CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

MARTA BUNGE
MAMUMKA JIBLADZE
THOMAS STREICHER

Definable completeness

Cahiers de topologie et géométrie différentielle catégoriques, tome 45, nº 4 (2004), p. 243-266

<http://www.numdam.org/item?id=CTGDC_2004__45_4_243_0>

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XLV-4 (2004)

DEFINABLE COMPLETENESS

by Marta BUNGE, Mamumka JIBLADZE and Thomas STREICHER

RESUME. Le but de cet article est d'isoler une notion de complétude, dite définissable, pour les morphismes géométriques sur un topos de base S telle que, si on ajoute la propriété d'étalement, on obtient la notion d'étalement complet de Bunge-Funk qui est, à la fois, la contrepartie géométrique de la notion de distribution de Lawvere sur un topos, aussi bien qu'une généralisation de la notion topologique de Fox. Pour cela, on utilise des méthodes de fibration pour analyser la factorisation, dite "compréhensive", d'un morphisme géométrique ayant un topos localement connexe comme source, en un morphisme pur suivi d'un étalement complet, le tout relatif à un topos de base S. On démontre qu'un morphisme géométrique sur S (à source localement connexe) est définissablement complet si et seulement si le facteur pur de sa factorisation compréhensive est surjective.

Introduction

The notions of spread and complete spread in topology are due to R. H. Fox [4]. On account of their connection with Lawvere distributions [8] in toposes it has become of interest to cast these notions within topos theory. This has been done by Funk [5] for locales in a topos $\mathcal{S}$, and by Bunge-Funk [2] for geometric morphisms over $\mathcal{S}$.

In the case of morphisms of locales in a topos $\mathcal{S}$ there is no loss of generality when considering (complete) spreads $Y \longrightarrow X$ in $\mathcal{S}$ either in ignoring the fibrational aspect, or in using a particular site for X, namely the canonical one associated with the frame $\mathcal{O}(X)$.

By contrast, in the case of geometric morphisms $\mathcal{F} \longrightarrow \mathcal{E}$ over $\mathcal{S}$, when revisiting the (complete) spreads we find that the fibrational theory associated with a geometric morphism (Moens [9], Streicher [10]) is advantageous, and that it is desirable to provide definitions not just of a spread (already done in [2]), but also of completeness.

The complete spread geometric morphism associated with a distribution μ is obtained by a topos pullback construction that we call the display topos of μ. The *generating* covering families for the topos pullback of a subtopos can easily be given an explicit description: in the case of the display topos these families are described in [2], which here we shall call μ-covers. Our purpose is first, in §1 and §2, to effectively describe the μ-covers within the fibrational theory associated with a geometric morphism. The new description we find involves a certain right adjoint from the distribution algebra associated with the distribution [3] to the display category of the distribution.

Then beginning in §3 we use the fibrational description of μ-covers to identify the definable completeness condition. Our main result states that a geometric morphism is definably complete iff the pure factor of its "comprehensive factorization" is a surjection. We prove this result by examining the comprehensive factorization in terms of a given site for $\mathscr{E}$ over $\mathscr{S}$ [2]. The necessity of the condition follows from a result in Johnstone [7] (Theorem C3.3.14) that describes the pullback topology. In the last section of the paper we prove that a geometric morphism is a complete spread iff it is a spread and definably complete.

1 μ-covers

Let $\mathscr{E}$ denote a Grothendieck topos over *Set*, with $\Delta \dashv \Gamma : \mathscr{E} \longrightarrow Set$. A distribution on $\mathscr{E}$ in Lawvere's sense is a functor $\mu : \mathscr{E} \longrightarrow Set$ that preserves colimits. We may consider *the display category of* μ, whose objects are (E, x), where $x \in \mu(E)$. A morphism $(E, x) \longrightarrow (F, y)$ in this category is a morphism $m : E \longrightarrow F$ in $\mathscr{E}$ such that $\mu(m)(x) = y$. There is a geometric morphism over $\mathscr{E}$ obtained by a topos pullback

$$\begin{array}{ccc}
\mathscr{Y} & \rightarrowtail & P(\mathbf{Y}) \\
\big\downarrow & \lrcorner & \big\downarrow \gamma \\
\mathscr{E} & \rightarrowtail & P(\mathbf{C})
\end{array}$$

that we call the complete spread geometric morphism associated with μ. Here $\mathbf{C}$ is a small site for $\mathscr{E}$ and $\mathbf{Y}$ is the restriction of the display category to $\mathbf{C}$: the forgetful functor $U : \mathbf{Y} \longrightarrow \mathbf{C}$ is a discrete opfibration (and a cosheaf) that gives the essential geometric morphism γ of presheaf toposes. The pullback topology in $\mathbf{Y}$ is *generated* by covering families ([2], page 19)

$$U^*R = \{(E_i, x) \longrightarrow (F, y) \mid E_i \longrightarrow F \in R\} \ , \tag{1}$$

where R is an epimorphic family in $\mathscr{E}$. This description of the pullback topology is valid for any functor $\mathbf{Y} \longrightarrow \mathbf{C}$, but in the case of a distribution μ we shall call these covering families *μ-covers*.

We wish to describe the μ-cover U^*R in fibrational terms. Let

$$C = \{(i, x) \mid x \in \mu(E_i), \mu(m_i)(x) = y\} \ ,$$

which is an object of *Set*. Let

$$P = \coprod_{(i,x)\in C} E_i \ ,$$

which is an object of $\mathscr{E}$. We have a map $s : C \longrightarrow \mu(P)$ such that $s(i, x) = x$, making (P, s) a C-indexed family of objects of the display category of μ. There is a morphism $(P, s) \longrightarrow (F, y)$ in the category of families of the display category that is precisely the μ-cover U^*R.

