one obtains an exact couple

$$\begin{array}{ccc} D_1^{**} & \xrightarrow[\alpha]{(-1,1)} & D_1^{**} \\ {\scriptstyle (0,0)\ \gamma}\nwarrow & & \swarrow{\scriptstyle \beta\ (1,0)} \\ & E_1^{**} & \end{array}$$

with

$$\begin{aligned} D_1^{s,t} &= F_t(\underset{f}{\mathrm{HoLKan}}(\tilde{R}^{-s}Y)) \\ E_1^{s,t} &= F_t(\underset{f}{\mathrm{HoLKan}}(\tilde{P}\tilde{R}^{-s}Y)). \end{aligned}$$

This gives rise to a (cohomological) spectral sequence in the usual way.

We want to discuss its **Convergence**: Let lgt D denote the length of D, and let $r > \mathrm{lgt}\,D$. The $(r-1)^{st}$ derived exact couple (D_r^{**}, E_r^{**}) is:

$$D_r^{s,t} = \mathrm{im}(\alpha^{r-1}\colon D_0^{s+r-1,t-r+1} \longrightarrow D_0^{s,t})$$

$$= \left\{ \begin{array}{ll} D_0^{0,s+t} & \text{if } s \geqslant 0, \\ \mathrm{im}(\alpha^{-s}) & \text{if } -\mathrm{lgt}\,D \leqslant s < 0, \\ 0 & \text{else.} \end{array} \right\}$$

It is of the form

$$\begin{array}{ccccccccccc} D_0^{1,*+1} & \overset{+}{=\!=} & D_0^{0,*} & \overset{+}{\twoheadrightarrow} & \mathrm{im}(\alpha) & \overset{+}{\twoheadrightarrow} & \mathrm{im}(\alpha^2) & \overset{+}{\twoheadrightarrow} & \ldots\,\mathrm{im}(\alpha^{\mathrm{lgt}(D)}) & \overset{+}{\twoheadrightarrow} & 0 \\ \uparrow & \swarrow_0 & \uparrow & \swarrow_0 & \uparrow & \swarrow_0 & \uparrow & & \uparrow & \swarrow_0 & \\ 0 & & E_r^{0,*} & & E_r^{-1,*} & & E_r^{-2,*} & & E_r^{-\mathrm{lgt}(D),*}. & & \end{array}$$

Therefore, the spectral sequence collapses after $(\mathrm{lgt}(D)+1)$ steps, and converges to

$$\begin{array}{ccccccc} F_t(\mathrm{HoLKan}_f Y) & \twoheadrightarrow & D_\infty^{-1,t+1} & \twoheadrightarrow & D_\infty^{-2,t+2} \ldots & \twoheadrightarrow\ D_\infty^{-\mathrm{lgt}\,D,t+\mathrm{lgt}\,D} & \twoheadrightarrow 0 \\ & \vdots & & \vdots & & & \\ & \ker = E_\infty^{0,t} & & \ker = E_\infty^{-1,t+1} & & & \end{array}$$

In order to **identify the E_2 term** with

$$\operatorname{LKan}_{f} {}_{-s}F_t(Y),$$

we need to show that

(12) $$F_t(Y) \twoheadleftarrow F_t(\tilde{P}\tilde{R}^\bullet Y)$$

is an LKan_f-acyclic resolution. It is obviously exact. Since

$$F_t(\tilde{P}Y') = \bigoplus_{d\in D} \operatorname{LKan}_{i_d} F_t(Y'_d),$$

the following lemma implies that the objects $F_t(\tilde{P}\tilde{R}^\bullet Y)$ are LKan_f-acyclic.

Lemma 3.11. *Let $\mathcal{A}$ be a Grothendieck category. Let D be a finite poset and assume that $X \in \mathcal{A}^D$ is such that for any $d \in D$ the map*

$$\varinjlim_{c<d} X_c \longrightarrow X_d,$$

given by the universal property of the colimit applied to the edges of X, is a monomorphism. Then for any map of finite posets $f\colon D \longrightarrow C$, the object X is LKan_f-acyclic. In particular, X is also $\varinjlim$-acyclic.

PROOF: We endow the category $\mathcal{C}(\mathcal{A})$ of cochain complexes in $\mathcal{A}$ with the injective model structure[8]. For the category of D-diagrams of chain complexes,

$$\mathcal{C}(\mathcal{A}^D) \cong \mathcal{C}(\mathcal{A})^D,$$

we choose the model structure whose fibrations and weak equivalences are defined vertex-wise, and whose cofibrations are characterized by (1). Then the satellite functors of the total derived functor

$$\mathcal{A}^D \longrightarrow \mathcal{D}(\mathcal{A}^D) \xrightarrow{\mathrm{HoLKan}_f} \mathcal{D}(\mathcal{A}^C)$$

satisfy the universal property of the (left-) derived left Kan extensions. Now the condition on X in the lemma says exactly that X, viewed as

[8]By that we mean that cofibrations are degree-wise monomorphisms, weak equivalences are the quasi isomorphisms, and fibrations are degree-wise epimorphisms with degree-wise injective cokernel. For the existence of such a model structure on the category of unbounded cochain complexes in a Grothendieck-category, see [Fra01] or [Hov01]. There one can also find a discussion of the connection between this story and derived functors in the sense of homological algebra.

an object of $\mathcal{C}(\mathcal{A}^{\mathsf{D}})$ (i.e. as a cochain complex which is concentrated in degree zero), is Reedy cofibrant. Thus in $\mathcal{D}(\mathcal{A}^{\mathsf{D}})$, we have

$$\operatorname{HoLKan}_{\mathsf{f}} \mathsf{X} \cong \operatorname{LKan}_{\mathsf{f}} \mathsf{X}.$$

The object $\operatorname{LKan}_{\mathsf{f}} \mathsf{X}$, however, is also concentrated in degree zero. Therefore all higher derived functors vanish. □

We have shown that the objects in the resolution (12) are $\operatorname{LKan}_{\mathsf{f}}$-acyclic. Further, by (10), we have

$$\mathsf{F}_t(\operatorname{HoLKan}_{\mathsf{f}} \tilde{\mathsf{P}}\mathsf{Y}')_{\mathsf{c}} = \mathsf{F}_t\Big(\bigoplus_{\mathsf{f}(\mathsf{d})\leqslant \mathsf{c}} \mathsf{Y}'_{\mathsf{d}}\Big) = \bigoplus_{\mathsf{f}(\mathsf{d})\leqslant \mathsf{c}} \mathsf{F}_t(\mathsf{Y}'_{\mathsf{d}}) = \Big(\operatorname{LKan}_{\mathsf{f}} \mathsf{F}_t(\tilde{\mathsf{P}}\mathsf{Y}')\Big)_{\mathsf{c}}.$$

Therefore we have shown that the cohomology of the complex

$$\mathsf{F}_t(\operatorname{HoLKan}_{\mathsf{f}} \tilde{\mathsf{P}}\tilde{\mathsf{R}}^\bullet \mathsf{Y})$$

computes the derived functors of the left Kan extensions.

In calculations, we sometimes write down the E_2-term directly. If the reader is not familiar with computing derived Kan extensions, (s)he can check its correctness by writing down the E_1-term.

4. Smash products for diagram categories

This section discusses the interaction of a monoidal structure with a system of triangulated diagram categories. Our first goal is to define a smash product between the (homotopy) diagram categories. We start by defining a strict smash product of diagrams.

Definition 4.1. Let $(\mathcal{M}, \wedge)$ be a (model) category with monoidal structure. Let C and D be finite posets, let

$$\mathsf{f}\colon \mathsf{X} \to \mathsf{Y} \in \operatorname{Mor}(\mathcal{M}^{\mathsf{C}}) \text{ and } \mathsf{g}\colon \mathsf{U} \to \mathsf{V} \in \operatorname{Mor}(\mathcal{M}^{\mathsf{D}}).$$

We define $\mathsf{X} \wedge \mathsf{U} \in \operatorname{Ob}(\mathcal{M}^{\mathsf{C}\times\mathsf{D}})$ by

$$(\mathsf{X} \wedge \mathsf{U})_{(a,b)\leqslant(c,d)} := \mathsf{X}_{a\leqslant c} \wedge \mathsf{U}_{b\leqslant d}$$

and $\mathsf{f} \wedge \mathsf{g} \in \operatorname{Hom}_{\mathcal{M}^{\mathsf{C}\times\mathsf{D}}}(\mathsf{X} \wedge \mathsf{U}, \mathsf{Y} \wedge \mathsf{V})$ by

$$(\mathsf{f} \wedge \mathsf{g})_{(c,d)} := \mathsf{f}_c \wedge \mathsf{g}_d.$$

The following definitions are special cases of the definitions in [Hov99, 4.2.].

Definition 4.2. In the situation of the previous definition the pushout smash product of f and g is defined to be the canonical map

$$f\Box g\colon X\wedge V \coprod_{X\wedge U} Y\wedge U \longrightarrow Y\wedge V \in \mathrm{Mor}(\mathcal{M}^{C\times D}).$$

Remark 4.3. If we view f and g as objects of $\mathcal{M}^{C\times \mathfrak{z}}$ and $\mathcal{M}^{D\times \mathfrak{z}}$, we have

$$f\Box g = \operatorname*{LKan}_{\mathrm{id}_{C\times D}\times p_{\diamond}} f\wedge g, \tag{13}$$

where $p_{\diamond}$ is the map

$$\begin{aligned} p_{\diamond}\colon \mathfrak{z}\times\mathfrak{z} &\to \mathfrak{z} \\ (1,1) &\mapsto 1 \\ (1,0),(0,0),(0,1) &\mapsto 0. \end{aligned}$$

Definition 4.4. We say that

$$-\wedge-\colon \mathcal{M}^C\times\mathcal{M}^D \longrightarrow \mathcal{M}^{C\times D}$$

satisfies the pushout product axiom, if for any two cofibrations f in $\mathcal{M}^C$ and g in $\mathcal{M}^D$, the map $f\Box g$ is a cofibration, which is trivial if f or g is.

Definition 4.5. A (symmetric) monoidal model category $(\mathcal{M},\wedge)$ is a model category $\mathcal{M}$ together with a closed (symmetric) monoidal structure $-\wedge-$, such that

- there are functorial factorizations of morphisms into a trivial cofibration followed by a fibration, and into a cofibration followed by a trivial fibration respectively,
- $-\wedge-\colon \mathcal{M}\times\mathcal{M}\longrightarrow\mathcal{M}$ satisfies the pushout product axiom, and
- the cofibrant replacement of the unit

  $$QS \xrightarrow{\ q\ } S$$

  satisfies the following condition: for any cofibrant object X,

  $$QS\wedge X \xrightarrow{q\wedge \mathrm{id}_X} S\wedge X$$

  is a weak equivalence (and

  $$X\wedge QS \xrightarrow{\mathrm{id}_X\wedge q} X\wedge S$$

  is, too).

Hovey showed in [Hov99, 4.3.1.] that for such a (symmetric) monoidal model category, the monoidal structure on $\mathcal{M}$ has a left derived functor, which is itself a (symmetric) monoidal structure on the homotopy category. For our application, only one of the following two examples will be relevant (compare Section 5).

Example 4.6. The model category $\mathcal{M}_R$ of strict modules over a strict, strictly commutative ring spectrum R is a symmetric monoidal model category [SS00].

Example 4.7. (I learned this from Stefan Schwede.) Let $(\mathcal{M}_S, \wedge_S)$ be a model for the stable homotopy category, and let E be an object of $\mathcal{M}_S$. Then

$$(\mathcal{M}[E^{-1}], \wedge_S)$$

is also a monoidal model category.

