CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 46, nº 2 (2005), p. 123-138

<http://www.numdam.org/item?id=CTGDC_2005__46_2_123_0>

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GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XLVI-2 (2005)

CONSTRUCTING ORDERED GROUPOIDS

by Mark V. LAWSON

Résumé. Nous montrons que chaque groupoïde ordonné est isomorphe à un groupoïde construit à partir d'une catégorie opérant de manière appropriée sur un groupoïde provenant d'une relation d'équivalence. Cette construction est utilisée, dans un article suivant, pour analyser le monoïde structurel ou géométrique de Dehornoy associé à une variété balancée.

Introduction

The theory of ordered groupoids was introduced by Ehresmann as a way of formalising the theory of pseudogroups of transformations [3].[1] During the 1990's, the author began a systematic investigation of the role of ordered groupoids in inverse semigroup theory. This work is summarised in [8] and forms a part of 'the ordered groupoid approach to inverse semigroups'. This approach has been substantially advanced in recent years; see [4, 10, 13, 19]. We may summarise by saying that ordered groupoids are an important tool in studying inverse semigroups, and that inverse semigroups are turning out to be natural mathematical objects; the books [8, 17, 18] provide many examples justifying this last claim.

The aim of this paper is to describe a way of constructing ordered groupoids. The construction grew out of concrete examples: the clause inverse semigroup introduced by Girard in [5] for applications in linear

[1] What we call 'ordered groupoids' were termed 'functorially ordered groupoids' by Ehresmann.

logic; my construction of inverse semigroups from category actions in [9] that was motivated by an analysis of Girard's semigroup; and Dehornoy's construction of the structural monoid of an algebraic variety defined by a set of balanced equations [2]. In this section, I describe the idea behind my construction.

Let G be a groupoid. I shall denote the right identity of $g \in G$ by $\mathbf{d}(g)$ and the left identity by $\mathbf{r}(g)$. I shall also denote g by an arrow $\mathbf{d}(g) \xrightarrow{g} \mathbf{r}(g)$. The partial product will be denoted by concatenation; note that the product gh is defined iff $\mathbf{d}(g) = \mathbf{r}(h)$. The set of identities of G is denoted G_o. A groupoid G is said to be *ordered* if it is equipped with a partial order $\leq$ in such a way that the following four axioms hold:

(OG1) $x \leq y$ implies $x^{-1} \leq y^{-1}$.

(OG2) If $x \leq y$ and $u \leq v$ and xu and yv are defined then $xu \leq yv$.

(OG3) Let $e \leq \mathbf{d}(x)$ where e is an identity. Then there exists a unique element $(x \mid e)$, called the *restriction of x to e*, such that $(x \mid e) \leq x$ and $\mathbf{d}(x \mid e) = e$.

(OG3)* Let $e \leq \mathbf{r}(x)$ where e is an identity. Then there exists a unique element $(e \mid x)$, called the *corestriction of x to e*, such that $(e \mid x) \leq x$ and $\mathbf{r}(e \mid x) = e$.

In fact, axiom (OG3)* is a consequence of the other axioms; see [8]. The homomorphisms between ordered groupoids are the *ordered functors*: those functors that are also order-preserving. An ordered functor $\alpha \colon G \to H$ is an *ordered embedding* if $g \leq h$ iff $\alpha(g) \leq \alpha(h)$.

In the class of groupoids, those that arise from equivalence relations deserve to be regarded as the simplest. They can be characterised by the property that if e and f are identities then there is at most one element g such that $e \xrightarrow{g} f$. We call such groupoids *combinatorial*. Our goal is to construct ordered groupoids from combinatorial groupoids together with some other data. The ingredients we need are as follows:

- A category C acts on a combinatorial groupoid H.

- This action induces a preorder $\preceq$ on H whose associated equivalence relation is $\equiv$.
- The quotient structure $H/\equiv$ is a groupoid on which the preorder induces an order.
- The groupoid $H/\equiv$ is ordered and every ordered groupoid is isomorphic to one constructed in this way.

In Sections 1 and 2, I shall show that this construction can be realised. In Section 3, I show how this construction contains the one described in [9]; in addition, I make some remarks of an historical nature. The construction is put to work in [14] in analysing Dehornoy's structural monoids [2].