These considerations can be described with diagrams. The epimorphic family R may be regarded as a collective epimorphism (below left) where the fiber $p^{-1}(i) = E_i$, so that $E = \coprod E_i$.

$$\begin{array}{ccc} E & \xrightarrow{\ m\ } & F \\ {\scriptstyle p}\downarrow & & \downarrow \\ \Delta I & \longrightarrow & \Delta 1 \end{array} \qquad \begin{array}{ccc} C & \longrightarrow & 1 \\ {\scriptstyle r}\downarrow & \lrcorner & \downarrow{\scriptstyle y} \\ \mu(E) & \xrightarrow{\ \mu(m)\ } & \mu(F) \\ {\scriptstyle t}\downarrow & & \downarrow \\ I & \longrightarrow & 1 \end{array} \qquad \begin{array}{ccc} P & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta C & \xrightarrow{\ \Delta(tr)\ } & \Delta I \end{array}$$

Then C is the pullback (above center), where $t^{-1}(i) = \mu(E_i)$. The left side of this pullback may not be an I-indexed family of objects of the display category of μ because the map $t \cdot r$ may not be an isomorphism. We rectify this by forming the object P previously defined: it is the pullback in $\mathscr{E}$, above right. Then $\mu(P) \longrightarrow C$ has a canonical section s making (P, s) a C-indexed family of objects of the display category of μ.

2 The display category

We proceed to define in fibrational terms the display category of a distribution, and the μ-covers in the display category.

We work over what we call a base topos, always denoted $\mathcal{S}$. Throughout, $e : \mathcal{E} \longrightarrow \mathcal{S}$ denotes a geometric morphism, $e^* \dashv e_*$. We usually just say that $\mathcal{E}$ is a topos over $\mathcal{S}$. Let $\mathcal{E}/e^*$ denote the category of objects $D \longrightarrow e^*A$ whose morphisms are pairs (k, α) such that

$$\begin{array}{ccc} D & \xrightarrow{k} & E \\ \downarrow & & \downarrow \\ e^*A & \xrightarrow{e^*\alpha} & e^*B \end{array}$$

commutes.

Let $\mu : \mathcal{E} \longrightarrow \mathcal{S}$ be a colimit preserving functor over $\mathcal{S}$. We define a category *Display*$_\mu$ associated with μ. An object of *Display*$_\mu$ is a pair of morphisms

$$(D \xrightarrow{d} e^*A, A \xrightarrow{s} \mu D) = (d, s) ,$$

where s is a section of $\mu D \xrightarrow{t} A$: $t \cdot s = 1_A$ and $t = \mu^A(d)$. (Note that $\Sigma_A \mu^A(d) \cong \mu\Sigma_A(d) \cong \mu D$ because we are assuming that μ preserves coproducts.) A morphism in *Display*$_\mu$ is a morphism (k, α) in $\mathcal{E}/e^*$ such that

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow & & \downarrow \\ \mu D & \xrightarrow{\mu(k)} & \mu E \end{array}$$

commutes.

Proposition 2.1 *The forgetful functor*

$$\begin{array}{ccc} \textit{Display}_\mu & \xrightarrow{U} & \mathcal{E}/e^* \\ & \searrow \quad \swarrow & \\ & \mathcal{S} & \end{array}$$

such that $U(d,s) = d$ is a Cartesian functor of $\mathcal{S}$-fibrations. U has a right adjoint (which is not Cartesian) *and the $\mathcal{S}$-fibers of U are discrete opfibrations.*

Proof. An explicit description of the right adjoint is available. Given $D \xrightarrow{d} e^*A$, let $t : \mu D \longrightarrow A$ denote $\mu^A(d)$. Form the following two pullbacks. The right-hand kernel pair arises because μ is a Cartesian functor.

$$\begin{array}{ccccccc} F & \longrightarrow & D & \qquad & \mu F & \longrightarrow & \mu D \\ {\scriptstyle z}\downarrow & \lrcorner & \downarrow{\scriptstyle d} & & \downarrow & \lrcorner & \downarrow{\scriptstyle t} \\ e^*\mu D & \xrightarrow{e^*t} & e^*A & & \mu D & \xrightarrow{t} & A \end{array}$$

Then the right adjoint assigns to d the pair (z, s), where $s : \mu D \longrightarrow \mu F$ is the section given by the kernel pair. □

Consider *the distribution algebra* [3] associated with a distribution μ: if $\mu \dashv \mu_*$, then $\mu_*(\Omega_{\mathcal{S}})$ is a partially ordered object of $\mathcal{E}$ that we call the distribution algebra associated with μ. We typically denote $\mu_*(\Omega_{\mathcal{S}})$ by H. (H has other properties that do not immediately concern us here.)

We may compare H with $\mathit{Display}_\mu$ by pulling back the fibration of H over $\mathcal{E}$ to $\mathcal{E}/e^*$, as in the following diagram of pullback categories.

$$\begin{array}{ccccc} \mathbf{H}_\mu & \longrightarrow & FAM(P_H) & \longrightarrow & [H] \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle P_H} \\ \mathcal{E}/e^* & \longrightarrow & \mathcal{E}^2 & \xrightarrow{\partial_0} & \mathcal{E} \\ {\scriptstyle P_e}\downarrow & & \downarrow{\scriptstyle \partial_1} & & \\ \mathcal{S} & \xrightarrow{e^*} & \mathcal{E} & & \end{array} \qquad (2)$$

An object of $[H]$ is a pair (D, S) such that D is an object of $\mathcal{E}$ and $S \rightarrowtail \mu D$ is a subobject. The functor ∂_1 is the codomain fibration. Objects of $\mathbf{H}_\mu$ are pairs (d, S) where $D \xrightarrow{d} e^*A$ is an object of $\mathcal{E}/e^*$ and $S \rightarrowtail \mu D$ is a subobject.