PROOF: Without loss of generality we may choose E to be cofibrant. Since Bousfield localization does not change the cofibrations, we only have to check the pushout product axiom for

$$f\colon X \to Y \text{ and } g\colon U \to V$$

in the case that g is an E-isomorphism, i.e. if

$$g \wedge \mathrm{id}_E : U \wedge E \to V \wedge E$$

is a week equivalence. The pushout product axiom for $(\mathcal{M}_S, \wedge_S)$ implies that

$$g \wedge \mathrm{id}_E = g \square (\, * \rightarrowtail E\,)$$

is a cofibration, because E is cofibrant. If we apply the pushout product axiom for

$$(\mathcal{M}_S, \wedge_S)$$

once more, this time to f and $g \wedge \mathrm{id}_E$, it follows that

$$f \square (g \wedge \mathrm{id}_E) = (f \square g) \wedge \mathrm{id}_E \colon \Big(X \wedge V \coprod_{X \wedge U} Y \wedge U\Big) \wedge E \longrightarrow Y \wedge V \wedge E$$

is also a weak equivalence. □

Proposition 4.8. *Let $(\mathcal{M}, \wedge)$ be a monoidal model category. Then*

$$-\wedge -\colon \mathcal{M}^{\mathrm{C}} \times \mathcal{M}^{\mathrm{D}} \longrightarrow \mathcal{M}^{\mathrm{C}\times\mathrm{D}}$$

has a total left derived functor

$$-\wedge^{\mathrm{L}} -\colon \mathrm{Ho}(\mathcal{M}^{\mathrm{C}}) \times \mathrm{Ho}(\mathcal{M}^{\mathrm{D}}) \longrightarrow \mathrm{Ho}(\mathcal{M}^{\mathrm{C}\times\mathrm{D}}).$$

PROOF: By [Hov99, 4.3.1.], it is enough to show that the pushout product axiom is satisfied. Let $\mathrm{U} \rightarrowtail \mathrm{V}$ be a Reedy cofibration in $\mathcal{M}^{\mathrm{C}}$ and $\mathrm{X} \rightarrowtail \mathrm{Y}$ a Reedy cofibration in $\mathcal{M}^{\mathrm{D}}$. I.e., we assume

$$\forall c \in \mathrm{C} : \mathrm{U}_c \coprod_{\varinjlim_{a<c} \mathrm{U}_a} \varinjlim_{a<c} \mathrm{V}_a \rightarrowtail \mathrm{V}_c \tag{14}$$

and

$$\forall d \in \mathrm{D} : \mathrm{X}_d \coprod_{\varinjlim_{b<d} \mathrm{X}_b} \varinjlim_{b<d} \mathrm{Y}_b \rightarrowtail \mathrm{Y}_d \quad .$$

to be cofibrations in $\mathcal{M}$. Since $- \wedge \mathrm{A}$ and $\mathrm{A} \wedge -$ are left adjoints and therefore preserve colimits, and because of the pushout product axiom in $\mathcal{M}$, it follows from (14) that the morphism with source the pushout of

$$\mathrm{U}_c\wedge\mathrm{Y}_d \coprod_{\varinjlim_{a<c} \mathrm{U}_a\wedge\mathrm{Y}_d} \varinjlim_{a<c} \mathrm{V}_a\wedge\mathrm{Y}_d \quad \text{and} \quad \mathrm{V}_c\wedge\mathrm{X}_d \coprod_{\varinjlim_{b<d} \mathrm{V}_c\wedge\mathrm{X}_b} \varinjlim_{b<d} \mathrm{V}_c\wedge\mathrm{Y}_b$$

over

$$\left(\mathrm{U}_c \coprod_{\varinjlim_{a<c} \mathrm{U}_a} \varinjlim_{a<c} \mathrm{V}_a\right) \wedge \left(\mathrm{X}_d \coprod_{\varinjlim_{b<d} \mathrm{X}_b} \varinjlim_{b<d} \mathrm{Y}_b\right)$$

and target $\mathrm{V}_c \wedge \mathrm{Y}_d$ is a cofibration for every pair (c, d). By the remark below this pushout is just the pushout of

$$\mathrm{U}_c \wedge \mathrm{Y}_d \coprod_{\mathrm{U}_c\wedge\mathrm{X}_d} \mathrm{V}_c \wedge \mathrm{X}_d \quad \text{and} \quad \varinjlim_{(a,b)<(c,d)} \mathrm{V}_a \wedge \mathrm{Y}_b$$

over

$$\left(\varinjlim_{(a,b)<(c,d)} \mathrm{U}_a \wedge \mathrm{Y}_b \coprod_{\varinjlim_{(a,b)<(c,d)} \mathrm{U}_a\wedge\mathrm{X}_b} \varinjlim_{(a,b)<(c,d)} \mathrm{V}_a \wedge \mathrm{X}_b\right),$$

and the map is the one you would expect there. We have shown that

$$\mathrm{U} \wedge \mathrm{Y} \coprod_{\mathrm{U}\wedge\mathrm{X}} \mathrm{V} \wedge \mathrm{Y} \to \mathrm{V} \wedge \mathrm{Y}$$

is a Reedy cofibration. If one of the two maps $U \rightarrowtail V$ and $X \rightarrowtail Y$ is a vertex-wise weak equivalence, it follows from the pushout product axiom in $\mathcal{M}$ that $U \wedge Y \coprod_{U \wedge X} V \wedge Y \to V \wedge Y$ is also a vertex-wise weak equivalence. Therefore the pushout product axiom is satisfied. □

Remark 4.9. The calculation uses the isomorphism

$$\varinjlim_{(a,b)<(c,d)} ?_a \wedge ?_b \cong \varinjlim_{a<c} ?_a \wedge ?_d \coprod_{\varinjlim_{\substack{a<c\\ b<d}} ?_a \wedge ?_b} \varinjlim_{b<d} ?_c \wedge ?_b,$$

which follows in a straightforward way from the various universal properties. Alternatively, by (18) the right hand side is

$$\varinjlim_{\circ\!\!\searrow\!\circ\!\!\nearrow\!\circ} \operatorname{LKan}_f(?_a \wedge ?_b)$$

with

$$\begin{aligned} f\colon \{(a,b) \mid (a,b) < (c,d)\} &\to \circ\!\!\searrow\!\circ\!\!\nearrow\!\circ \\ (c,b) &\mapsto (1,0) \\ (a,d) &\mapsto (0,1) \\ (a,b) \mid a < c \text{ and } b < d &\mapsto (0,0) \end{aligned}$$

(compare notation 1).

Remark 4.10. In the proof of the preceding proposition we are working with the model structure that has Reedy cofibrations as cofibrations. We do so for later reference. Of course, if we work with vertex-wise cofibrations and weak equivalences instead, the pushout product axiom is straight forward. The universal property of $- \wedge^L -$ implies that up to canonical isomorphism both constructions give the same result.

In the following, $(\mathcal{M}, \wedge)$ is a monoidal, stable model category. We discuss compatibility of $- \wedge^L -$ with the various other structures of the system $\operatorname{Ho}(\mathcal{M}^C)$. In order to do so, we need a lemma about the composition of derived functors.

Lemma 4.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, $\mathcal{E}$ an arbitrary category, and let*

$$F\colon \mathcal{C} \longrightarrow \mathcal{D} \quad \textit{and} \quad G\colon \mathcal{D} \longrightarrow \mathcal{E}$$

be functors, such that F *sends trivial cofibrations between cofibrant objects to weak equivalences between cofibrant objects, and* G *sends trivial cofibrations between cofibrant objects to isomorphisms. Then the derived functors* LG, LF *and* $\mathsf{L(G \circ F)}$ *exist, and in the commutative diagram of natural transformations*

$$\begin{array}{ccc} \mathsf{LG} \circ \mathsf{LF} & \xrightarrow{\iota} & \mathsf{L(G \circ F)} \\ & \searrow \quad \swarrow & \\ & \mathsf{G} \circ \mathsf{F}, & \end{array} \tag{15}$$

induced by the universal properties of the various derived functors, ι *is a functor isomorphism. Moreover,* ι *is associative up to canonical natural equivalence.*

PROOF: The existence of the derived functors is for example discussed in [Hov99]. By construction, the diagonal morphisms in (15) are isomorphisms on cofibrant objects. Therefore ι_X is an isomorphism for cofibrant X. For arbitrary X we use the diagram

$$\begin{array}{ccc} \mathsf{LF} \circ \mathsf{LG}(X) & \xrightarrow{\iota_X} & \mathsf{L(G \circ F)}(X) \\ \uparrow \cong & & \uparrow \cong \\ \mathsf{LF} \circ \mathsf{LG}(QX) & \xrightarrow[\cong]{\iota_{QX}} & \mathsf{L(G \circ F)}(QX), \end{array}$$

where for the moment $Q(-)$ denotes the cofibrant replacement functor. The argument also shows the associativity of ι. □

Example 4.12. Left Quillen functors preserve trivial cofibrations, cofibrations and initial objects. Therefore they also preserve cofibrant objects (and trivial cofibrations between them).

Corollary 4.13. *There is a functor isomorphism*

$$\underrightarrow{\mathrm{Holim}}_{\mathsf{C}\times\mathsf{D}}(A \wedge^{\mathsf{L}} B) \cong (\underrightarrow{\mathrm{Holim}}_{\mathsf{C}} A) \wedge^{\mathsf{L}} (\underrightarrow{\mathrm{Holim}}_{\mathsf{D}} B).$$

PROOF: As in [Hov99, 4.3.1], it follows from the pushout product axiom (proof of Proposition 4.8) that $- \wedge -$ sends Reedy cofibrant objects of $\mathcal{M}^{\mathsf{C}} \times \mathcal{M}^{\mathsf{D}}$ to diagram cofibrant objects of $\mathcal{M}^{\mathsf{C}\times\mathsf{D}}$ and preserves (vertex-wise) trivial Reedy cofibrations between Reedy cofibrant objects. As a left Quillen functor, $\varinjlim$ also satisfies the conditions of the lemma. Further, for $A \in \mathcal{M}$, we know that $A \wedge -$ is a left adjoint and therefore commutes with colimits, which implies the analogous strict formula. □

More generally, we have

Corollary 4.14. *There is a functor isomorphism*

$$\underset{f\times g}{\mathrm{HoLKan}}(A \wedge^{L} B) \cong (\underset{f}{\mathrm{HoLKan}}\, A) \wedge^{L} (\underset{g}{\mathrm{HoLKan}}\, B).$$

PROOF: The proof is analogous to the proof of Corollary 4.13. The strict formula follows, because left Kan extensions commute with left adjoints. □

Corollary 4.15. *There is a functor isomorphism*

$$f^{*}A \wedge^{L} g^{*}B \cong (f \times g)^{*}(A \wedge^{L} B).$$

PROOF: This time we work with the model structure whose cofibrations and weak equivalences are defined vertex-wise. Then $- \wedge -$ and pulling back both preserve cofibrant objects and trivial cofibrations between them. The analogous strict statement follows directly from the definition. □

Corollary 4.16. *The pushout smash product has a left derived functor*

$$-\square^{L}-\colon \mathrm{Ho}(\mathcal{M}^{\wr}) \times \mathrm{Ho}(\mathcal{M}^{\wr}) \longrightarrow \mathrm{Ho}(\mathcal{M}^{\wr}).$$

PROOF: According to (13), we have

$$f \square g = \underset{p_{\diamond}}{\mathrm{LKan}}(f \wedge g).$$

But

$$- \wedge -\colon \mathcal{M}^{\wr} \times \mathcal{M}^{\wr} \longrightarrow \mathcal{M}^{\wr \times \wr}$$

and $\mathrm{LKan}_{p_{\diamond}}$ both preserve Reedy cofibrant objects and vertex-wise trivial Reedy cofibrations between them. Therefore the left derived pushout smash product exists and is given by

$$-\square^{L}- = \underset{p_{\diamond}}{\mathrm{HoLKan}}(- \wedge^{L} -). \tag{16}$$

□

There is one further corollary, that has nothing to do with the monoidal structure, but will turn out to be useful.