Finally, I need to say a few words about the relationship between ordered groupoids and inverse semigroups. Let G be an ordered groupoid. If $x, y \in G$ are such that $e = \mathbf{d}(x) \wedge \mathbf{r}(y)$ exists, then Ehresmann defined

$$x \otimes y = (x \mid e)(e \mid y),$$

called the *pseudoproduct* of x and y. It can be proved [8] that if $x \otimes (y \otimes z)$ and $(x \otimes y) \otimes z$ are both defined, then they are equal. An ordered groupoid is said to be *inductive* if the order on the set of identities is a meet semilattice.[2] An inductive groupoid gives rise to an inverse semigroup $(G, \otimes)$ using the pseudoproduct, and every inverse semigroup arises in this way. Ordered functors between inductive groupoids that preserve the meet operation on the set of identities give rise to homomorphisms between the associated inverse semigroups. An ordered groupoid is said to be $*$-*inductive* if the following condition holds for each pair of identities: if they have a lower bound, they have a greatest lower bound. A $*$-inductive groupoid gives rise to an inverse semigroup with zero $(G^0, \otimes)$: adjoin a zero to the set G, and extend the pseudoproduct on G to G^0 in such a way that if $s, t \in G$ and $s \otimes t$ is not defined then put $s \otimes t = 0$, and define all products with 0 to be 0. Every inverse semigroup with zero arises in this way. The details of the ordered groupoid approach to inverse semigroup theory are described in [8].

[2] The term 'inductive' is used in inverse semigroup theory in a way quite different from that used in Ehresmann's work.

1 Categories acting on groupoids

In this section, I shall define a class of actions of categories on combinatorial groupoids, and show that they can be used to construct ordered groupoids.

We begin by recalling the definition of a category acting on another category, in this case, a groupoid.[3] Let C be a category and G a groupoid. Let $\pi\colon G \to C_o$ be a function to the set of identities of C. Define

$$C * G = \{(a, x) \in C \times G \colon \mathbf{d}(a) = \pi(x)\}.$$

We say that C *acts on* G if there is a function from $C * G$ to G, denoted by $(a, x) \mapsto a \cdot x$, which satisfies the axioms (A1)–(A6) below. Note that I write $\exists a \cdot x$ to mean that $(a, x) \in C * G$. I shall also use $\exists$ to denote the existence of products in the categories C and G.

(A1) $\exists \pi(x) \cdot x$ and $\pi(x) \cdot x = x$.

(A2) $\exists a \cdot x$ implies that $\pi(a \cdot x) = \mathbf{r}(a)$.

(A3) $\exists a \cdot (b \cdot x)$ iff $\exists (ab) \cdot x$, and if they exist they are equal.

(A4) $\exists a \cdot x$ iff $\exists a \cdot \mathbf{d}(x)$, and if they exist then $\mathbf{d}(a \cdot x) = a \cdot \mathbf{d}(x)$;
$\exists a \cdot x$ iff $\exists a \cdot \mathbf{r}(x)$, and if they exist then $\mathbf{r}(a \cdot x) = a \cdot \mathbf{r}(x)$.

(A5) If $\pi(x) = \pi(y)$ and $\exists xy$ then $\pi(xy) = \pi(x)$.

(A6) If $\exists a \cdot (xy)$ then $\exists (a \cdot x)(a \cdot y)$ and $a \cdot (xy) = (a \cdot x)(a \cdot y)$.

We write (C, G) to indicate the fact that C acts on G. If C acts on G and $x \in G$ then define

$$C \cdot x = \{a \cdot x \colon \exists a \cdot x\}.$$

Define $x \preceq y$ in G iff there exists $a \in C$ such that $x = a \cdot y$. It is easy to check that $\preceq$ is a preorder on G. Let $\equiv$ be the associated equivalence: $x \equiv y$ iff $x \preceq y$ and $y \preceq x$. Denote the $\equiv$-equivalence class containing x by $[x]$, and denote the set of $\equiv$-equivalence classes by $J(C, G)$. The set $J(C, G)$ is ordered by $[x] \leq [y]$ iff $x \preceq y$.