Proposition 2.2 *The full inclusion of fibrations over* $\mathcal{S}$

$$\begin{array}{ccc} \mathit{Display}_\mu & \hookrightarrow & \mathbf{H}_\mu \\ & \searrow \quad \swarrow & \\ & \mathcal{S} & \end{array}$$

is Cartesian, and it has a right adjoint (which is not Cartesian), *which we shall call* the D-coreflection.

Proof. The right adjoint assigns to a pair $(D \xrightarrow{d} e^*A, S \overset{r}{\rightarrowtail} \mu D)$ the pullback m

$$\begin{array}{ccc} F & \longrightarrow & D \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle d} \\ e^*S & \xrightarrow{e^*(tr)} & e^*A \end{array} \qquad \begin{array}{ccc} \mu F & \longrightarrow & \mu D \\ {\scriptstyle \mu^S(m)}\downarrow & \lrcorner & \downarrow{\scriptstyle t} \\ S & \xrightarrow{tr} & A \end{array}$$

in $\mathcal{E}$, where $t : \mu D \longrightarrow A$ is equal to $\mu^A(d)$. Then $\mu^S(m)$ is equipped with a section $s : S \longrightarrow \mu F$ induced by r and the pullback above right. □

Definition 2.3 A morphism (k, α) in $\mathcal{E}/e^*$ for which k is an epimorphism in $\mathcal{E}$ is called *a collective epimorphism.*

Suppose we are given a collective epimorphism

$$\begin{array}{ccc} D & \overset{k}{\twoheadrightarrow} & E \\ \downarrow & & \downarrow{\scriptstyle z} \\ e^*A & \xrightarrow{e^*\alpha} & e^*B \end{array}$$

in $\mathcal{E}/e^*$, and an object (z, s) of $\mathit{Display}_\mu$. The D-coreflection of the Cartesian lifting along the fibration $\mathbf{H}_\mu \longrightarrow \mathcal{E}/e^*$ of (k, α) is formed from the pullbacks

$$\begin{array}{ccc} C & \twoheadrightarrow & B \\ {\scriptstyle r}\downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ \mu D & \overset{\mu(k)}{\twoheadrightarrow} & \mu E \end{array} \qquad \begin{array}{ccc} F & \longrightarrow & D \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle d} \\ e^*C & \xrightarrow{e^*(tr)} & e^*A \end{array}$$

in $\mathscr{S}$ and $\mathscr{E}$, where $t : \mu D \longrightarrow A$ is equal to $\mu^A(d)$. Then $\mu^C(m)$ is equipped with a section $s' : C \longrightarrow \mu F$ induced by r and the pullback

$$\begin{array}{ccc} \mu F & \longrightarrow & \mu D \\ {\scriptstyle \mu^C(m)}\downarrow & \lrcorner & \downarrow{\scriptstyle t} \\ C & \xrightarrow{tr} & A \end{array}$$

in $\mathscr{S}$. (This square is a pullback because μ is a Cartesian functor.) There is a morphism $(m, s') \longrightarrow (z, s)$ in $\mathit{Display}_\mu$, which is a counit of the D-coreflection composed with the Cartesian lifting of (k, α).

Definition 2.4 The D-coreflection of the Cartesian lifting along

$$\mathbf{H}_\mu \longrightarrow \mathscr{E}/e^*$$

of a collective epimorphism in $\mathscr{E}/e^*$ and a given object of $\mathit{Display}_\mu$ is called a *μ-cover* of the given object.

We recall that *a generating family* for a geometric morphism $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ is a morphism $G \xrightarrow{g} \psi^* D$ with the property that every $Y \xrightarrow{x} \psi^* E$ can be put in a diagram of the following form, where the right hand square is a pullback.

$$\begin{array}{ccccc} Y & \twoheadleftarrow & P & \longrightarrow & G \\ {\scriptstyle x}\downarrow & & \downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ \psi^* E & \xleftarrow[\psi^* k]{} & \psi^* F & \xrightarrow[\psi^* h]{} & \psi^* D \end{array}$$

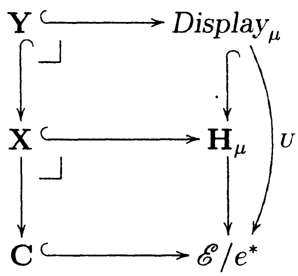
A geometric morphism ψ is said to be *bounded* if it has a generating family.

Suppose that $G \xrightarrow{g} e^* B$ is a generating family for $\mathscr{E}$ over $\mathscr{S}$. Let $\mathbf{C}$ denote the full subcategory of $\mathscr{E}/e^*$ on objects $D \xrightarrow{d} e^* A$ for which there is a pullback

$$\begin{array}{ccc} D & \longrightarrow & G \\ {\scriptstyle d}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ e^* A & \xrightarrow{e^* \alpha} & e^* B \end{array}$$

in $\mathscr{E}$. It follows that $\mathbf{C}$ is equivalent to a small (or internal or representable) $\mathscr{S}$-fibration: $\mathbf{C}$ is said to be *essentially small.* Every object of $\mathscr{E}/e^*$ can be covered by a collective epimorphism with domain in $\mathbf{C}$.