Notation 4.17. Let $f\colon C \longrightarrow D$ be a map of posets. For $d \in D$, we let $C \to d$ denote the subposet

$$\{c \in C \mid f(c) \leqslant d\}$$

of C, and we let

$$j_d\colon (C \to d) \longrightarrow C$$

denote its inclusion into C. For an edge $d \leqslant d' \in D$, let

$$\begin{aligned} p_d^{d'}\colon (C \to d') &\longrightarrow \mathfrak{I} \\ c &\mapsto 0 \quad \text{if } f(c) \leqslant d, \\ c &\mapsto 1 \quad \text{else.} \end{aligned}$$

The vertices of a left homotopy Kan extension are given by [Fra96, Prop.1.4.2]:

$$(\mathrm{HoLKan}_f(X))_d \cong \underset{C\to d}{\underrightarrow{\mathrm{Holim}}}\, j_d^* X. \tag{17}$$

In an algebraic situation, where it makes sense to speak about satellite functors, this becomes

$$\left(\mathrm{LKan}_f{}_s X\right)_d \cong \left(\underset{C\to d}{\underrightarrow{\lim}}\right)_s (X|_{C\to d})\,. \tag{18}$$

The following corollary says that the edges of a homotopy Kan extension are also what we would expect.

Corollary 4.18 (Edges of homotopy Kan extensions). *There is a functor isomorphism*

$$(d \leqslant d')^* \,\mathrm{HoLKan}_f X \cong \mathrm{HoLKan}_{p_d^{d'}} j_{d'}^* X.$$

This implies that the edges of

$$\left(\mathrm{LKan}_f{}_s X\right)$$

are given by the various universal properties on the right hand side of (18).

PROOF: Look at the analogous strict diagram. Its vertices are given by (18) with $s = 0$, and at the edge $d \leqslant d'$ there is the map that is obtained by applying the universal property of $\varinjlim_{c \in C \to d} X_c$ to the colimit inclusions

$$X_c \hookrightarrow \varinjlim_{c' \in C \to d'} X_{c'}.$$

This diagram satisfies the universal property of the strict left Kan extension of X along f in $\mathcal{M}^C$. Therefore the analogous strict statement is true. Since $j_{d'}^*$ and left Kan extensions preserve diagram cofibrations, the claim follows by Lemma 4.11. □

We also need a variation of Corollary 4.18.

Notation 4.19. In the situation of Notation 4.17, let

$$B := (C \to d) \times \wr \coprod_{(C \to d) \times \{1\}} (C \to d').$$

Let

$$r_B \colon B \longrightarrow (C \to d')$$

be the projection onto the first factor on $(C \to d) \times \wr$ and the identity on the rest. Let l_B be the left adjoint to r_B, i.e. l_B sends $(C \to d)$ to $(C \to d) \times \{0\}$. Let further

$$p_B \colon B \longrightarrow \wr$$

send $(C \to d')$ to 1 and $(C \to d) \times \{0\}$ to 0, and let

$$j_B := j_{d'} \circ r_B.$$

Corollary 4.20. *With Notation 4.19, we have a functor isomorphism*

$$(d \leqslant d')^* \circ \operatorname{HoLKan}_f \cong \operatorname{HoLKan}_{p_B} j_B^*.$$

PROOF: We have

$$p_d^{d'} = p_B \circ l_B.$$

Therefore

$$\operatorname{HoLKan}_{p_d^{d'}} j_{d'}^* \cong \operatorname{HoLKan}_{p_B} \operatorname{HoLKan}_{l_B} j_{d'}^* \cong \operatorname{HoLKan}_{p_B} r_B^* j_{d'}^*.$$

□

Next we discuss the compatibility of the smash product with the triangulated structure.

Proposition 4.21. *There is a functor isomorphism*

$$\mathrm{cone}(-) \wedge^{L} \mathrm{cone}(-) \cong \mathrm{cone}(-\square^{L}-).$$

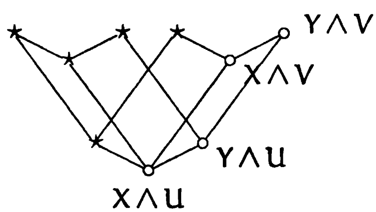
PROOF: We have

$$\begin{aligned}\mathrm{cone}(f) \wedge^{L} \mathrm{cone}(g) &= \underrightarrow{\mathrm{Holim}}_{\vee} \circ \overset{\star}{\vee}^{\circ}(f) \wedge^{L} \underrightarrow{\mathrm{Holim}}_{\vee} \circ \overset{\star}{\vee}^{\circ}(g) \\ &\cong \underrightarrow{\mathrm{Holim}}_{\vee \times \vee}(\overset{\star}{\vee}^{\circ}(f) \wedge^{L} \overset{\star}{\vee}^{\circ}(g)),\end{aligned}$$

where the first isomorphism is Definition 3.3 and the second isomorphism is Corollary 4.13. The diagram

$$(\overset{\star}{\vee}^{\circ} f) \wedge^{L} (\overset{\star}{\vee}^{\circ} g)$$

has the form (for $f\colon X \to Y$ and $g\colon U \to V$)



In particular, Corollary 4.15 implies

$$(\overset{\star}{\vee}^{\circ}(f) \wedge^{L} \overset{\star}{\vee}^{\circ}(g))\,|_{((0,0)\leqslant(1,0))\times((0,0)\leqslant(0,1))} \cong f \wedge^{L} g \in \mathrm{Ho}(\mathcal{M}^{8\times 8}).$$

We consider the map

$$\begin{aligned} \mathrm{pr}\colon \vee \times \vee &\to \vee \\ ((0,1),(0,1)) &\mapsto (0,1) \\ ((0,1),(0,0)),((0,0),(0,0)),((0,0),(0,1)) &\mapsto (0,0) \\ (1,0) \times \vee \cup \vee \times (1,0) &\mapsto (1,0). \end{aligned}$$

By Corollary 4.18 and with $p_{\diamond}$ as in (13), we have

$$\begin{aligned}\underset{\mathrm{pr}}{\mathrm{HoLKan}}\,(\overset{\star}{\vee}^{\circ} f \wedge^{L} \overset{\star}{\vee}^{\circ} g)\,|_{(0,0)\leqslant(0,1)} &\cong \underset{p_{\diamond}}{\mathrm{HoLKan}}\,(f \wedge^{L} g) \\ &\cong f \square^{L} g,\end{aligned}$$

where the second isomorphism is (16). Further, by (17), we have

$$\underset{\mathrm{pr}}{\mathrm{HoLKan}}\,(\overset{\star}{\vee}^{\circ} f \wedge^{L} \overset{\star}{\vee}^{\circ} g)\,|_{(1,0)} \cong \star.$$

Together, we obtain

$$\operatorname{HoLKan}_{\mathrm{pr}}\left(\overset{\star}{\curlyvee}{}^{\circ} f \wedge^{L} \overset{\star}{\curlyvee}{}^{\circ} g\right) \cong \overset{\star}{\curlyvee}{}^{\circ}\left(f \square^{L} g\right)$$

If we apply $\underrightarrow{\operatorname{Holim}}_{\curlyvee}$ to this equation, the claim follows. □

For future reference, we recall another functor isomorphism [Fra96, Thm.2]: We have

$$\mathfrak{Cone} \circ \operatorname{HoLKan}_{\mathrm{pr}_{\wr}}(-) \cong \operatorname{HoLKan}_{\mathrm{pr}_{\wr}} \circ \mathfrak{Cone}_{C}(-), \tag{19}$$

where

$$\mathrm{pr}_{\wr} \colon C \times \wr \longrightarrow \wr$$

denotes the projection to the second factor. It follows that

$$\operatorname{cone} \circ \operatorname{HoLKan}_{\mathrm{pr}_{\wr}}(-) \cong \underrightarrow{\operatorname{Holim}}_{C} \circ \operatorname{cone}_{C}(-). \tag{20}$$

The remainder of this section is about the interaction of $\mathfrak{Cone}$ with the monoidal structure. It needs some preparation and is not needed in the proof of Theorem 1.1. The reader only interested in this theorem can skip ahead to Section 5. Before we can discuss the compatibility of the smash product with the "cone inclusion" functor $\mathfrak{Cone}$ from (3), we need an alternative description of $\mathfrak{Cone}$. For completeness, we also give a similar description of the "cone map"

$$\mathfrak{Cone}(\mathfrak{Cone}(f)) \colon \operatorname{cone}(f) \longrightarrow \Sigma X \in \operatorname{Ho}(\mathcal{M}^{\wr}).$$

For our next definition, we make an exception from our conventions about posets, and let the arrows point to the right and down.

Definition 4.22. We define the functors

$$\overset{\star}{\square}, \underset{\star}{\square} \colon \mathcal{M}^{\wr} \longrightarrow \mathcal{M}^{\wr \times \wr},$$

that map

$$X \xrightarrow{f} Y \in \mathcal{M}^{\wr}$$

to

$$\begin{array}{ccc} \star & \longrightarrow & Y \\ \downarrow & & \| \\ X & \xrightarrow{f} & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \| & & \downarrow \\ X & \longrightarrow & \star \end{array} \in \mathcal{M}^{\wr \times \wr}$$

respectively.

In these pictures, horizontal arrows correspond to the second factor of $\mathbb{I} \times \mathbb{I}$ whereas vertical arrows correspond to the first factor. Both of these functors preserve weak equivalences and therefore induce functors on the homotopy categories. We use the same names for these induced functors.

Lemma 4.23. *There are functor isomorphisms*

$$\mathfrak{Cone}(-) \cong \mathrm{cone}_{\mathbb{I}} \circ \overset{\star}{\square}(-)$$

and

$$\mathfrak{Cone} \circ \mathfrak{Cone}(-) \cong \mathrm{cone}_{\mathbb{I}} \circ \underset{\star}{\square}(-).$$

Here we used the following notation: Let C be a poset. We write

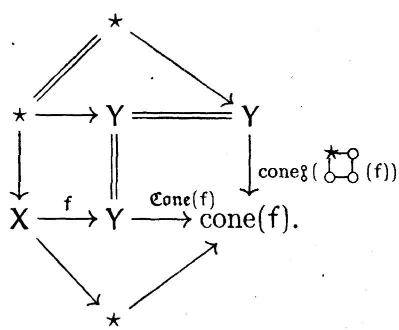
$$\mathrm{cone}_{\mathsf{C}}\colon \mathrm{Ho}(\mathcal{M}^{\mathsf{C}\times\mathbb{I}}) = \mathrm{Ho}((\mathcal{M}^{\mathsf{C}})^{\mathbb{I}}) \longrightarrow \mathrm{Ho}(\mathcal{M}^{\mathsf{C}})$$

for the cone functor with $\mathcal{M}^{\mathsf{C}}$ playing the role of $\mathcal{M}$. In our situation, $\mathsf{C} = \mathbb{I}$, the first factor of $\mathbb{I} \times \mathbb{I}$.

PROOF: We write $\overset{\star}{\vee}{}_{\mathsf{C}}$ for the functor $\overset{\star}{\vee}$ with $\mathcal{M}^{\mathsf{C}}$ playing the role of $\mathcal{M}$. We claim that

$$\underset{\mathrm{id}_{\mathbb{I}} \times (\vee_{\mathsf{C}}(\mathbb{I}\times\mathbb{I}))}{\mathrm{HoLKan}} \circ \overset{\star}{\vee}{}_{\mathbb{I}} \circ \overset{\star}{\square}(\mathsf{X} \xrightarrow{\mathsf{f}} \mathsf{Y})$$

is of the form

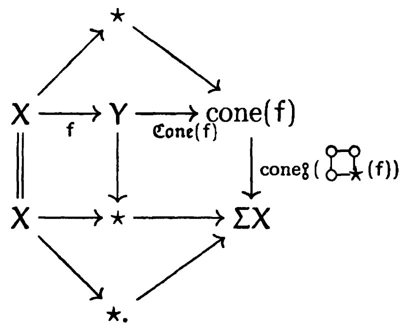


Indeed, this follows from Definition 3.3, (3), and the fact that for a diagram $\mathsf{X} \in \mathrm{Ho}(\mathcal{M}^{\mathsf{C}\times\mathbb{I}})$ the vertex $(\mathrm{cone}_{\mathsf{C}}(\mathsf{X}))_{\mathsf{c}}$ is isomorphic to the cone of the corresponding restriction $\mathsf{X}|_{\mathsf{c}\times\mathbb{I}}$ (see [Fra96, 1.4.2]).