[3] I am grateful to the referee for pointing out that Ehresmann originated this notion; see [3], volume III-3, page 439.

Remarks

(1) The usual definition of a category acting on a set is a special case of the definition of a category acting on a groupoid: a set can be regarded as a groupoid in which each element is an identity. In this case, axioms (A4)–(A6) are automatic.

(2) Let C act on the groupoid G. Then C acts on the set G_o. To prove this, it is enough to show that if x is an identity in G and $\exists a \cdot x$ then $a \cdot x$ is an identity in G. This follows by (A4), since $\mathbf{d}(a \cdot x) = a \cdot \mathbf{d}(x) = a \cdot x$.

(3) Let C act on the groupoid G. If $x \in G$ and $a \in C$ then $\exists a \cdot x$ iff $\exists a \cdot x^{-1}$, in which case $(a \cdot x)^{-1} = a \cdot x^{-1}$. It is straightforward to check that $G_e = \pi^{-1}(e)$ is a subgroupoid of G, and that if $f \xleftarrow{a} e$ in C, then the function $x \mapsto a \cdot x$ from G_e to G_f is a functor.

(4) Let C act on the groupoids G and G'. We say that (C, G) is *isomorphic* to (C, G') iff there is an isomorphism $\alpha \colon G \to G'$ such that $\exists a \cdot x$ iff $\exists a \cdot \alpha(x)$ in which case $\alpha(a \cdot x) = a \cdot \alpha(x)$.

(5) Observe that $x \preceq y$ iff $C \cdot x \subseteq C \cdot y$. Thus $x \equiv y$ iff $C \cdot x = C \cdot y$.

(6) If $x \equiv y$ then $\mathbf{d}(x) \equiv \mathbf{d}(y)$ and $\mathbf{r}(x) \equiv \mathbf{r}(y)$ by axiom (A4).

We shall be interested in actions of categories C on groupoids G that satisfy two further conditions:

(A7) G is combinatorial.

(A8) $\mathbf{d}(a \cdot x) = \mathbf{d}(b \cdot x)$ iff $\mathbf{r}(a \cdot x) = \mathbf{r}(b \cdot x)$.

Condition (A7) is to be expected; condition (A8) will make everything work, as will soon become clear.

Theorem 1.1 *Let C be a category acting on the groupoid G, and suppose in addition that both (A7) and (A8) hold. Then*

(i) *$J(C, G)$ is an ordered groupoid.*

(ii) $J(C,G)$ *is* $*$*-inductive iff for all identities* $e, f \in G$ *we have that* $C \cdot e \cap C \cdot f$ *non empty implies there exists an identity* i *such that* $C \cdot e \cap C \cdot f = C \cdot i$.

Proof (i) Define

$$\mathbf{d}[x] = [\mathbf{d}(x)] \text{ and } \mathbf{r}[x] = [\mathbf{r}(x)].$$ [4]

These are well-defined by Result (4).

We claim that $\mathbf{d}[x] = \mathbf{r}[y]$ iff there exists $x' \in [x]$ and $y' \in [y]$ such that $\exists x'y'$. To prove this, suppose first that $\mathbf{d}[x] = \mathbf{r}[y]$. Then $\mathbf{d}(x) \equiv \mathbf{r}(y)$. There exist elements $a, b \in C$ such that $\mathbf{d}(x) = a \cdot \mathbf{r}(y)$ and $\mathbf{r}(y) = b \cdot \mathbf{d}(x)$. Thus by (A3) and (A4), we have that

$$\mathbf{r}(b \cdot (a \cdot y)) = \mathbf{r}(y).$$

By (A8), this implies that

$$\mathbf{d}(b \cdot (a \cdot y)) = \mathbf{d}(y).$$

By (A7), this means that $y = b \cdot (a \cdot y)$. Hence $y \equiv a \cdot y$ and $\exists x(a \cdot y)$, as required. The converse follows by Result (4).