Let $\mathbf{Y}$ and $\mathbf{X}$ denote the full subcategories of $\mathit{Display}_\mu$, respectively $\mathbf{H}_\mu$, on the objects (d, s), respectively (d, S), such that d is an object of $\mathbf{C}$. It follows that $\mathbf{Y}$ and $\mathbf{X}$ are also essentially small. The forgetful functor $\mathbf{Y} \longrightarrow \mathbf{C}$ forms part of the category pullbacks.



Since U is an $\mathscr{S}$-Cartesian functor whose fibers are discrete opfibrations (Prop. 2.1), the left vertical $\mathbf{Y} \longrightarrow \mathbf{C}$ is a discrete opfibration when considered as a functor internal to $\mathscr{S}$. $\mathbf{X} \longrightarrow \mathbf{C}$ is an internal fibration.

Lemma 2.5 *We have the following.*

1. *The D-coreflection restricts to a right adjoint* $\mathbf{X} \longrightarrow \mathbf{Y}$. *In particular, a μ-cover associated with a collective epimorphism of* $\mathbf{C}$ *lies in* $\mathbf{Y}$.

2. *A μ-cover of an object of* $\mathbf{Y}$ *may be refined by a μ-cover associated with a collective epimorphism in* $\mathbf{C}$.

Proof. 1. The D-coreflection is defined by a pullback in $\mathscr{E}$ along a morphism $e^*A \longrightarrow e^*B$. Such a pullback of an object of $\mathbf{C}$ is again an object of $\mathbf{C}$. Therefore, the D-coreflection of an object of $\mathbf{X}$ is an object of $\mathbf{Y}$.

2. The domain of the collective epimorphism associated with the given μ-cover may be covered by another collective epimorphism whose domain is an object of $\mathbf{C}$. The composite of the two collective epimorphisms therefore lies in $\mathbf{C}$ because $\mathbf{C}$ is full. The μ-cover associated with the composite refines the one with which we started. □

The complete spread $\psi : \mathcal{Y} \longrightarrow \mathcal{E}$ associated with μ is the topos pullback

$$\begin{array}{cccccc} & \mathcal{Y} & \rightarrowtail & P(\mathbf{Y}) & \\ & \downarrow & & \downarrow & \\ \psi & \mathcal{E}^{H^{\mathrm{op}}} & \rightarrowtail & P(\mathbf{X}) & \gamma \\ & \downarrow & & \downarrow & \\ & \mathcal{E} & \rightarrowtail & P(\mathbf{C}) & \end{array} \tag{3}$$

induced by the functors $\mathbf{Y} \longrightarrow \mathbf{X} \longrightarrow \mathbf{C}$. The topos $\mathcal{Y}$ is locally connected ($y_! \dashv y^*$), and we recover μ as $y_! \cdot \psi^*$. Hence, $\mu_* \cong \psi_* \cdot y^*$.

Remark 2.6 ψ is localic, and $\mathcal{Y}$ may be considered as $Sh_{\mathcal{E}}(X)$, for the locale X in $\mathcal{E}$ whose frame is $\mathcal{O}(X) = \psi_*(\Omega_{\mathcal{Y}})$. This frame may be described as a sheaf on $\mathbf{C}$: if C is an object of $\mathbf{C}$, then

$$\mathcal{O}(X)(C) = \{ \text{ subobjects of } \gamma^*(C) \text{ that are closed for } \mu\text{-covers } \} ,$$

where $\gamma^*(C)(C', s') = \mathbf{C}(C', C)$. There is a canonical order preserving map

$$\psi_*(\tau) : H \longrightarrow \mathcal{O}(X)$$

in $\mathcal{E}$, where $\tau : y^*(\Omega_{\mathcal{S}}) \rightarrowtail \Omega_{\mathcal{Y}}$ classifies $y^*(\top_{\mathcal{S}})$ in $\mathcal{Y}$ (τ is a subobject as y is subopen). We may describe $\psi_*(\tau)$ explicitly. For any object C, a subobject $S \rightarrowtail \mu C$ defines a subobject $T \rightarrowtail \gamma^*(C)$ that is closed for μ-covers:

$$T(C', s') = \{C' \xrightarrow{m} C \mid \mu(m)(s') \in S\} .$$

One of our goals is to describe in $\mathcal{E}$ the covering families $\{h_\alpha \leq h\}$ in H that give $\mathcal{Y}$ as $Sh_{\mathcal{E}}(H)$.

The following result is due to Peter Johnstone, called C3.3.14 in the Elephant. (Our proof of Theorem 8.3 depends on this result.)

Theorem 2.7 *The pullback topology along a geometric morphism $\gamma : \mathcal{Y} \longrightarrow \mathcal{X}$ of a topology J in $\mathcal{X}$ is given by the upclosure of the image-object K in the diagram:*

$$\begin{array}{ccc} \gamma^* J & \twoheadrightarrow & K \\ \downarrow & & \downarrow \\ \gamma^* \Omega_{\mathcal{X}} & \xrightarrow{\chi} & \Omega_{\mathcal{Y}} \end{array}$$

where χ *classifies* $\gamma^*\top$.

We have a presheaf J on $\mathbf{C}$ such that

$$J(d) = \{\text{sieves on } d \text{ engendered by a collective epimorphism in } \mathbf{C}\} \ ,$$

where $D \xrightarrow{d} e^*A$ is an object of $\mathbf{C}$. Then J is a topology in $P(\mathbf{C})$ whose sheaf subtopos is $\mathscr{E}$: $(\mathbf{C}, J)$ is a site for $\mathscr{E}$. Define a presheaf K on $\mathbf{Y}$ such that

$$K(d,s) = \{ \text{ sieves on } (d,s) \text{ engendered by a } \mu\text{-cover associated with a collective epimorphism in } \mathbf{C} \ \}.$$

We have the following.