For the cone map, we look at

$$\underset{\mathrm{id}_{\mathbb{I}} \times (\vee_{\mathsf{C}}(\mathbb{I}\times\mathbb{I}))}{\mathrm{HoLKan}} \circ \overset{\star}{\vee}{}_{\mathbb{I}} \circ \underset{\star}{\square}(\,\mathsf{X} \xrightarrow{\ \mathsf{f}\ } \mathsf{Y}\,).$$

By the same argument as above and by (17) this is of the shape



The right vertical edge is given by (17) with $\mathcal{M}^{\mathfrak{g}}$ playing the role of $\mathcal{M}$. It is

$$\mathrm{cone}_{\mathfrak{g}} \circ \boxdot(\mathsf{f}).$$

We have to show that the right square is homotopy bi-cartesian. But the top square is homotopy bi-cartesian, and so is the square that we obtain by putting the top square and the right square next to each other. Therefore, [Fra96, Prop.1.4.6] implies that the right square is also homotopy bi-cartesian. □

Now let $\mathcal{N}$ stand for the model category $\mathcal{M}^{\mathfrak{g}}$ with vertex-wise cofibrations and weak equivalences. Then

$$\boxdot : \mathcal{M}^{\mathfrak{g}} \longrightarrow \mathcal{N}^{\mathfrak{g}}$$

preserves Reedy cofibrations. Here we identified the first exponent in $\mathcal{N}^{\mathfrak{g}} = (\mathcal{M}^{\mathfrak{g}})^{\mathfrak{g}}$ with the vertical arrows. Therefore Lemma 4.11 implies

Corollary 4.24. *There is a functor isomorphism*

$$\boxdot \circ (-\square^{\mathsf{L}}-) \cong (\boxdot(-)) \square^{\mathsf{L}}_{\mathfrak{g}} (\boxdot(-)).$$

Here $\square_{\mathfrak{g}}$ denotes the pushout with $\mathcal{N}$ playing the role of $\mathcal{M}$. Proposition 4.21 now implies

Corollary 4.25. *There is a functor isomorphism*

$$\mathfrak{Cone}(-\square^{\mathsf{L}}-) \cong \mathfrak{Cone}(-) \wedge^{\mathsf{L}}_{\mathfrak{g}} \mathfrak{Cone}(-).$$

Here $\wedge_{\mathfrak{g}}$ denotes the (internal) smash product in $\mathcal{N}$.

PROOF: We have

$$
\begin{aligned}
\mathfrak{Cone}(f\Box^{L}g) &\cong \mathrm{cone}_{\mathfrak{g}}\left(\boxdot(f\Box^{L}g)\right)\\
&\cong \mathrm{cone}_{\mathfrak{g}}\left((\boxdot(f))\,\Box^{L}_{\mathfrak{g}}\,(\boxdot(g))\right)\\
&\cong \mathrm{cone}_{\mathfrak{g}}\left(\boxdot(f)\right)\wedge^{L}_{\mathfrak{g}}\mathrm{cone}_{\mathfrak{g}}\left(\boxdot(g)\right)\\
&\cong \mathfrak{Cone}(f)\wedge^{L}_{\mathfrak{g}}\mathfrak{Cone}(g),
\end{aligned}
$$

where the third isomorphism is Proposition 4.21. □

5. WHICH MODEL?

There are four canonical choices of a model for $\mathcal{S}_{E(1)}$, all of which are equally well suited for our purposes. Firstly, we can work either in the world of symmetric spectra [HSS00] or in the world of S-modules [EKMM97]. Let $\mathcal{M}_S$ be one of these two models for the stable homotopy category. One possible model for $\mathcal{S}_{E(1)}$ is $\mathcal{M}_S[E(1)^{-1}]$ (compare Example 4.7). To obtain the other model, we recall that Hopkins and Ravenel have shown that the localization at $E(n)$ is smashing, i.e. that for all X in $\mathcal{S}$, one has

$$X_{E(n)} \cong X \wedge^{L}_{S} S_{E(n)}$$

[Rav92]. If the localization at a spectrum E is smashing, the fibrant replacement S_E of the sphere spectrum in $\mathcal{M}_S[E^{-1}]$ can be chosen to be a strict, strictly commutative ring spectrum, and

$$-\wedge_S S_E\colon \mathcal{M}_S[E^{-1}] \longrightarrow \mathcal{M}_{S_E}$$

is a Quillen equivalence. In other words,

$$\mathrm{Ho}(\mathcal{M}_{S_E})$$

is the localization of $\mathcal{M}_S$ at E_*, and

$$-\wedge^{L}_{S} S_E$$

is the localization functor[9]. For Franke's methods it is irrelevant which model for $\mathcal{S}_{E(1)}$ one likes to choose. The only property of the model category that is relevant for him is that it induces a system of *triangulated* diagram categories, i.e. that the model category is stable[10]. Moreover, Schwede [Sch01a] has constructed a functor[11]

$$\Phi\colon \mathcal{M}_S \longrightarrow Sp^{\Sigma},$$

that maps (strict) ring spectra to (strict) ring spectra, and induces for any strict (strictly commutative) ring spectrum R a monoidal equivalence

$$Ho(\mathcal{M}_R) \xrightarrow{\sim} Ho(\Phi(R)\text{-mod}).$$

We also know that for any strict, strictly commutative ring spectrum R, the functor

$$-\wedge_S R\colon (\mathcal{M}_S, \wedge_S) \longrightarrow (\mathcal{M}_R, \wedge_R)$$

is strictly monoidal. It follows that up to equivalence all four models mentioned above give rise to the same system of triangulated diagram categories, and the same smash product on it.

6. The Derived Tensor Product

In this section we define the derived tensor product $\otimes^{L}_{E(1)_*}$ on the derived category of quasi-periodic cochain complexes in $\mathrm{Comod}^{0}_{E(1)_*E(1)}$ and show that $\otimes^{L}_{E(1)_*}$ is a monoidal structure. Since $\mathrm{Comod}_{E(1)_*E(1)}$ does not have enough projectives, we need to work with flat replacement

[9]For symmetric spectra, this statement is [SS03, 3.2.(iii)]. For S-modules this is a result of Wolbert [Wol98], which can also be found in [EKMM97]. In order to get this precise statement from [EKMM97], one actually has to combine a few propositions: it follows from VIII.2.1., VIII.3.2., and from the fact that by III.4.2. and VII.4.9. the derived categories of ΛS-modules and of S-modules are equivalent as monoidal categories. Here $\Lambda S \to S$ denotes the q-cofibrant replacement of the sphere spectrum.

[10]A model category is called stable, if the suspension functor is invertible in the homotopy category. In our example this follows from the fact that the suspension in $Ho(\mathcal{M}_R)$ is given by smashing over R with $S^1 \wedge^{L}_{S} R$. Therefore smashing over R with $S^{-1} \wedge^{L}_{S} R$ is an inverse of the suspension. Also Quillen equivalent models give rise to the same suspension functor. The fact that the homotopy category of a stable model category is triangulated, is proved in [Hov99, 7.1].

[11]just for the moment, $\mathcal{M}_S$ denotes the S-modules from [EKMM97] and Sp^{Σ} denotes the symmetric spectra from [HSS00].

rather than projective replacement. Our definition of a flat complex is object-wise, forcing us to replace both sides of $\otimes_{E(1)_*}$ with flat objects. The following lemma is essentially due to Christensen and Hovey.

Lemma 6.1. *Let* E *be a Landweber exact cohomology theory, and let* C *be a quasi-periodic cochain complex of* (E_*, E_*E)*-comodules. Then there exist a flat quasi-periodic cochain complex* P *of* (E_*, E_*E)*-comodules and a quasi-periodic quasi-isomorphism* i *from* P *to* C*. Moreover, the pair* (P, i) *depends on* C *in a functorial way.*

PROOF: We claim that cofibrant replacement in Christensen and Hovey's projective model structure[12] on $\mathcal{C}h(E_*, E_*E)$ can be done in such a way that these conditions are satisfied. We have to show three things: (a) Cofibrant objects in the projective model structure are flat over E_*, (b) weak equivalences in the projective model structure are quasi-isomorphisms, and (c) if C is quasi-periodic, one can choose a quasi-periodic cofibrant replacement of C (functorially in such C).

(a) Let P be cofibrant in the projective model structure. By [CH02, 4.4], P is a retract of a (transfinite) colimit of a diagram of complexes

$$0 = P_0 \to P_1 \to \cdots \to P_\alpha \to P_{\alpha+1} \to \cdots,$$

where each $P_\alpha \to P_{\alpha+1}$ is a degree-wise split monomorphism whose cokernel is a complex of so called "relative projectives" with no differential, and if α is a limit ordinal, P_α is the colimit over all P_β with $\beta < \alpha$. It is also pointed out in [Hov04, p.15] that every "relative projective" comodule is projective as an E_*-module. We can therefore prove by transfinite induction that the limit over all the P_α is flat: The complex P_0 is flat. In every degree, $P_{\alpha+1}$ is the direct sum of P_α with a projective E_*-module, thus if P_α is degree-wise flat, so is $P_{\alpha+1}$. If α is a limit ordinal, P_α is a direct limit of flat objects and hence flat. For the same reason the colimit over the entire diagram is flat. Moreover, in an abelian category, retracts are direct summands, and thus retracts of flat modules are flat. This proves that P is degree-wise flat.

[12] Cf. [Hov04, 2] and [CH02, 4.4].

(b) Since E_* is Landweber exact, it follows from [Hov04, Sec.1.4] that (E_*, E_*E) satisfies the conditions of [Hov04, 2.1.5]. Therefore, weak equivalences are quasi-isomorphisms.
(c) Now let C be quasi-periodic. In the proof of [CH02, 4.2] an explicit cofibrant replacement is constructed. Observe that for a quasi-periodic complex C the complexes P_i and Q_i may be chosen (functorially) in such a way that the "partial cofibrant replacement" of C is again quasi-periodic. None of the other steps in [CH02, 4.2] (colimits, pullbacks, path objects, cofibres) destroy quasi-periodicity. This proves the claim.

□

Remark 6.2. Flatness in our case means flatness over $E(1)_*$. Therefore flat objects are exactly the p-torsion free objects, and flat replacement in

$$\mathcal{C}^1(\mathrm{Comod}_{E(1)_*E(1)})$$

translates into flat replacement in

$$\mathcal{C}^{2p-2}(\mathrm{Comod}^0_{E(1)_*E(1)}).$$

Note also that for the same reason subobjects of flat objects are again flat.

The following is a corollary of [GM96, III.2.10].

Proposition 6.3. *Let $\mathcal{C}$ be a category, S a left-localizing system of morphisms in $\mathcal{C}$ and $\mathcal{B} \subseteq \mathcal{C}$ a full subcategory. Suppose that for all $X \in \mathrm{Ob}\,\mathcal{C}$ there exist $Y \in \mathrm{Ob}\,\mathcal{B}$ and $s\colon Y \to X$ in S. Then*

$$S_{\mathcal{B}} := S \cap \mathrm{Mor}\,\mathcal{B}$$

is left-localizing in $\mathcal{B}$, and the canonical map

$$\mathcal{B}[S_{\mathcal{B}}^{-1}] \to \mathcal{C}[S^{-1}]$$

is an equivalence of categories.

Notation 6.4. Let

$$\mathcal{C}_{\mathrm{flat}} \subset \mathcal{C}^{2p-2}(\mathrm{Comod}^0_{E(1)_*E(1)})$$

be the full subcategory of flat objects, and let

$$\gamma_{\mathrm{flat}}\colon \mathcal{C}_{\mathrm{flat}} \longrightarrow \mathcal{D}^{2p-2}(\mathrm{Comod}^0_{E(1)_*E(1)})$$

be the composition of its inclusion with the localization functor. Let $\mathsf{K}_{\mathsf{flat}}$ denote the homotopic category (in the sense of homological algebra) of $\mathcal{C}_{\mathsf{flat}}$, and let S denote the class of quasi-isomorphisms in $\mathsf{K}_{\mathsf{flat}}$.