We define a partial product on $J(C,G)$ as follows: if $\mathbf{d}[x] = \mathbf{r}[y]$ then

$$[x][y] = [x'y'] \text{ where } x' \in [x],\ y' \in [y] \text{ and } \exists x'y',$$

otherwise the partial product is not defined. To show that it is well-defined we shall use (A7) and (A8). Let $x'' \in [x]$ and $y'' \in [y]$ be such that $\exists x''y''$. We need to show that $x'y' \equiv x''y''$. By definition there exist $a, b, c, d \in C$ such that

$$x' = a \cdot x, \quad x = b \cdot x', \quad x'' = c \cdot x, \quad x = d \cdot x''$$

and there exist $s, t, u, v \in C$ such that

$$y' = s \cdot y, \quad y = t \cdot y', \quad y'' = u \cdot y, \quad y = v \cdot y''.$$

[4] Strictly speaking, I should write $\mathbf{d}([x])$ but I shall omit the outer pair of brackets.

Now $x = b \cdot x'$ and $x'' = c \cdot x$. Thus $x'' = (cb) \cdot x'$ by (A3). Now $\exists x'y'$ and so $\pi(x'y') = \pi(x')$ by (A5). Thus $\exists (cb) \cdot (x'y')$. Hence $(cb) \cdot (x'y') = [(cb) \cdot x'][(cb) \cdot y']$ by (A6) which is $x''[(cb) \cdot y']$. We shall show that $(cb) \cdot y' = y''$, which proves that $x''y'' \preceq x'y'$; the fact that $x'y' \preceq x''y''$ holds by a similar argument so that $x'y' \equiv x''y''$ as required. It therefore only remains to prove that $(cb) \cdot y' = y''$. We have that $y'' = (ut) \cdot y'$ and $\mathbf{d}(x'') = \mathbf{r}(y'')$. Thus $\mathbf{d}(x'') = \mathbf{r}(y'') = (ut) \cdot \mathbf{r}(y')$ by (A4). But $\mathbf{d}(x'') = (cb) \cdot \mathbf{r}(y')$. Thus $(ut) \cdot \mathbf{r}(y') = (cb) \cdot \mathbf{r}(y')$. Hence

$$\mathbf{r}((ut) \cdot y') = \mathbf{r}((cb) \cdot y')$$

by (A4). Therefore

$$\mathbf{d}((ut) \cdot y') = \mathbf{d}((cb) \cdot y')$$

by (A8). It follows that the elements $(ut) \cdot y'$ and $(cb) \cdot y'$ have the same domains and codomains, and so are equal by (A7). It follows that $(cb) \cdot y' = (ut) \cdot y' = y''$. Thus the partial product is well-defined.

It is now easy to check that $J(C, G)$ is a groupoid: $[x]^{-1} = [x^{-1}]$, and the identities are the elements of the form $[x]$ where $x \in G_o$.

The order on $J(C, G)$ is defined by $[x] \leq [y]$ iff $x = a \cdot y$ for some $a \in C$. It remains to show that $J(C, G)$ is an ordered groupoid with respect to this order.

(OG1) holds by Result (3).

(OG2) holds: let $[x] \leq [y]$ and $[u] \leq [v]$ and suppose that the partial products $[x][u]$ and $[y][v]$ exist. Then there exist $x' \in [x]$, $u' \in [u]$, $y' \in [y]$ and $v' \in [v]$ such that $[x][u] = [x'u']$ and $[y][v] = [y'v']$. By assumption, $[x'] \leq [y']$ and $[u'] \leq [v']$ so that there exist $a, b \in C$ such that $x' = a \cdot y'$ and $u' = b \cdot v'$. We need to show that $x'u' \preceq y'v'$. Now $\mathbf{d}(x') = \mathbf{r}(u')$ and so $a \cdot \mathbf{d}(y') = b \cdot \mathbf{r}(v')$. But $\mathbf{d}(y') = \mathbf{r}(v')$. Thus $a \cdot \mathbf{d}(y') = b \cdot \mathbf{d}(y')$. Hence

$$\mathbf{d}(a \cdot y') = \mathbf{d}(b \cdot y').$$

By (A8), we therefore have that

$$\mathbf{r}(a \cdot y') = \mathbf{r}(b \cdot y'),$$

and so $a \cdot y' = b \cdot y'$ by (A7). Thus $x'u' = (a \cdot y')(b \cdot v') = (b \cdot y')(b \cdot v')$. Now $\exists y'v'$ and so by (A5) and (A6) we have that $(b \cdot y')(b \cdot v') = b \cdot (y'v')$. Thus $x'u' = b \cdot (y'v')$ and so $x'u' \preceq y'v'$, as required.