Proposition 2.8 *The above presheaf* K *is isomorphic to the object also denoted* K *in 2.7 for the geometric morphism* γ *in (3). The topology in* $\mathbf{Y}$ *giving the complete spread topos* $\mathscr{Y}$ *is therefore equal to the upclosure of* K.

Proof. By virtue of the D-coreflection (Lemma 2.5) we may equivalently describe the above presheaf K as

$$K(d,s) = \{ \text{ sieves on } (d,s) \text{ engendered by the Cartesian lifting to } \mathbf{H}_\mu \text{ of a collective epimorphism in } \mathbf{C} \ \}.$$

This presheaf is the K in 2.7 for the topos pullback (3). □

3 The category CAT_ψ of ψ-families

We work in what we shall call *the category of* ψ*-families* associated with a geometric morphism over a base topos $\mathscr{S}$.

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\ \psi\ } & \mathscr{E} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle e} & \\ & \mathscr{S} & \end{array}$$

Usually we assume that the domain topos $\mathscr{F}$ is *locally connected*, in the sense that there is a left adjoint $f_! \dashv f^*$ over $\mathscr{S}$. A ψ-family (or just family) is a pair $(D \xrightarrow{d} e^*A, X \longrightarrow \psi^*D)$, which we usually depict as a diagram

$$X \longrightarrow \psi^*D \xrightarrow{\psi^*d} f^*A \tag{4}$$

in $\mathscr{F}$. A morphism of such families is a triple of morphisms $X \longrightarrow Y$ in $\mathscr{F}$, $D \longrightarrow E$ in $\mathscr{E}$, and $A \longrightarrow B$ in $\mathscr{S}$ making the obvious squares commute. We denote the category of ψ-families by CAT_ψ.

CAT_ψ is the total category of an $\mathscr{S}$-fibration $CAT_\psi \longrightarrow \mathscr{S}$. Briefly, if $P_\psi : \mathscr{F}/\psi^* \longrightarrow \mathscr{E}$ denotes the fibration corresponding to the geometric morphism ψ, then CAT_ψ arises from an $\mathscr{E}$-fibration $FAM(P_\psi)$ by change of base along e^*. This is described in the following diagram of category pullbacks.

$$\begin{array}{ccccc}
CAT_\psi & \longrightarrow & FAM(P_\psi) & \longrightarrow & \mathscr{F}/\psi^* \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle P_\psi} \\
\mathscr{E}/e^* & \longrightarrow & \mathscr{E}^2 & \xrightarrow{\partial_0} & \mathscr{E} \\
\downarrow{\scriptstyle P_e} & & \downarrow{\scriptstyle \partial_1} & & \\
\mathscr{S} & \xrightarrow{e^*} & \mathscr{E} & &
\end{array} \tag{5}$$

If $\mathscr{E}$ is locally connected, then the top horizontal $CAT_\psi \longrightarrow \mathscr{F}/\psi^*$ has a full and faithful left adjoint.

We mention specially the functor $CAT_\psi \longrightarrow \mathscr{E}/e^*$, depicted vertically in (5), which associates with an object (4) the object d. This functor is a fibration. It has a right adjoint that associates with an object d the object $\psi^*D \xrightarrow{1} \psi^*D \longrightarrow f^*A$.

We also have a functor, *not* depicted in (5),

$$\begin{array}{ccc}
CAT_\psi & \longrightarrow & \mathscr{F}/f^* \\
& \searrow \quad \swarrow{\scriptstyle P_f} & \\
& \mathscr{S} &
\end{array}$$

that forgets ψ^*D in an object (4). This functor has a full and faithful right adjoint that associates with $X \longrightarrow f^*A$ the object $X \longrightarrow \psi^*e^*A \cong f^*A$. Thus, $\mathscr{F}/f^*$ is a full reflective subcategory of CAT_ψ.

4 Families of components: FC_ψ

We recall some basic notions from topos theory that we need [1, 6, 7]. Let $\mathscr{F} \xrightarrow{f} \mathscr{S}$ denote a geometric morphism, with $f^* \dashv f_*$.

Definition 4.1 A morphism $X \xrightarrow{m} Y$ in $\mathscr{F}$ is said to be *definable* if it can be put in a pullback square as follows.

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ \big\downarrow & \lrcorner & \big\downarrow \\ f^*A & \xrightarrow{f^*\alpha} & f^*B \end{array}$$

A *definable subobject* is a monomorphism that is definable.

Proposition 4.2 *We have the following.*

1. *Definable morphisms are pullback stable.*
2. *A definable subobject is defined by a monomorphism in* $\mathscr{S}$.
3. *The inverse image functor of a geometric morphism preserves definable morphisms.*
4. *If* f *is locally connected* ($f_! \dashv f^*$), *then* m *is definable iff the adjunction square*

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ \big\downarrow & & \big\downarrow \\ f^*f_!X & \xrightarrow{f^*f_!m} & f^*f_!Y \end{array}$$

   is a pullback.

5. *In a locally connected topos, definable morphisms compose, and if* $n \cdot m$ *and* n *are definable morphisms, then so is* m (by the above).

Let $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ denote a geometric morphism over a base topos $\mathscr{S}$.

Definition 4.3 A family $X \longrightarrow \psi^* D \xrightarrow{\psi^* d} f^* A$ is *a definable family* if $X \longrightarrow \psi^* D$ is a definable subobject. Let FD_ψ denote the full subcategory of CAT_ψ on the definable families.

