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ORDER CONVERGENCE AND COMPACTNESS

by D. van der ZYPEN

Résumé. Soit $(P, \leq)$ un ensemble partiellement ordonné et soit $\tau$ une topologie compacte sur $P$ qui est plus fine que la topologie d'intervalles. Alors $\tau$ est contenu dans la topologie de convergence d'ordre.

1 Topologies on a given poset

On any given partially ordered set $(P, \leq)$ there are topologies arising from the given order in a natural way (see also [2]). Perhaps the best known such topology is the interval topology. Set $S^- = \{P \setminus \{x\} : x \in P\}$, and $S^+ = \{P \setminus \{x\} : x \in P\}$ where $(x) = \{y \in P : y \leq x\}$ and $[x) = \{y \in P : y \geq x\}$. Then $S = S^- \cup S^+$ is a subbase for the interval topology $\tau_i(P)$ on $P$.

There is another natural way to endow an arbitrary poset $(P, \leq)$ with a topology. We want to describe this topology in the following.

A (set) filter $\mathcal{F}$ on $(P, \leq)$ is a nonempty subset of the powerset of $P$ such that

- $\emptyset \notin \mathcal{F}$
- $U, V \in \mathcal{F}$ implies $U \cap V \in \mathcal{F}$
- $U \in \mathcal{F}$ and $V \supseteq U$ imply $V \in \mathcal{F}$.

(Note that the above concept can of course be defined for arbitrary sets.) For any subset $A \subseteq P$ let the set of lower bounds of $A$ be denoted by $A^l = \{x \in P : x \leq a \text{ for all } a \in A\}$ and the set of upper bounds by $A^u = \{x \in P : x \geq a \text{ for all } a \in A\}$. If $\mathcal{S}$ is a collection of subsets of $P$ then we set $S^l = \bigcup\{S^l : S \in \mathcal{S}\}$, similarly set $S^u = \bigcup\{S^u : S \in \mathcal{S}\}$.

Let $A \subseteq P$ be a subset of a poset $P$ and $y \in P$. We say that $y$ is the infimum of $A$ if $y$ is the greatest element of $A^l$ and write $\bigwedge A = y$. Dually we define...
the supremum of $A$, written $\bigvee A$. Note that in general, suprema and infima need not exist.

Let $\mathcal{F}$ be a filter on a poset $P$ and let $x \in P$. We say that $\mathcal{F}$ order-converges to $x$, in symbols $\mathcal{F} \rightarrow x$, if $\bigwedge \mathcal{F}^u = x = \bigvee \mathcal{F}$. Note that the principal ultrafilter consisting of the subsets of $P$ that contain $x$ order-converges to $x$.

Now we are able to define the order convergence topology $\tau_o(P)$ (called order topology in [1]) on any given poset $P$ by:

$$\tau_o(P) = \{U \subseteq P : \text{for any } x \in U \text{ and any filter } \mathcal{F} \text{ with } \mathcal{F} \rightarrow x \text{ we have } U \in \mathcal{F}\}.$$  

It is straightforward to verify that this is a topology. Indeed, $\tau_o(P)$ is the finest topology on $P$ such that order convergence implies topological convergence (which is not hard to prove either). We will make constant use of the following facts:

**FACT 1.1.** Let $P$ be a poset, let $\mathcal{F}$ be a filter on $P$. Then:

1. $x \in \mathcal{F} \iff (x] \in \mathcal{F}$ and $x \in \mathcal{F}$, $\iff [x) \in \mathcal{F}.$

2. If $\mathcal{F} \rightarrow x$ then $\mathcal{F}^u \neq \emptyset \neq \mathcal{F}$.

3. Suppose $\mathcal{F} \rightarrow x$. If $x \nleq a$ then $P \setminus (a] \in \mathcal{F}$. Dually if $x \nleq b$ then $P \setminus [b) \in \mathcal{F}$.

4. If $\mathcal{F} \rightarrow x$ and $\mathcal{G}$ is a filter on $P$ with $\mathcal{G} \supseteq \mathcal{F}$ then $\mathcal{G} \rightarrow x$.