(OG3) holds: let $[e] \leq \mathbf{d}[x]$ where $e \in G_o$. Then $e \preceq \mathbf{d}(x)$ and so $e = a \cdot \mathbf{d}(x)$ for some $a \in C$. Now $\exists a \cdot x$ by (A4). Define

$$([x] \mid [e]) = [a \cdot x].$$

Clearly $[a \cdot x] \leq [a]$, and $\mathbf{d}[a \cdot x] = [a \cdot \mathbf{d}(x)] = [e]$. It is also unique with these properties as we now show. Let $[y] \leq [x]$ such that $\mathbf{d}[y] = [e]$. Then $y = b \cdot x$ for some $b \in C$ and $\mathbf{d}(y) \equiv e$. Because of the latter, there exists $c \in C$ such that $e = c \cdot \mathbf{d}(y)$. Thus $e = (cb) \cdot \mathbf{d}(x)$. But $e = a \cdot \mathbf{d}(x)$. Thus $(cb) \cdot \mathbf{d}(x) = a \cdot \mathbf{d}(x)$. Hence $(cb) \cdot \mathbf{r}(x) = a \cdot \mathbf{r}(x)$ by (A8). So by (A7), $c \cdot (b \cdot x) = a \cdot x$, giving $c \cdot y = a \cdot x$. It follows that we have shown that $a \cdot x \preceq y$. From $\mathbf{d}(y) \equiv e$, there exists $d \in C$ such that $\mathbf{d}(y) = d \cdot e$. Using (A7) and (A8), we can show that $y = d \cdot (a \cdot x)$, and so $y \preceq a \cdot x$. We have therefore proved that $y \equiv a \cdot x$. Hence $[y] = [a \cdot x]$, as required.

(OG3)* holds: although this axiom follows from the others, we shall need an explicit description of the corestriction. Let $[e] \leq \mathbf{r}[x]$ where $e \in G_o$. Then $e \preceq \mathbf{r}(x)$ and so $e = b \cdot \mathbf{r}(x)$ for some $b \in C$. Now $\exists b \cdot x$ by (A4). Define

$$([e] \mid [x]) = [b \cdot x].$$

The proof that this has the required properties is similar to the one above.

(ii) We now turn to the properties of the pseudoproduct in $J(C, G)$. Let $[e], [f]$ be a pair of identities in $J(C, G)$. It is immediate from the definition of the partial order that $[e]$ and $[f]$ have a lower bound iff $C \cdot e \cap C \cdot f \neq \emptyset$. Next, a simple calculation shows that $[i] \leq [e], [f]$ iff $C \cdot i \subseteq C \cdot e \cap C \cdot f$. It is now easy to deduce that $[i] = [e] \wedge [f]$ iff $C \cdot i = C \cdot e \cap C \cdot f$.

It will be useful to have a description of the pseudoproduct itself. If $C \cdot i = C \cdot e \cap C \cdot f$ then denote by

$$e * f \text{ and } f * e$$

elements of C, not necessarily unique, such that

$$i = (e * f) \cdot f = (f * e) \cdot e.$$

Suppose that $[x], [y]$ are such that the pseudoproduct $[x] \otimes [y]$ exists. Then by definition $[\mathbf{d}(x)] \wedge [\mathbf{r}(y)]$ exists. Thus $C \cdot \mathbf{d}(x) \cap C \cdot \mathbf{r}(y) = C \cdot e$ for some $e \in G_o$. It follows that

$$[x] \otimes [y] = ([x] \mid [e])([e] \mid [y]).$$

Now

$$([x] \mid [e]) = [(\mathbf{r}(y) * \mathbf{d}(x)) \cdot x]$$

and

$$([e] \mid [y]) = [(\mathbf{d}(x) * \mathbf{r}(y)) \cdot y].$$

Hence

$$[x] \otimes [y] = [((\mathbf{r}(y) * \mathbf{d}(x)) \cdot x)((\mathbf{d}(x) * \mathbf{r}(y)) \cdot y)].$$

■

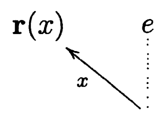
The condition that if $C \cdot e \cap C \cdot f$ is non-empty, where e and f are identities, then there exists an identity i such that $C \cdot e \cap C \cdot f = C \cdot i$ will be called the *orbit condition* for the pair (C, G). Part (ii) of Theorem 2.1 can therefore be stated thus: $J(C, G)$ is $*$-inductive iff (C, G) satisfies the orbit condition.