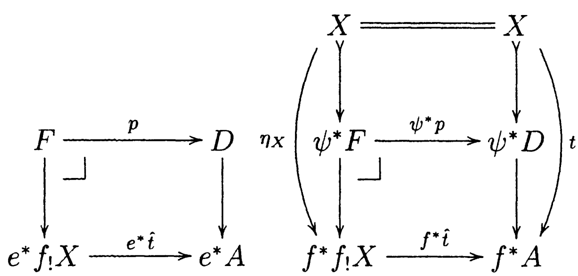
$FD_\psi \longrightarrow \mathscr{E}/e^*$ is a fibration with a right adjoint. The inclusion functor $FD_\psi \hookrightarrow CAT_\psi$ is Cartesian over $\mathscr{E}/e^*$ because definable morphisms are pullback stable.

Definition 4.4 When $\mathscr{F}$ is locally connected, we may consider what we shall call *a family of components*: this is a definable family with the property that the transpose $f_! X \longrightarrow A$ under $f_! \dashv f^*$ is an isomorphism. Let FC_ψ denote the full subcategory of FD_ψ on these families, for $\mathscr{F}$ locally connected.

Informally speaking, a family of components has the property that there is exactly one component of X in every fiber $d^{-1}(a)$. For instance, if $A = 1$, then $X \rightarrowtail \psi^* D \xrightarrow{\psi^* d} f^* A$ is a family of components just when X is connected.

Remark 4.5 A family $X \xrightarrow{m} \psi^* D \xrightarrow{\psi^* d} f^* A$ is a family of components iff m is a definable morphism and the transpose $f_! X \longrightarrow A$ is an isomorphism. Indeed, if m is definable and the transpose is an isomorphism, then m is a subobject as it is a pullback of the (split) monomorphism $f^* f_!(m)$.

Proposition 4.6 *The inclusion $FC_\psi \hookrightarrow FD_\psi$ is $\mathscr{S}$-Cartesian. Moreover, FC_ψ is coreflective in FD_ψ* (but this right adjoint is not Cartesian)*: the coreflection is formed by first considering the following pullback in $\mathscr{E}$ and then lifting back to $\mathscr{F}$ under ψ^*. Let $\hat{t}$ denote the transpose under $f_! \dashv f^*$ of a definable family t, below right.*



The left vertical $X \rightarrowtail \psi^* F \longrightarrow f^* f_! X$ *is then the coreflection, which we call* the FC-coreflection. η *denotes the unit of* $f_! \dashv f^*$.

Proof. FC_ψ is $\mathscr{S}$-Cartesian in FD_ψ because the transpose under $f_! \dashv f^*$ of a pullback square is a pullback. The coreflection of a definable family is again definable because evidently the morphism p, and hence $\psi^* p$, is definable (Prop. 4.2). □

5 The display category of ψ

Let $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ have locally connected domain. Let *Display*$_\psi$ denote *Display*$_{f_!\psi^*}$, as defined in §2. For instance, an object of *Display*$_\psi$ is a pair of morphisms

$$(D \xrightarrow{d} e^* A, A \xrightarrow{s} f_! \psi^* D) = (d, s) \; ,$$

where s is a section of the transpose $f_! \psi^* D \xrightarrow{t} A$ of $\psi^* D \xrightarrow{\psi^* d} f^* A$: $t \cdot s = 1_A$. Similarly, we have $\mathbf{H}_\psi$.

For the reader's convenience we again describe the μ-cover of an object (z, s) of *Display*$_\psi$ associated with a collective epimorphism

$$\begin{array}{ccc} D & \xrightarrow{x} & E \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle z} \\ e^* A & \xrightarrow{e^* \alpha} & e^* B \end{array}$$

in $\mathscr{E}/e^*$. We form the pullbacks

$$\begin{array}{ccc} C & \longrightarrow & B \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ f_! \psi^* D & \xrightarrow{f_! \psi^* x} & f_! \psi^* E \end{array} \qquad \begin{array}{ccc} F & \longrightarrow & D \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle d} \\ e^* C & \xrightarrow{e^*(tr)} & e^* A \end{array}$$

in $\mathscr{S}$ and $\mathscr{E}$, where $t : f_! \psi^* D \longrightarrow A$ is the transpose of $\psi^* d$. Then m is paired with a section $s' : C \longrightarrow f_! \psi^* F$ induced by r and the transpose pullback

$$\begin{array}{ccc} f_! \psi^* F & \longrightarrow & f_! \psi^* D \\ \downarrow & & \downarrow{\scriptstyle t} \\ C & \xrightarrow{tr} & A \end{array}$$

in $\mathscr{S}$. We have a morphism $(m, s') \longrightarrow (z, s)$ that is the μ-cover associated with (x, α).

Proposition 5.1 *There are functors V and V'*

$$\begin{array}{ccc} \mathit{Display}_\psi & \xrightarrow{\ V\ } & FC_\psi \\ \downarrow & & \downarrow \\ \mathbf{H}_\psi & \xrightarrow{\ V'\ } & FD_\psi \\ & \searrow \quad \swarrow & \\ & \mathscr{E}/e^* & \end{array}$$

defined as follows. $V(d, s)$ is the following pullback.

$$\begin{array}{ccccc} V(d,s) & \xrightarrow{\ m\ } & \psi^* D & & \\ \downarrow & & \downarrow \eta & \searrow^{\psi^* d} & \\ f^*A & \xrightarrow{\ f^*s\ } & f^* f_!(\psi^* D) & \xrightarrow{\ f^*t\ } & f^*A \end{array} \qquad (6)$$

(Really $V(d, s)$ is the top row $V(d,s) \xrightarrow{m} \psi^* D \xrightarrow{\psi^* d} f^*A$.) *$V'(d, S)$ is the definable subobject of $\psi^* D$ with characteristic morphism $\psi^* D \longrightarrow f^*\Omega_{\mathscr{S}}$ transposed from the characteristic morphism of $S \rightarrowtail f_!\psi^* D$.*

$V(d, s)$ is a family of components, and both V and V' are equivalences. In particular, $FC_\psi \longrightarrow \mathscr{E}/e^*$ is $\mathscr{S}$-Cartesian with a right adjoint, and its fibers are discrete opfibrations (Prop. 2.1).