Corollary 6.5. *The functor γ_{flat} induces an equivalence of localized categories*

$$\mathsf{K}_{\mathsf{flat}}[\mathsf{S}^{-1}] \to \mathcal{D}^{2p-2}(\mathrm{Comod}^{0}_{\mathsf{E}(1)_*\mathsf{E}(1)}).$$

In other words, flat replacement can be done in such a way that it is functorial on morphisms in the derived category, not just the strict category: let ϕ be an equivalence of categories inverse to the equivalence in Corollary 6.5, then ϕ is such a flat replacement functor on the derived category.

Corollary 6.6. *The tensor product in $\mathcal{C}^{2p-2}(\mathrm{Comod}^{0}_{\mathsf{E}(1)_*\mathsf{E}(1)})$ has a left derived functor $\otimes^{\mathsf{L}}_{\mathsf{E}(1)_*}$, defining a symmetric monoidal structure on the derived category.*

PROOF: The functor

$$(-\otimes_{\mathsf{E}(1)_*}-)\colon \mathsf{K}_{\mathsf{flat}} \times \mathsf{K}_{\mathsf{flat}} \longrightarrow \mathsf{K}_{\mathsf{flat}}[\mathsf{S}^{-1}]$$

takes pairs of acyclic complexes to acyclic complexes. Therefore, by the same argument as in [GM96, III.2.23], it factors over the localization

$$(\mathsf{K}_{\mathsf{flat}} \times \mathsf{K}_{\mathsf{flat}})[(\mathsf{S}\times\mathsf{S})^{-1}] = \mathsf{K}_{\mathsf{flat}}[\mathsf{S}^{-1}] \times \mathsf{K}_{\mathsf{flat}}[\mathsf{S}^{-1}].$$

As in the proof of [GM96, III.6.8], we precompose $(-\otimes_{\mathsf{E}(1)_*}-)$ with $\phi\times\phi$, to obtain a functor $(-\otimes^{\mathsf{L}}_{\mathsf{E}(1)_*}-)$ satisfying the universal property of the left derived functor of $(-\otimes_{\mathsf{E}(1)_*}-)$. As in the proof of [Hov99, 4.3.2]) the structure diagrams making $\otimes_{\mathsf{E}(1)_*}$ a monoidal structure can be translated into the respective diagrams for $\otimes^{\mathsf{L}}_{\mathsf{E}(1)_*}$. □

7. PROOF OF THE MAIN THEOREM

In this section we prove the following theorem:

Theorem 7.1. *There is a functor isomorphism*

$$(-\wedge^{\mathsf{L}}_{\mathsf{S}_{\mathsf{E}(1)}}-)\circ \mathcal{R}ec\circ\gamma_{\mathsf{flat}} \cong \mathcal{R}ec\circ\gamma_{\mathsf{flat}}\circ(-\otimes_{\mathsf{E}(1)_*}-).$$

Combined with the results of Section 6 this implies Theorem 1.1. Let C and $\tilde{\mathsf{C}}$ be in $\mathcal{C}_{\mathrm{flat}}$. Recall that $\mathcal{R}ec$ was defined as

$$\mathcal{R}ec(\mathsf{C}) = \underset{\mathsf{C_N}}{\underrightarrow{\mathrm{Holim}}}\, \mathsf{Q}^{-1}(\mathsf{C}),$$

(cf. Notation 3.7 and Equation (8)). We wish to find an isomorphism

$$\mathcal{R}ec^{-1}(\mathcal{R}ec(\mathsf{C}) \wedge^{\mathsf{L}}_{\mathsf{S}_{\mathsf{E}(1)}} \mathcal{R}ec(\tilde{\mathsf{C}})) \cong \mathsf{C} \otimes_{\mathrm{E}(1)_*} \tilde{\mathsf{C}}.$$

The computation of the left-hand side consists of two steps: First, we need to find an object of $\mathcal{L}$ whose homotopy colimit is

$$\mathcal{R}ec(\mathsf{C}) \wedge^{\mathsf{L}}_{\mathsf{S}_{\mathsf{E}(1)}} \mathcal{R}ec(\tilde{\mathsf{C}}).$$

Then, we need to apply Q to this object.

Notation 7.2. In what follows, we abbreviate $- \wedge^{\mathsf{L}}_{\mathsf{S}_{\mathsf{E}(1)}} -$ by $\quad - \wedge -$.

7.1. An object of $\mathcal{L}$, whose homotopy colimit is $\mathcal{R}ec(\mathsf{C}) \wedge \mathcal{R}ec(\tilde{\mathsf{C}})$.

Notation 7.3. We write A for $\mathsf{Q}^{-1}(\mathsf{C})$ and $\tilde{\mathsf{A}}$ for $\mathsf{Q}^{-1}(\tilde{\mathsf{C}})$.

We have

$$\mathcal{R}ec(\mathsf{C}) \wedge \mathcal{R}ec(\tilde{\mathsf{C}}) \cong (\underset{\mathsf{C_N}}{\underrightarrow{\mathrm{Holim}}}\, \mathsf{A}) \wedge (\underset{\mathsf{C_N}}{\underrightarrow{\mathrm{Holim}}}\, \tilde{\mathsf{A}}) \cong \underset{\mathsf{C_N} \times \mathsf{C_N}}{\underrightarrow{\mathrm{Holim}}}(\mathsf{A} \wedge \tilde{\mathsf{A}}),$$

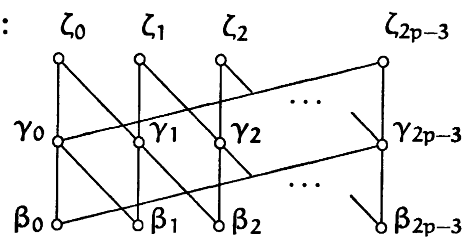
where the first isomorphism is (8) and the second is Corollary 4.13. Let $\mathsf{D_N}$ be the poset

$$\mathsf{D_N} := \{\beta_n, \gamma_n, \zeta_n \mid n \in \mathbf{Z}/_{2p-2}\}$$

with relations generated by

$$\beta_{n+1} \leqslant \gamma_n, \quad \beta_n \leqslant \gamma_n, \quad \gamma_{n+1} \leqslant \zeta_n \text{ and } \gamma_n \leqslant \zeta_n.$$

$\mathsf{D_N}$:



Consider the map of posets

$$\begin{aligned} \mathrm{pr}\colon C_N \times C_N &\to D_N \\ (\beta_s, \beta_t) &\mapsto \beta_{s+t} \\ (\beta_s, \zeta_t), (\zeta_s, \beta_t) &\mapsto \gamma_{s+t} \\ (\zeta_s, \zeta_t) &\mapsto \zeta_{s+t}. \end{aligned}$$

Notation 7.4. With A and $\tilde{A}$ as in Notation 7.3, let

$$E := \operatorname*{HoLKan}_{\mathrm{pr}}(A \wedge \tilde{A}).$$

This is an object of $\mathrm{Ho}(\mathcal{M}^{D_N}_{S_{E(1)}})$.

Proposition 7.5. *The objects*

$$E(1)_*(E_{\alpha_n}) \text{ with } \alpha \in \{\beta, \gamma, \zeta\}$$

are concentrated in degrees congruent to n *modulo* $2p-2$. *The morphisms*

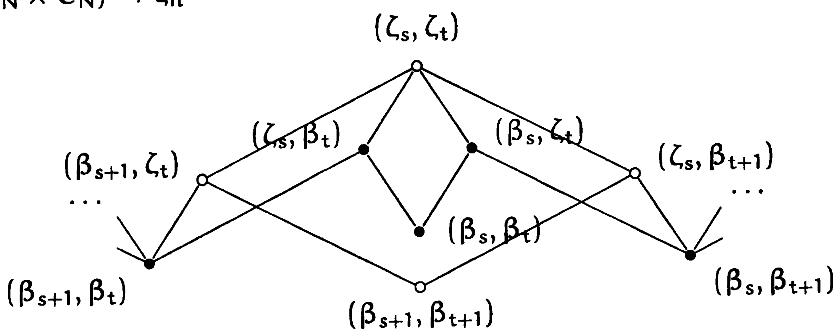
$$E(1)_*(E_{\gamma_n}) \to E(1)_*(E_{\zeta_n}), \tag{21}$$

induced by the corresponding edges in E, *are monomorphisms.*

PROOF: By Corollary 4.18,

$$E|_{\gamma_n \leqslant \zeta_n} \cong \operatorname*{HoLKan}_{p^{\zeta_n}_{\gamma_n}} j^*_{\zeta_n}(A \wedge \tilde{A}) \tag{22}$$

(Notation 4.17). Still using Notation 4.17,
$(C_N \times C_N) \to \zeta_n =$



The black vertices are the ones mapped to zero by $p^{\zeta_n}_{\gamma_n}$. In other words, the source of (22) is the homotopy colimit over the black subdiagram, whereas the target is the homotopy colimit over the entire diagram. In order to compute the $E(1)_*$-homology of (22), we apply the spectral sequence (9) to the right hand side of (22). In order to compute the E_2-term, which involves left derived Kan extensions along $p^{\zeta_n}_{\gamma_n}$, we will need three lemmas.

Notation 7.6. Let

$$\Diamond_n \subset (C_N \times C_N \to \zeta_n)$$

denote the sub-poset with elements

$$\{(\zeta_s, \zeta_t), (\zeta_s, \beta_t), (\beta_s, \zeta_t), (\beta_s, \beta_t), (\beta_{s+1}, \beta_t) \mid s + t = n\}.$$

Let

$$j_{\Diamond} : \Diamond_n \longrightarrow (C_N \times C_N \to \zeta_n),$$

denote its inclusion, and

$$l_{\Diamond} : (C_N \times C_N \to \zeta_n) \longrightarrow \Diamond_n$$

denote the left adjoint to $j_{\Diamond}$. Let further

$$p_{\Diamond} := p^{\zeta_n}_{\gamma_n} \circ j_{\Diamond} : \Diamond_n \longrightarrow \mathbf{2}.$$

Lemma 7.7. *There are functor isomorphisms*

$$\underset{p^{\zeta_n}_{\gamma_n}}{(\mathrm{LKan})_s} \cong \underset{p_{\Diamond}}{(\mathrm{LKan})_s} \circ j^*_{\Diamond}.$$

PROOF: Consider the Grothendieck spectral sequence for the composition of derived functors, where the composition is

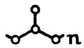
$$p^{\zeta_n}_{\gamma_n} = p_{\Diamond} \circ l_{\Diamond}.$$

Since $j_{\Diamond}$ is right adjoint to $l_{\Diamond}$,

$$\underset{l_{\Diamond}}{\mathrm{LKan}} = j^*_{\Diamond},$$

and higher derived left Kan extensions along $l_{\Diamond}$ vanish. Therefore, the spectral sequence collapses, and the claim follows. □

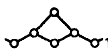
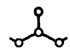
Notation 7.8. Consider the poset

 $_n := \{\alpha_{s,t}, (\zeta_s, \zeta_t), (\beta_{s+1}, \beta_t) \mid s + t = n\}$

with relations

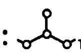
$$\begin{aligned} (\beta_{s+1}, \beta_t), (\beta_s, \beta_{t+1}) &\leqslant \alpha_{s,t} \\ \alpha_{s,t} &\leqslant (\zeta_s, \zeta_t). \end{aligned}$$