2 Universality of the construction

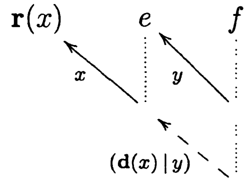
In this section, I shall show that every ordered groupoid is isomorphic to one of the form $J(C, H)$ for some action of a category C on a combinatorial groupoid H.

Let G be an ordered groupoid. There are three ingredients needed to construct $J(C, H)$: a category, which I shall denote by $C'(G)$, a combinatorial groupoid, which I shall denote by $R(G)$, and a suitable action of the former on the latter. We define these as follows:

- We define the category $C'(G)$ as follows: an element of $C'(G)$ is an ordered pair (x, e) where $(x, e) \in G \times G_o$ and $\mathbf{d}(x) \leq e$. This element can be represented thus



We define a partial product on $C'(G)$ as follows: if $(x, e), (y, f) \in C'(G)$ and $e = \mathbf{r}(y)$ then $(x, e)(y, f) = (x \otimes y, f)$. This product can be represented thus



since in this case $x \otimes y = x(\mathbf{d}(x) \mid y)$. It is easy to check that in this way $C'(G)$ becomes a right cancellative category with identities $(e, e) \in G_o \times G_o$. Further details of this construction may be found in [12].

- We define the groupoid $R(G)$ as follows: its elements are pairs (x, y) where $\mathbf{r}(x) = \mathbf{r}(y)$. Define $\mathbf{d}(x, y) = (y, y)$ and $\mathbf{r}(x, y) = (x, x)$. The partial product is defined by $(x, y)(y, z) = (x, z)$. Evidently, $R(G)$ is the groupoid associated with the equivalence relation that relates x and y iff $\mathbf{r}(x) = \mathbf{r}(y)$.

- We shall now define what will turn out to be an action of $C'(G)$ on $R(G)$. Define $\pi\colon R(G) \to C'(G)_o$ by $\pi(x, y) = (\mathbf{r}(x), \mathbf{r}(y))$, a well-defined function. Define $(g, e) \cdot (x, y) = (g \otimes x, g \otimes y)$ iff $e = \mathbf{r}(x) = \mathbf{r}(y)$. This is a well-defined function from $C'(G) * R(G)$ to $R(G)$.

Proposition 2.1 *Let G be an ordered groupoid. With the above definition, the pair $(C'(G), R(G))$ satisfies axioms (A1)–(A8).*

Proof The verification of axioms (A1)–(A7) is routine. We show explicitly that (A8) holds. Suppose that

$$\mathbf{r}[(s, e) \cdot (x, y)] = \mathbf{r}[(t, e) \cdot (x, y)].$$

Then $s \otimes x = t \otimes x$. The groupoid product $x^{-1}y$ is defined, and the two ways of calculating the pseudoproduct of the triple $(s, x, x^{-1}y)$ are defined, and the two ways of calculating the pseudoproduct of the triple $(t, x, x^{-1}y)$ are defined. It follows that $s \otimes y = t \otimes y$; that is,

$$\mathbf{d}[(s,e)\cdot(x,y)] = \mathbf{d}[(t,e)\cdot(x,y)].$$

The converse is proved similarly. ∎

The next theorem establishes what we would hope to be true is true.