Proof. In diagram (6) the composite morphism $\psi^* d \cdot m$ is equal to the left vertical since the bottom horizontal is the identity on f^*A. Thus the transpose of $V(d, s)$ under $f_! \dashv f^*$ is an isomorphism because it is equal to the left vertical of the transpose pullback.

$$\begin{array}{ccc} f_! V(d,s) & \xrightarrow{\ f_! m\ } & f_!(\psi^* D) \\ \downarrow & & \downarrow 1 \\ A & \xrightarrow{\ s\ } & f_!(\psi^* D) \end{array}$$

It follows easily that V is an equivalence. V' is also obviously an equivalence and it is not difficult to check that the V, V'-diagram commutes. □

The FC-coreflection and D-coreflection are thus identified under the V, V' equivalence.

6 Definable completeness

Let $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ be a geometric morphism over $\mathscr{S}$. We shall sometimes denote a typical object $Y \longrightarrow \psi^*E \xrightarrow{\psi^*x} f^*B$ of CAT_ψ just by Y.

Definition 6.1 *A ψ-cover of Y* is a commutative diagram in $\mathscr{F}$ of the following form.

$$\begin{array}{ccc} X & \longrightarrow\!\!\!\!\!\rightarrow & Y \\ \downarrow & & \downarrow \\ \psi^*D & \xrightarrow{\psi^*h} & \psi^*E \\ \downarrow & & \downarrow \\ f^*A & \xrightarrow{f^*\alpha} & f^*B \end{array}$$

If the top square in a ψ-cover is a pullback, then we say that the ψ-cover is *Cartesian*: we call it *the Cartesian ψ-cover of Y associated with the morphism (h, α) of $\mathscr{E}/e^*$.*

Remark 6.2 When $\mathscr{F}$ is locally connected we have the following.

1. The counit of the FC-coreflection is a ψ-cover.

2. If the codomain object of a ψ-cover is a family of components, then the morphism $A \xrightarrow{\alpha} B$ from $\mathscr{S}$ must be an epimorphism. This follows by transposing under $f_! \dashv f^*$ the ψ-cover to $\mathscr{S}$.

The domain object of a Cartesian ψ-cover of a family of components must be a definable family, although it may not be a family of components. However, we may always consider the FC-coreflection of the domain object. The FC-coreflection produces the following diagram.

$$\begin{array}{ccccc}
X & = & X & \twoheadrightarrow & Y \\
\downarrow & & \downarrow & \lrcorner & \downarrow \\
\psi^* F & \xrightarrow{\psi^* p} & \psi^* D & \xrightarrow{\psi^* k} & \psi^* E \\
\downarrow & \lrcorner & \downarrow & & \downarrow \\
f^* f_! X & \xrightarrow{f^* \hat{t}} & f^* A & \xrightarrow{f^* \alpha} & f^* B
\end{array}$$

The upper right and lower left squares are pullbacks. The outer square is a morphism in FC_ψ.

Proposition 6.3 *A morphism in FC_ψ corresponds under the equivalence V to a μ-cover iff it is the FC-coreflection of the Cartesian ψ-cover of a family of components $Y \rightarrowtail \psi^* E \longrightarrow f^* B$ associated with a collective epimorphism (k, α) of $\mathscr{E}/e^*$.*

Proof. Use Proposition 5.1 and the description of μ-covers in §5. □

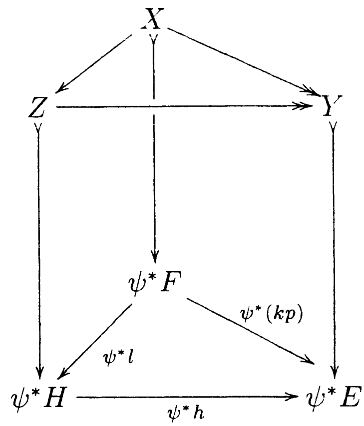
Remark 6.4 Thus, we may speak of μ-covers in FC_ψ. A μ-cover in FC_ψ is a diagram of the following kind, where (k, α) is a collective epimorphism.

$$\begin{array}{ccc}
X & \twoheadrightarrow & Y \\
\downarrow & & \downarrow \\
\psi^* F & \xrightarrow{\psi^*(kp)} & \psi^* E \\
\downarrow & & \downarrow \\
f^* f_! X & \xrightarrow{f^*(\alpha \hat{t})} & f^* B
\end{array}$$

A μ-cover in FC_ψ is a ψ-cover.

Definition 6.5 Let $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ have locally connected domain. We say ψ is *definably complete* if in FC_ψ every ψ-cover can be refined by a

μ-cover. The prism diagram



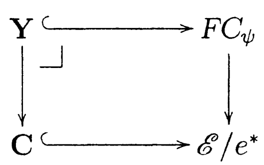
depicts such a refinement, where the front face of the prism is a typical ψ-cover, and the refining back right face is a μ-cover, associated with a collective epimorphism (k, α) in $\mathscr{E}/e^*$. We have not depicted the $\mathscr{S}$-fibering data in the above prism.