Let g and p be the maps of posets

$g:$  $_n \to$  $_n$

$$\begin{aligned} (\zeta_s, \zeta_t) &\mapsto (\zeta_s, \zeta_t) \\ (\zeta_s, \beta_t), (\beta_s, \zeta_t), (\beta_s, \beta_t) &\mapsto \alpha_{s,t} \\ (\beta_{s+1}, \beta_t) &\mapsto (\beta_{s+1}, \beta_t) \end{aligned}$$

and

p$:$  $_n \to$ 

$$\begin{aligned} (\zeta_s, \zeta_t) &\mapsto 1 \\ (\beta_{s+1}, \beta_t), \alpha_{s,t} &\mapsto 0. \end{aligned}$$

Lemma 7.9. *Let* $X \in \mathrm{Ho}(\mathcal{M}$$)$ *be such that*

$$X|_{\{(\zeta_s,\beta_t),(\beta_s,\zeta_t),(\beta_s,\beta_t)\}}$$

is $\varinjlim$*-acyclic. Then there is an isomorphism (natural in such* X*)*

$(\mathrm{LKan})_s X = (\mathrm{LKan})_s \circ \mathrm{LKan}_g X.$ (subscripts: p, p)

PROOF: We have

p $= p$ $\circ\, g.$

By (18), X is LKan_g-acyclic. The Grothendieck spectral sequence implies the claim. □

Consider a diagram $\mathcal{Y}$ with underlying poset ⅄:

(23)
$$
\begin{array}{ccccccccccc}
 & & Z_1 & & & & Z_2 & & & & Z_3 & & \\
 & & \uparrow{\scriptstyle h_1} & & & & \uparrow{\scriptstyle h_2} & & & & \uparrow{\scriptstyle h_3} & & \\
\cdots & {\scriptstyle g_1}\nearrow & X_1 & \nwarrow & & {\scriptstyle g_2}\nearrow & X_2 & \nwarrow & & {\scriptstyle g_3}\nearrow & X_3 & \nwarrow{\scriptstyle f_3} & \cdots \\
Y_1 & & & {\scriptstyle f_1} & Y_2 & & & {\scriptstyle f_2} & Y_3 & & & & Y_4
\end{array}
$$

Lemma 7.10. *Let $\mathcal{Y}$ be as in (23). Then there are functor isomorphisms*

$$
\underset{p_{⅄}}{\mathrm{LKan}}(\mathcal{Y}) \cong \mathrm{coeq}_{\S}\left(\begin{array}{ccc}
\bigoplus Y_i & \overset{\oplus h_i f_i}{\underset{\oplus h_i g_i}{\rightrightarrows}} & \bigoplus Z_i \\
\Vert & & \uparrow{\scriptstyle \oplus h_i} \\
\bigoplus Y_i & \overset{\oplus f_i}{\underset{\oplus g_i}{\rightrightarrows}} & \bigoplus X_i
\end{array}\right),
$$

$$
\underset{p_{⅄}}{(\mathrm{LKan})_1}(\mathcal{Y}) \cong \mathrm{eq}_{\S}\left(\begin{array}{ccc}
\bigoplus Y_i & \overset{\oplus h_i f_i}{\underset{\oplus h_i g_i}{\rightrightarrows}} & \bigoplus Z_i \\
\Vert & & \uparrow{\scriptstyle \oplus h_i} \\
\bigoplus Y_i & \overset{\oplus f_i}{\underset{\oplus g_i}{\rightrightarrows}} & \bigoplus X_i
\end{array}\right),
$$

where (co)eq$_{\S}$ *denotes the (co)equalizers of the pairs of horizontal arrows (together with the induced map between them). The higher derived left Kan extensions along* $p_{⅄}$ *vanish.*

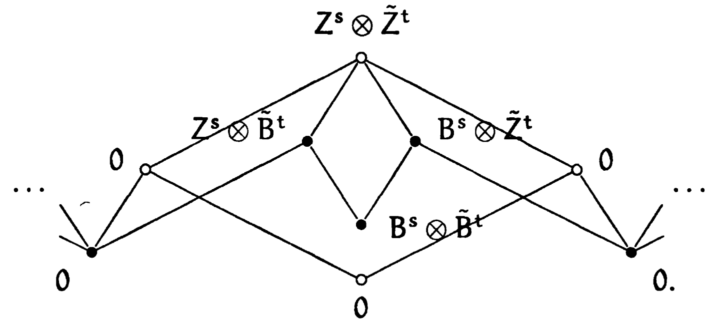
PROOF: The first statement is (18), the third statement follows from (18) and the fact that for a diagram of length one the second and all higher derived colimits vanish. The second statement follows from the other two together with the universal property of derived functors and the snake lemma applied to an LKan-acyclic resolution. □

We are now ready to complete the proof of Proposition 7.5. As subobjects of flat objects, $E(1)_*(A_{\alpha_s})$ and $E(1)_*(\tilde{A}_{\alpha'_t})$ are flat, and the Knneth spectral sequence[13] reduces to

$$E(1)_*(A_{\alpha_s} \wedge \tilde{A}_{\alpha'_t}) \cong E(1)_*(A_{\alpha_s}) \otimes_{E(1)_*} E(1)_*(\tilde{A}_{\alpha'_t}).$$

[13]The Knneth spectral sequence for non connective spectra can for example be found in [EKMM97].

Firstly, we have
$E(1)_{-n}(j^*_{\zeta_n}(A \wedge \tilde{A})) =$



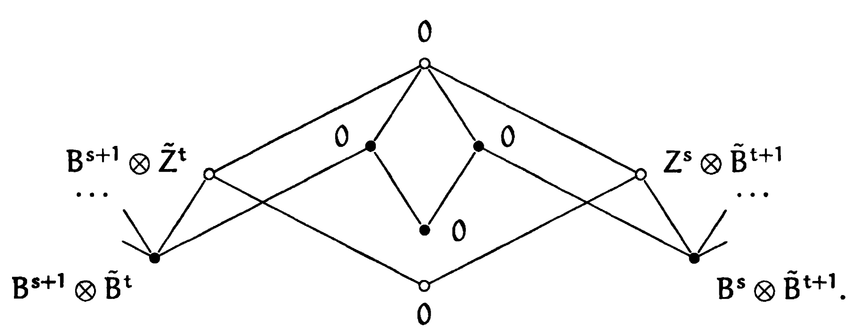
(24)

Since B^s and $\tilde{B}^t$ are flat, the maps $B^s \otimes \tilde{B}^t \to B^s \otimes \tilde{Z}^t$ and $B^s \otimes \tilde{B}^t \to Z^s \otimes \tilde{B}^t$ are monomorphisms. Thus $j_{\diamond}$ applied to (24) satisfies the condition of Lemma 7.9. We are reduced to the situation of Lemma 7.10, with $\mathcal{Y}$ the left Kan-extension along g of $j^*_{\diamond}$ applied to the diagram (24). Computing $\mathcal{Y}$ with (18) and then applying Lemma 7.10, we obtain

$$\operatorname{LKan}_{p^{\zeta_n}_{\gamma_n}} E(1)_{-n}(j^*_{\zeta_n}(A \wedge \tilde{A})) = \Big(\bigoplus_{s+t=n} Z^s \otimes \tilde{B}^t \coprod_{B^s \otimes \tilde{B}^t} B^s \otimes \tilde{Z}^t \hookrightarrow \bigoplus_{s+t=n} Z^s \otimes \tilde{Z}^t \Big),$$

where the map is the canonical inclusion. The higher derived left Kan extensions of (24) along $p^{\zeta_n}_{\gamma_n}$ vanish.

Secondly, we consider
$E(1)_{-n-1}(j^*_{\zeta_n}(A \wedge \tilde{A})) =$



By the same argument as above, we get:

$$\begin{aligned}\underset{p^{\zeta_n}_{\gamma_n}}{\mathrm{LKan}}\, E(1)_{-n-1}(j^*_{\zeta_n}(A\wedge\tilde{A})) &= 0\colon 0 \to 0,\\ \underset{p^{\zeta_n}_{\gamma_n}}{\mathrm{LKan}}{}_1 E(1)_{-n-1}(j^*_{\zeta_n}(A\wedge\tilde{A})) &= \mathrm{id}_{\bigoplus_{s+t=n+1} B^s\otimes\tilde{B}^t}\,.\end{aligned}$$

The higher derived left Kan extensions vanish.

Thirdly,

$$E(1)_{-n-k}((A\wedge\tilde{A})|_{\vartriangle}) = 0,$$

if k is not congruent to 0 or 1 modulo $2p-2$. This completes the calculation of the E_2-term. It is concentrated in degrees $(0,m)$ and $(-1,m+1)$, with $m\equiv n \mod 2p-2$. Therefore the spectral sequence collapses at the E_2-term and becomes a short exact sequence (of morphisms)

(25)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{s+t=n} Z^s\otimes\tilde{Z}^t & \longrightarrow & E(1)_{-n}(E_{\zeta_n}) & \to & \bigoplus_{s+t=n+1} B^s\otimes\tilde{B}^t & \to & 0\\ & & \uparrow & & \uparrow{\scriptstyle E(1)_{-n}(\underrightarrow{\mathrm{Holim}}(p^{\zeta_n}_{\gamma_n}))} & & \| & & \\ 0 & \to & \bigoplus_{s+t=n} Z^s\otimes\tilde{B}^t \coprod_{B^s\otimes\tilde{B}^t} B^s\otimes\tilde{Z}^t & \to & E(1)_{-n}(E_{\gamma_n}) & \to & \bigoplus_{s+t=n+1} B^s\otimes\tilde{B}^t & \to & 0.\end{array}$$

In particular, $E(1)_*(E_{\gamma_n})$ and $E(1)_*(E_{\zeta_n})$ are concentrated in the correct degrees, and

$$E(1)_{-n}(\underrightarrow{\mathrm{Holim}}(p^{\zeta_n}_{\gamma_n}))$$

is a monomorphism. This completes the proof of Proposition 7.5. □

Note that

$$E(1)_*(E_{\beta_n}) = E(1)_*(\bigoplus_{s+t=n} A_{\beta_s}\wedge\tilde{A}_{\beta_t}) = \bigoplus_{s+t=n} B^s\otimes\tilde{B}^t$$

is concentrated in the same degrees. We need one more step to obtain an object of $\mathcal{L}$ with the correct homotopy colimit.

Remark 7.11. The assumptions of the following proposition are superfluous. We only state them to simplify the proof.

Proposition 7.12. *Let* $E\in \mathrm{Ho}(\mathcal{M}^{D_N}_{S_{E(1)}})$ *be such that for all* $n\in \mathbf{Z}_{/2p-2}$ *and all* $\alpha\in\{\beta,\gamma,\zeta\}$ *the object* $E(1)_*(E_{\alpha_n})$ *is concentrated in degrees* *congruent to* $-n$ *modulo* $2p-2$. *Let* $i\colon C_N \to D_N$ *send* β_n *to* γ_n *and* ζ_n *to* ζ_n. *Then there is an isomorphism, natural in* E,

$$\underrightarrow{\mathrm{Holim}}_{D_N} E \cong \underrightarrow{\mathrm{Holim}}_{C_N} i^*E.$$

Corollary 7.13. *Let* E *be as in Notation 7.4. Then* i^*E *is an object of* $\mathcal{L}$ *with*

$$\underrightarrow{\mathrm{Holim}}_{C_N} i^*E \cong \mathcal{R}ec(C) \wedge \mathcal{R}ec(\tilde{C}).$$

Moreover this isomorphism is natural in C *and* $\tilde{C}$.