Theorem 2.2 *Let G be an ordered groupoid. Then $J(C'(G), R(G))$ is isomorphic to G.*

Proof Define $\alpha\colon\ G \to J(C'(G), R(G))$ by $\alpha(g) = [(\mathbf{r}(g), g)]$. We show first that α is a bijection. Suppose that $\alpha(g) = \alpha(h)$. Then $(\mathbf{r}(g), g) \equiv (\mathbf{r}(h), h)$. Thus $(a, \mathbf{r}(g)) \cdot (\mathbf{r}(g), g) = (\mathbf{r}(h), h)$ and $(b, \mathbf{r}(h)) \cdot (\mathbf{r}(h), h) = (\mathbf{r}(g), g)$ for some category elements $(a, \mathbf{r}(g))$ and $(b, \mathbf{r}(h))$. Hence

$$a \otimes \mathbf{r}(g) = \mathbf{r}(h), \quad b \otimes \mathbf{r}(h) = \mathbf{r}(g), \quad a \otimes g = h, \text{ and } b \otimes h = g.$$

It follows that a and b are identities and so $h \leq g$ and $g \leq h$, which gives $g = h$. Thus α is injective. To prove that α is surjective, observe that if $[(x,y)]$ is an arbitrary element of $J(C'(G), R(G))$, then $(x,y) \equiv (\mathbf{d}(x), x^{-1}y)$ because

$$(x^{-1}, \mathbf{r}(x)) \cdot (x,y) = (\mathbf{d}(x), x^{-1}y) \text{ and } (x, \mathbf{d}(x)) \cdot (\mathbf{d}(x), x^{-1}y) = (x,y).$$

Next we show that α is a functor. It is clear that identities map to identities. Suppose that gh is defined in G. Now $\alpha(g) = [(\mathbf{r}(g), g)]$ and $\alpha(h) = [(\mathbf{r}(h), h)]$. We have that $\mathbf{d}[(\mathbf{r}(g), g)] = [(g,g)]$ and $\mathbf{r}[(\mathbf{r}(h), h)] = [(\mathbf{r}(h), \mathbf{r}(h))]$. Now $(g,g) \equiv (\mathbf{d}(g), \mathbf{d}(g))$ because

$$(g^{-1}, \mathbf{d}(g)) \cdot (g,g) = (\mathbf{d}(g), \mathbf{d}(g))$$

and

$$(g, \mathbf{d}(g)) \cdot (\mathbf{d}(g), \mathbf{d}(g)) = (g,g).$$

Thus $\alpha(g)\alpha(h)$ is also defined. Now $(\mathbf{r}(h), h) \equiv (g, gh)$ because

$$(g, \mathbf{r}(h)) \cdot (\mathbf{r}(h), h) = (g, gh)$$

and

$$(g^{-1}, \mathbf{r}(g)) \cdot (g, gh) = (\mathbf{r}(h), h).$$

Thus $\alpha(g)\alpha(h) = [(\mathbf{r}(g), gh)] = \alpha(gh)$. It follows that α is a functor.

Finally, we prove that α is an order isomorphism. Suppose first that $g \leq h$ in G. Then $g^{-1} \leq h^{-1}$ and $(\mathbf{d}(g)|h^{-1}) \leq h^{-1}$ and $\mathbf{r}(\mathbf{d}(g)|h^{-1}) = \mathbf{d}(g) = \mathbf{r}(g^{-1})$. Thus $(\mathbf{d}(g)|h^{-1}) = g^{-1}$. It is now easy to check that $(\mathbf{r}(g), g) = (g \otimes h^{-1}, \mathbf{r}(h)) \cdot (\mathbf{r}(h), h)$. Thus $\alpha(g) \leq \alpha(h)$. Now suppose that $\alpha(g) \leq \alpha(h)$. Then $(\mathbf{r}(g), g) = (a, \mathbf{r}(h)) \cdot (\mathbf{r}(h), h)$. It follows that a is an identity and that $g = a \otimes h$ and so $g \leq h$. We have proved that α is an order isomorphism.

Hence α is an isomorphism of ordered groupoids. ■

As an application, I shall show how the theory can be used to describe inverse semigroups with zero; the case of inverse semigroups without zero is similar. Let S be an inverse semigroup with zero. We denote by S^* the set $S \setminus \{0\}$ regarded as an ordered groupoid: the partial product of s and t is defined iff $s^{-1}s = tt^{-1}$ in which case it is equal to the usual product st; the partial order is the natural partial order.

The category $C' = C'(S^*)$ consists of those ordered pairs (s, e) where $s \in S^*$ and $e \in E(S^*)$, the set of non-zero idempotents of S, such that $s^{-1}s \leq e$. The product of (s, e) and (t, f) is defined iff $e = tt^{-1}$ in which case $(s, e)(t, f) = (st, f)$.