*Proof.* The proofs of assertions 1 and 2 are straightforward, and assertion 3 follows directly from [1], p. 3. We prove assertion 4. Since $\mathcal{G}^u \supseteq \mathcal{F}^u$ it suffices to show that $\mathcal{G}^u \subseteq [x)$ in order to get $\bigwedge \mathcal{G}^u = x$. Assume that there is $y \in \mathcal{G}^u \setminus [x)$. By assertion 1, $(y] \in \mathcal{G}$. Since we have $x \nleq y$, we get $P \setminus (y] \in \mathcal{F} \subseteq \mathcal{G}$ (by assertion 3). So $(y] \cap (P \setminus (y]) = \emptyset \in \mathcal{G}$, which is a contradiction. The statement $\bigvee \mathcal{G}^l = x$ is proved similarly. $\square$

2 The result

Note that 1.1, assertion 3 implies that for any poset $P$, the interval topology $\tau_i(P)$ is contained in the order convergence topology $\tau_o(P)$. The following theorem connects the concepts of interval topology, order convergence and compactness.

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THEOREM 2.1. Let \((P, \leq)\) be a poset. If \(\tau\) is a compact topology on \(P\) such that \(\tau_1(P) \subseteq \tau\), then \(\tau \subseteq \tau_0(P)\).

Proof. Suppose that \(W \in \tau \setminus \tau_0(P)\). Then there is \(x \in W\) and a filter \(\mathcal{F}\) on \(P\) such that \(\mathcal{F} \to x\) and \(W \not\subseteq \mathcal{F}\).

The strategy now is to find an ultrafilter on the closed set \(Q := P \setminus W\) of \((P, \tau)\) that does not converge to any point of \(Q\) with respect to the subspace topology of \((P, \tau)\) on \(Q\). This will imply that \(Q\) is a non-compact closed subset of \((P, \tau)\), which in turn implies that \((P, \tau)\) is not compact.

Note that every element of \(\mathcal{F}\) intersects \(Q\) (otherwise we would have \(W \in \mathcal{F}\)). So \(\mathcal{F} \cup \{Q\}\) is a filter base which is contained in some ultrafilter \(\mathcal{U}\). Moreover, by 1.1, assertion 4, the ultrafilter \(\mathcal{U}\) order-converges to \(x\).

It is easy to check that

\[
\mathcal{U}|_Q = \{U \cap Q : U \in \mathcal{U}\}
\]

is an ultrafilter on \(Q\) (this uses of course the fact that \(Q\) is a member of \(\mathcal{U}\)).

Claim: \(\mathcal{U}|_Q\) does not converge to any \(y \in Q\) with respect to \(\tau|_Q\), the topology on \(Q\) induced by \(\tau\).

Proof of Claim: Pick any \(y \in Q\). First, we know that \(x \in W\) and \(y \in Q\), whence \(x \neq y\). Suppose that the following holds in \(P\):

(A) For all \(z \in \mathcal{U}^u\) we have \(y \leq z\) and for all \(z' \in \mathcal{U}^l\) we have \(y \geq z'\).

Then by definition of order convergence this would imply \(y \leq x\), since \(x = \bigwedge \mathcal{U}^u\), and similarly we would get \(y \geq x\), a contradiction to \(x \neq y\). So, (A) must be false, and without loss of generality we may assume that there is a \(z_0 \in \mathcal{U}^u\) with \(y \nleq z_0\). By 1.1, assertion 1, we get \((z_0) \in \mathcal{U}\) which implies

\[
B := (z_0] \cap Q \in \mathcal{U}|_Q.
\]

Since \(y \nleq z_0\) we also have

\[
y \in P \setminus (z_0]. \quad (*)
\]

Because \(\tau\) contains the interval topology \(\tau_1(P)\), statement (*) above implies that the set

\[
V := (P \setminus (z_0]) \cap Q = Q \setminus B
\]
is an open neighborhood of \( y \) in \((Q, \tau|_Q)\). But since \( B \in U|_Q \) and \( V = Q \setminus B \), we have \( V \notin U|_Q \), so \( U|_Q \) does not converge to \( y \) with respect to \( \tau|_Q \). Since \( y \in Q \) was arbitrary, the claim is proved.

The claim now shows that \( Q = P \setminus W \) is a closed, non-compact subset of \((P, \tau)\). So \((P, \tau)\) cannot be compact. \(\Box\)

This theorem has a direct consequence for Priestley spaces, i.e., compact totally order-disconnected ordered spaces as introduced in ([3], [4]).

**COROLLARY 2.2.** If \((P, \tau, \leq)\) is a Priestley space, then \( \tau \subseteq \tau_o(P) \).

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**References**


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