PROOF (OF PROPOSITION 7.12): We show that the counit of the adjunction

$$\varepsilon_E\colon \mathrm{HoLKan}_i\, i^*E \longrightarrow E$$

induces an isomorphism of homotopy colimits. Since $E(1)_*(-)$ is faithful on $\mathcal{S}_{E(1)}$, it is enough to show that

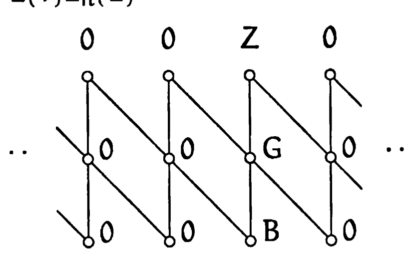
$$E(1)_*(\underrightarrow{\mathrm{Holim}}_{D_N}\, \varepsilon_E)$$

is an isomorphism. We compute this using spectral sequence (9), or rather [Fra96, 1.4.35]. Recall that

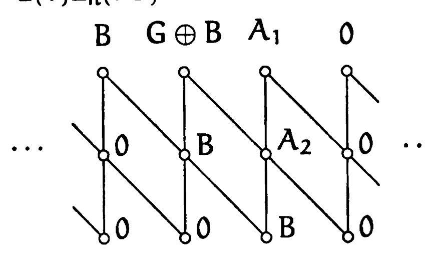
$$E \leftarrow \tilde{P}E \leftarrow \tilde{R}E \leftarrow \Omega E \tag{26}$$

is an exact triangle whose $E(1)_*$-homology is the first step of a $\underrightarrow{\lim}$-acyclic resolution of $E(1)_*(E)$. More precisely, it is the resolution

$E(1)_{-n}(E) =$



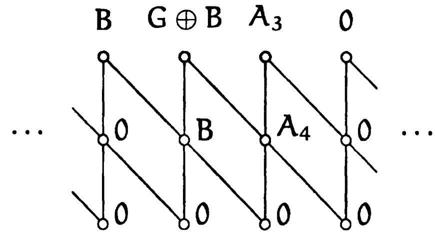
$E(1)_{-n}(\tilde{P}E) =$



,

where $A_1 = Z \oplus G \oplus B$ and $A_2 = G \oplus B$.

$$E(1)_{-n}(\tilde{R}E) =$$



where

$$A_3 = \ker(i_B^Z + i_G^Z + 1_Z) \cong B \oplus G$$

and

$$A_4 = \ker(i_B^G + 1_G) \cong B.$$

Note that

$$E(1)_{-n}\tilde{R}E$$

is already $\varinjlim$-acyclic, so that the construction stops here. Note also that the restrictions to $i(C_N)$ form a $\varinjlim$-acyclic resolution[14] of

$$E(1)_*(i^*E).$$

The functor $(\mathrm{Ho})\mathrm{LKan}_i$ simply adds a bottom row filled with zeros[15] and therefore preserves the property of a diagram to have $\varinjlim$-acyclic $E(1)_*$-homology. Thus, for computing the homotopy colimit of

$$\mathrm{HoLKan}_i\, i^*E,$$

we may replace (26) by

$$\mathrm{HoLKan}_i\, i^*E \leftarrow \mathrm{HoLKan}_i\, i^*\tilde{P}E \leftarrow \mathrm{HoLKan}_i\, i^*\tilde{R}E.$$

Then ε_E, $\varepsilon_{\tilde{P}E}$ and $\varepsilon_{\tilde{R}E}$ induce a map of exact couples. Since the image of i is cofinal in D_N, this map becomes an isomorphism in the E_2-term. □

---

[14]The $\varinjlim_1$'s of diagrams of the shape C_N are computed as equalizers in a way similar to Lemma 7.10.

[15]In Franke's setup, the statement about HoLKan_i follows from the statement about LKan_i using the spectral sequence (9). With model categories, it follows because LKan_i preserves weak equivalences.

7.2. **Computing** $Q(i^*E)$. We have found an object of $\mathcal{L}$, whose homotopy colimit equals $\mathcal{R}ec(C) \wedge \mathcal{R}ec(\tilde{C})$, namely

$$i^*E = i^* \operatorname*{HoLKan}_{pr}(A \wedge \tilde{A}).$$

In order to compute

$$\mathcal{R}ec^{-1}(\mathcal{R}ec(C) \wedge \mathcal{R}ec(\tilde{C})),$$

we apply Q to i^*E as explained in Construction 3.8. There are two steps: In 7.2.1 we determine the objects, in 7.2.2 the differentials of the complex $Q(i^*E)$.

7.2.1. *The objects.*

Notation 7.14. Write

$$\begin{aligned} (\mathcal{B}^{s+1} \to \mathcal{Z}^s) &:= A|_{\beta_{s+1} \leqslant \zeta_s} \\ \mathcal{C}^s &:= \operatorname{cone}(\mathcal{B}^{s+1} \to \mathcal{Z}^s), \end{aligned}$$

and similarly $\tilde{\mathcal{B}}^{t+1}$, $\tilde{\mathcal{Z}}^t$ and $\tilde{\mathcal{C}}^t$.

Proposition 7.15. *The cone of a "diagonal" edge* $(\beta_{n+1} \leqslant \zeta_n)$ *of* i^*E *is isomorphic to*

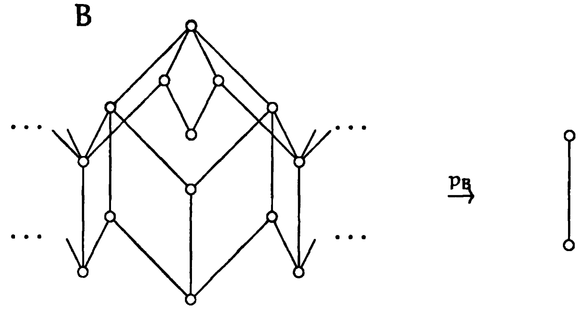
$$\bigoplus_{s+t=n} \mathcal{C}^s \wedge \tilde{\mathcal{C}}^t,$$

and this isomorphism is natural in A *and* $\tilde{A}$.

PROOF: In order to apply the cone, we need to compute $(i^*E)|_{\beta_{n+1} \leqslant \zeta_n}$ as object of $\operatorname{Ho}(\mathcal{M}^{\wr}_{S_{E(1)}})$. By Corollary 4.20

$$(i^*E)|_{\beta_{n+1} \leqslant \zeta_n} \cong \operatorname*{HoLKan}_{p_B} j_B^*((A \wedge \tilde{A})), \tag{27}$$

where B, p_B and j_B are as in Notation 4.19:



Let

$$i_B\colon B \hookrightarrow (C_N \times C_N \to \zeta_n) \times \mathcal{I}$$

be the canonical inclusion, and let

$$pr_{\mathcal{I}}\colon (C_N \times C_N \to \zeta_n) \times \mathcal{I} \longrightarrow \mathcal{I}$$

be the projection onto the second factor. Then

$$\operatorname*{HoLKan}_{p_B} j_B^*(A \wedge \tilde{A}) \cong \operatorname*{HoLKan}_{pr_{\mathcal{I}}} \operatorname*{HoLKan}_{i_B} j_B^*(A \wedge \tilde{A}). \tag{28}$$

We need to compute the cone of the right hand side of (28). This right hand side is the homotopy Kan extension along $pr_{\mathcal{I}}$ of a diagram which is by (17) of the form displayed in Figure 1. Here $- \underline{\amalg} -$ denotes the derived pushout, the vertices in B are black, and those edges that belong to the factor $\mathcal{I}$, are abbreviated by vertical arrows.

Remark 7.16. If we would forget some information and think of Figure 1 as a morphism in

$$\mathrm{Ho}\left(\mathcal{M}_{S_{E(1)}}^{C_N \times C_N \to \zeta_n}\right),$$

we would obtain the morphisms $E|_{\beta_{n+1} \leqslant \zeta_n}$ as homotopy colimit over

$$C_N \times C_N \to \zeta_n$$

of the map in Figure 1.

By (20) together with (27) and (28),

$$\mathrm{cone}((i^*E)|_{\beta_{n+1} \leqslant \zeta_n}) \cong \operatorname*{\underrightarrow{Holim}}_{C_N \times C_N \to \zeta_n} \mathrm{cone}_{C_N \times C_N \to \zeta_n}(\operatorname*{HoLKan}_{i_B} j_B^*(A \wedge \tilde{A})).$$

By [Fra96, 1.4.2], we know that for a diagram $X \in \mathrm{Ho}(\mathcal{M}^{C \times \mathcal{I}})$ the vertex

$$(\mathrm{cone}_C(X))_c$$

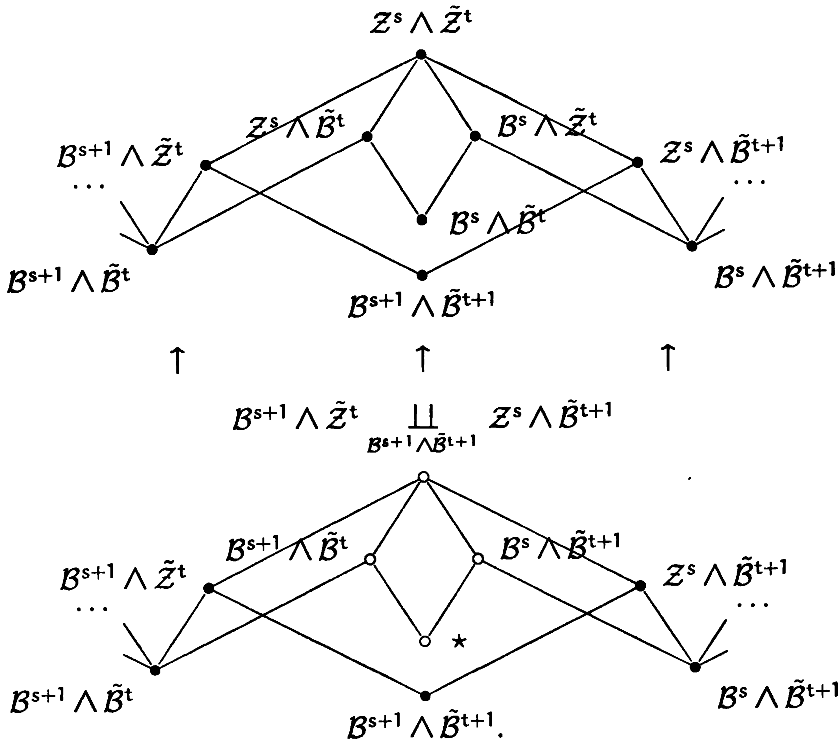


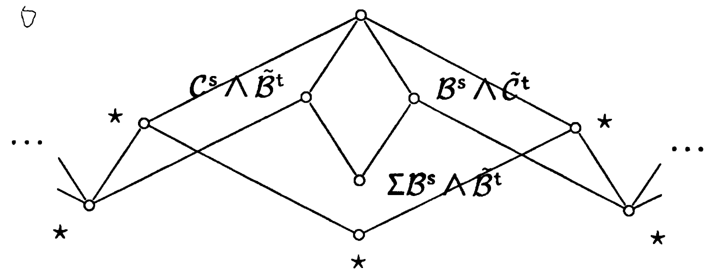
Figure 1

of the diagram $\text{cone}_C(X)$ is isomorphic to the cone of the corresponding restriction $X|_{c \times \mathcal{S}}$. Therefore,

$$\text{cone}_{C_N \times C_N \to \zeta_n} \operatorname*{HoLKan}_{i_B} j_B^*(A \wedge \tilde{A})$$

has the form

$$\operatorname{cone}\left(\mathcal{B}^{s+1}\wedge\tilde{\mathcal{Z}}^{t}\coprod_{\mathcal{B}^{s+1}\wedge\tilde{\mathcal{B}}^{t+1}}\mathcal{Z}^{s}\wedge\tilde{\mathcal{B}}^{t+1}\to\mathcal{Z}^{s}\wedge\tilde{\mathcal{Z}}^{t}\right)$$



(29)

Consider the subposet of $C_N\times C_N\to\zeta_n$

$$\vee\!\wedge\!\vee_n:=\{(\zeta_s,\zeta_t),(\beta_{s+1},\beta_t)\mid s+t=n\},$$

and let $j_{\vee\wedge\vee}$ denote its inclusion. Since $j_{\vee\wedge\vee}$ has a left adjoint, there is a functor isomorphism

$$\underset{C_N\times C_N\to\zeta_n}{\underrightarrow{\operatorname{Holim}}}\cong\underset{\vee\wedge\vee}{\underrightarrow{\operatorname{Holim}}}\,j^*_{\vee\wedge\vee}.$$