The combinatorial groupoid $R = R(S^*)$ consists of those pairs (s, t) such that s and t are both non-zero and $ss^{-1} = tt^{-1}$. Now the relation $\mathcal{R}$ is defined on S by $s \,\mathcal{R}\, t$ iff $ss^{-1} = tt^{-1}$ and is one of Green's relations.

By Theorem 2.2, the ordered groupoid S^* is isomorphic to $J(C', R)$. Thus the inverse semigroup S is isomorphic to $J(C', R)^0$ equipped with the pseudoproduct. We may summarise these results as follows.

Theorem 2.3 *Every inverse semigroup with zero S is determined upto isomorphism by three ingredients: the category $C'(S^*)$, Green's $\mathcal{R}$-relation, and the action of the category on the groupoid determined by Green's $\mathcal{R}$-relation.* ■

3 Final remarks

I begin by explaining how the construction of ordered groupoids from categories acting on sets described in [9] can be viewed as a special case of the construction of this paper.[5] Let (C, X) be a pair consisting of a category C acting on a set X where we denote by $\pi\colon X \to C_o$ the function used in defining the action. Define the relation $\mathcal{R}^*$ on the set X as follows: $x \, \mathcal{R}^* \, y$ iff $\pi(x) = \pi(y)$ and for all $a, b \in C$ we have that, when defined, $a \cdot x = b \cdot x \Leftrightarrow a \cdot y = b \cdot y$. Observe that $\mathcal{R}^*$ is an equivalence relation on the set X. In addition, $x \, \mathcal{R}^* \, y$ implies that $c \cdot x \, \mathcal{R}^* \, c \cdot y$ for all $c \in C$ where $c \cdot x$ and $c \cdot y$ are defined. Consequently, we get a combinatorial groupoid

$$G(C, X) = \{(x, y)\colon \; x \, \mathcal{R}^* \, y\}.$$

Define $\pi'\colon \; G(C, X) \to C_o$ by $\pi'(x, y) = \pi(x)$, and define an action of C on $G(C, X)$ by $a \cdot (x, y) = (a \cdot x, a \cdot y)$ when $\mathbf{d}(a) = \pi'(x, y)$. It is easy to check that axioms (A1)–(A8) hold. We may therefore construct an ordered groupoid from the pair $(C, G(C, X))$. This is identical to the ordered groupoid constructed in [9] directly from the pair (C, X).

To conclude, I would like to say a few words about the origins of the constructions described in Sections 1 and 2. Let G be an ordered groupoid. The category $C'(G)$ is one of a pair of categories that can be associated with an ordered groupoid G. The other, denoted $C(G)$, is left rather than right cancellative. The origin of these categories goes back to one of the founding papers of inverse semigroup theory written by Clifford [1]. However the explicit connection between Clifford's work and category theory was discovered by Leech [15]. He showed that in the case of inverse monoids, the whole structure of the semigroup could be reconstituted from either of these two categories. The importance of these categories was further underlined in the discovery by Loganathan [16] that the cohomology of inverse semigroups introduced by Lausch [7] was the same as the usual cohomology of one of its categories. Further applications of these categories can be found in [11, 12, 13]. As I indicated above, these categories completely determine the structure of the

[5] In fact, in [9] I look only at inverse semigroups, but the construction generalises easily.

semigroup in the case of inverse monoids. This raises the question of what can be said in general. The semigroup background to this question is discussed in [9]. As a result of reading a paper by Girard on linear logic, I was led to the construction described in [9], which shows how inverse semigroups can be constructed from categories acting on *sets.* I thought this was the final word on this construction until Claas Röver pointed out to me the paper by Dehornoy [2]. Dehornoy constructs an inverse semigroup from any variety, in the sense of universal algebra, that is described by equations which are balanced, meaning that the same variables occur on either side of the equation. This construction was clearly related to my construction in [9], but I felt the fit was not quite good enough. It was an analysis of the connections between the two that led me to the construction of this paper.

Acknowledgements

I would like to thank Claas Röver for referring me to Dehornoy's paper [2] which led directly to this new description of ordered groupoids.

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