But $\underrightarrow{\operatorname{Holim}}_{\vee\wedge\vee}\,j^*_{\vee\wedge\vee}$ applied to (29) becomes

$$\bigoplus_{s+t=n}\operatorname{cone}\left(\left(\operatorname*{HoLKan}_{i_B}j^*_B(A\wedge\tilde{A})\right)_{(\zeta_s,\zeta_t)\times 8}\right).$$

We have already suggested in the pictures, that

$$\left(\operatorname*{HoLKan}_{i_B}j^*_B(A\wedge\tilde{A})\right)_{(\zeta_s,\zeta_t)\times 8}$$

is of the form

$$\mathcal{B}^{s+1}\wedge\tilde{\mathcal{Z}}^{t}\coprod_{\mathcal{B}^{s+1}\wedge\tilde{\mathcal{B}}^{t+1}}\mathcal{Z}^{s}\wedge\tilde{\mathcal{B}}^{t+1}\longrightarrow\mathcal{Z}^{s}\wedge\tilde{\mathcal{Z}}^{t}.$$

We need to make this more precise. By Corollary 4.18, we have

$$\left(\operatorname*{HoLKan}_{i_B}j^*_B(A\wedge\tilde{A})\right)_{(\zeta_s,\zeta_t)\times 8}\cong\operatorname*{HoLKan}_{p_{B'}}j^*_{B'}(A\wedge\tilde{A}),$$

where

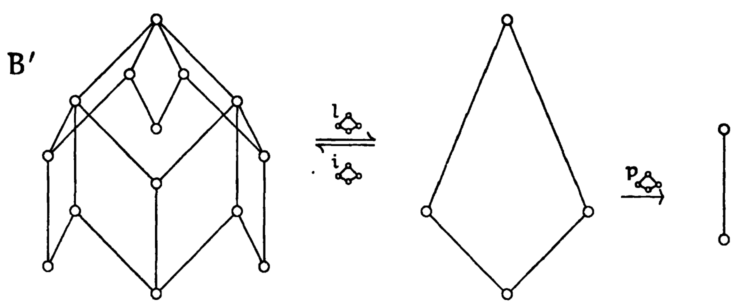
$$\begin{aligned} B' &:= B \to (\zeta_s, \zeta_t) \\ p_{B'} &:= p_B|_{B'} \\ j_{B'} &:= j_B|_{B'}. \end{aligned}$$

This expression can be simplified as follows. Consider the map of posets

$$\begin{aligned} i_{\Diamond} \colon \mathbf{I} \times \mathbf{I} &\to B' \\ (0,0) &\mapsto (\beta_{s+1}, \beta_{t+1}) \times \{0\} \\ (1,0) &\mapsto (\beta_{s+1}, \zeta_t) \times \{0\} \\ (0,1) &\mapsto (\zeta_s, \beta_{t+1}) \times \{0\} \\ (1,1) &\mapsto (\zeta_s, \zeta_t). \end{aligned}$$

It has a left adjoint $l_{\Diamond}$, satisfying

$$p_{B'} \circ i_{\Diamond} \circ l_{\Diamond} = p_{B'}.$$



Therefore, with

$$j_{\Diamond} := j_{B'} \circ i_{\Diamond} \quad \text{and } p_{\Diamond} = P_{B'} \circ i_{\Diamond}$$

as in (13),

$$\underset{p_{B'}}{\mathrm{HoLKan}}\, j^*_{B'} \cong \underset{p_{\Diamond}}{\mathrm{HoLKan}}\, j^*_{\Diamond}(A \wedge \tilde{A}).$$

By Corollary 4.15,

$$j^*_{\Diamond}(A \wedge \tilde{A}) \cong (A|_{\beta_{s+1} \leqslant \zeta_s}) \wedge (\tilde{A}|_{\beta_{t+1} \leqslant \zeta_t}).$$

Therefore, by (16),

$$\operatorname{HoLKan}_{p_{\diamond}} j_{\diamond}^{*}(A \wedge \tilde{A}) \cong (A|_{\beta_{s+1} \leqslant \zeta_s}) \square (\tilde{A}|_{\beta_{t+1} \leqslant \zeta_t}).$$

Finally, Proposition 4.21 implies the claim:

$$\operatorname{cone}\left(\operatorname{HoLKan}_{p_{\diamond}} j_{\diamond}^{*}(A \wedge \tilde{A})\right) \cong \operatorname{cone}\left(A|_{\beta_{s+1} \leqslant \zeta_s}\right) \wedge \operatorname{cone}\left(\tilde{A}|_{\beta_{t+1} \leqslant \zeta_t}\right).$$

□

7.2.2. *The differentials.* We have determined the objects of $Q(i^*E)$, and are ready to compute the differentials. It turns out that it is enough to consider a simple special case:

Remark 7.17 (Franke). Let $C^\bullet$ be a cochain complex, and let $s \in \mathbf{Z}$. Consider the map of cochain complexes

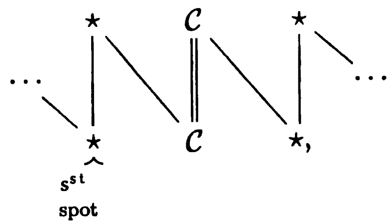
$$\begin{array}{ccccccccccc} \cdots \longrightarrow & 0 & \longrightarrow & C^s & = & C^s & \longrightarrow & 0 & \longrightarrow \cdots \\ & \downarrow & & \| & & \downarrow d^s & & \downarrow & \\ \cdots \longrightarrow & C^{s-1} & \longrightarrow & C^s & \xrightarrow{d^s} & C^{s+1} & \longrightarrow & C^{s+2} & \longrightarrow \cdots . \end{array} \tag{30}$$

We will write $f_{C,s}$ for this map and $f_{\tilde{C},t}$ for the analogous map for $\tilde{C}$ and t. We apply Propositions 7.5, 7.12 and 7.15 to the maps $f_{C,s}$ and $f_{\tilde{C},t}$ and obtain

$$\begin{array}{ccccc} \cdots \longrightarrow & C^s \otimes \tilde{C}^t & \xrightarrow{?} & C^s \otimes \tilde{C}^t \oplus C^s \otimes \tilde{C}^t & \longrightarrow \cdots \\ & \downarrow & & \downarrow (d^s \otimes 1, 1 \otimes \tilde{d}^t) & \\ \cdots \longrightarrow & \bigoplus_{s'+t'=n} C^{s'} \otimes \tilde{C}^{t'} & \xrightarrow{?} & \bigoplus_{s'+t'=n+1} C^{s'} \otimes \tilde{C}^{t'} & \longrightarrow \cdots . \end{array}$$

The left vertical arrow is the inclusion of the $(s,t)^{\text{th}}$ summand, and the horizontal arrows are the differentials we are looking for. Therefore, in order to find the differential in the general case, i.e. the lower horizontal arrow, it is sufficient to consider the much simpler case that C and $\tilde{C}$ are both as in the top row of (30). Note that Q^{-1} maps a cochain complex of this form into an object of $\mathcal{L}$ that looks like

(31)



where

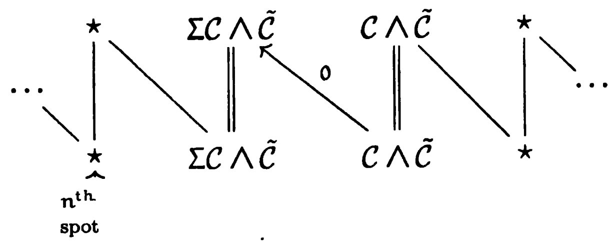
$$C^s = E(1)_*(\Sigma C).$$

Proposition 7.18. *Let* A *and* $\tilde{A} \in \mathrm{Ob}\,\mathcal{L}$ *be two objects of the form (31) for* s, C *and* t, $\tilde{C}$. *Then the map (7) that induces the* $(s+t)^{\text{th}}$ *differential in* $Q(i^*E)$ *is*

$$\text{diag}\colon \Sigma^2 C \wedge \tilde{C} \longrightarrow \Sigma^2 C \wedge \tilde{C} \oplus \Sigma^2 C \wedge \tilde{C}.$$

PROOF: Let $n = s + t$. By Corollary 4.18, i^*E looks like

(32)



The first morphism of (7) is

$$\mathfrak{Cone}(\mathfrak{Cone}(i^*E_{\beta_{n+1} \leqslant \zeta_n})).$$

In our case this is the identity of $\Sigma^2 C \wedge \tilde{C}$.

The second morphism of (7) is the suspension of

$$(i^*E)_{\beta_{n+1} \leqslant \zeta_{n+1}},$$

and is therefore also equal to the identity of $\Sigma^2 C \wedge \tilde{C}$.

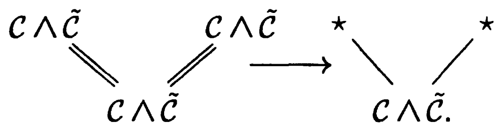
The third morphism of (7) is the suspension of the cone inclusion belonging to

$$(i^*E)_{\beta_{n+2} \leqslant \zeta_{n+1}}.$$

Corollary 4.18 implies that $(i^*E)_{\beta_{n+2} \leqslant \zeta_{n+1}}$ is the "equatorial embedding" of $\Sigma C \wedge \tilde{C}$ into its suspension, i.e. it is the left homotopy Kan extension along

of

(33)



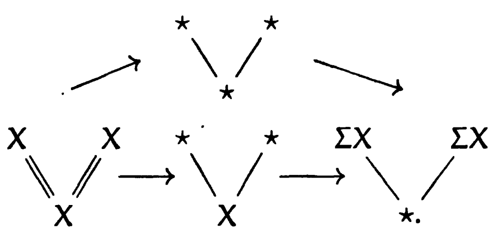
The claim now follows from the following lemma about the cone inclusion of the equatorial embedding. □

Lemma 7.19. *The cone inclusion of the equatorial embedding of* X *into* ΣX*, defined as in (33), is equal to*

$$\text{diag}\colon \Sigma X \longrightarrow \Sigma X \oplus \Sigma X.$$

PROOF: Taking cones commutes with homotopy colimits. Therefore the morphism that we want to identify is equal to the homotopy colimit over $\vee$ of the lower right hand arrow of the diagram

(34)



This arrow is a baby-phantom: restricted to any vertex of $\vee$, it becomes zero, yet its homotopy colimit is not equal to zero. For simplicial sets, and therefore for finite spectra, the claim of the lemma is well known. It can for example be seen by looking at explicit cofibrant replacements in the last diagram. But this special case implies the general case: In [Fra96, Cor.1.6.1.], Franke defines a family of bi-functors

$$-\wedge -\colon \mathcal{K}_{\mathsf{C}} \times \mathcal{S}^{\mathrm{fin}}_{\mathsf{C}} \longrightarrow \mathcal{K}_{\mathsf{C}\times\mathsf{D}}$$

that commutes with homotopy colimits and cones, and is associative. It has the properties that $-\wedge\mathbb{S}^0$ is the identity, $-\wedge\mathbb{S}^1$ is the suspension, and that $- \wedge \star$ is zero. Here $\mathcal{K}_{\mathsf{D}}$ is an arbitrary system of triangulated diagram categories, in our case

$$\mathcal{K}_{\mathsf{D}} = \mathrm{Ho}(\mathcal{M}^{\mathsf{D}}_{\mathsf{S}_{\mathsf{E}(1)}}),$$

and $\mathsf{S}^{\mathrm{fin}}_{\mathsf{C}}$ denotes the homotopy category of C-shaped diagrams of finite spectra. In order to obtain the two diagrams (33) and (34) for arbitrary X, we take the corresponding diagrams for $\mathbb{S}^0$ and smash it with X. Since the claim of the lemma is true for $\mathbb{S}^0$, and by the properties of Franke's smash product, it follows that it is also true for X. □

Together with Remark 7.17 this completes the proof of Theorem 1.1.

$$\Sigma^2 C \wedge \tilde{C} \cong \Sigma C \wedge \Sigma \tilde{C}.$$

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