We allow for reindexing over $\mathscr{S}$: ψ is definably complete if there is $\alpha : B' \twoheadrightarrow B$ in $\mathscr{S}$ such that α^* of the given ψ-cover is refined by a μ-cover over B'.

7 Comprehensive factorization revisited

In order to state and prove Theorem 8.3 we must revisit the comprehensive factorization of a geometric morphism, but now from a fibrational point of view. This section is directly related to §2.

Suppose that the codomain topos $\mathscr{E}$ of $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ is bounded, and let $\mathbf{C}$ denote the essentially small full subcategory of $\mathscr{E}/e^*$ determined by a generating family for $\mathscr{E}$, as in §2. Let $\mathbf{Y}$ denote the category pullback.



The right vertical is $\mathscr{S}$-Cartesian whose fibers are discrete opfibrations (Prop. 5.1). Therefore, the left vertical is a discrete opfibration internal to $\mathscr{S}$.

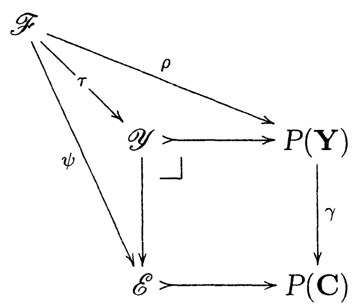
We rephrase Lemma 2.5.

Lemma 7.1 *We have the following.*

1. *If $D \longrightarrow e^*A$ is an object of $\mathbf{C}$, then the FC-coreflection of a definable family $X \rightarrowtail \psi^*D \longrightarrow f^*A$ is an object of $\mathbf{Y}$. In particular, a μ-cover associated with a collective epimorphism of $\mathbf{C}$ lies in $\mathbf{Y}$.*

2. *A μ-cover of a family of components in $\mathbf{Y}$ may be refined by a μ-cover associated with a collective epimorphism in $\mathbf{C}$.*

The comprehensive factorization of ψ may then be defined by the following topos pullback diagram (over $\mathscr{S}$).



(7)

The discrete opfibration $\mathbf{Y} \longrightarrow \mathbf{C}$ induces the essential geometric morphism γ. The geometric morphism ρ comes from the functor composite

$$\mathbf{Y} \longrightarrow FC_\psi \longrightarrow CAT_\psi \longrightarrow \mathscr{F}/f^* \tag{8}$$

that sends an object $X \rightarrowtail \psi^*D \xrightarrow{\psi^*d} f^*A$ of $\mathbf{Y}$ to the composite $X \longrightarrow f^*A$. This $\mathscr{S}$-Cartesian functor is flat, so that it corresponds to the inverse image functor of a geometric morphism ρ.

It is known that the above factorization of ψ is essentially unique and does not depend on the generating family chosen for the codomain topos $\mathscr{E}$.

We recall the following definition.

Definition 7.2 The above factorization of ψ (for which $\mathscr{F}$ is locally connected and $\mathscr{E}$ is bounded) is called its *comprehensive factorization*: the first factor τ is called its *pure factor*, and the second its *complete spread factor*. A geometric morphism is said to be a *complete spread* if its pure factor is an equivalence.

8 Main theorem

We have $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ over $\mathscr{S}$ such that $\mathscr{F}$ is locally connected and $\mathscr{E}$ is bounded. As always, $\mathbf{C}$ denotes the full subfibration of P_e for a generating family for $\mathscr{E}$ over $\mathscr{S}$. We have a presheaf J on $\mathbf{C}$ such that

$$J(d) = \{\text{sieves on } d \text{ engendered by a collective epimorphism in } \mathbf{C}\} ,$$

where $D \xrightarrow{d} e^*A$ is an object of $\mathbf{C}$. Then J is a topology in $P(\mathbf{C})$ whose sheaf subtopos is $\mathscr{E}$: $(\mathbf{C}, J)$ is a site for $\mathscr{E}$.

Define a presheaf $\widehat{J}$ on $\mathbf{Y}$ such that

$$\widehat{J}(Y) = \{\text{sieves on } Y \text{ engendered by a } \psi\text{-cover in } \mathbf{Y}\} ,$$

where Y denotes a typical object of $\mathbf{Y}$. Then $\widehat{J}$ is subpresheaf of Ω in $P(\mathbf{Y})$.

Proposition 8.1 *$\widehat{J}$ is a topology in $P(\mathbf{Y})$: its topos of sheaves is the subtopos of $P(\mathbf{Y})$ given by the image of ρ in diagram* (7).

Remark 8.2 As in §2, we have a presheaf $K \rightarrowtail \widehat{J}$ on $\mathbf{Y}$ such that

$$K(Y) = \{ \text{ sieves on } Y \text{ engendered by a } \mu\text{-cover associated with a collective epimorphism in } \mathbf{C} \} .$$

By Proposition 2.8, this K is isomorphic to the K in Johnstone's C3.3.14 (herein Theorem 2.7) for the geometric morphism γ in (7). Hence, if the pure factor τ of ψ is a surjection, then $\widehat{J}$ equals the up-closure of K.

Our main result is the following counterpart of Proposition 9.2.

Theorem 8.3 *Let $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ have locally connected domain and bounded codomain. Then ψ is definably complete iff the pure factor of the comprehensive factorization of ψ is a surjection.*

Proof. Suppose that ψ is definably complete. Consider the comprehensive factorization in terms of a chosen $\mathbf{Y} \longrightarrow \mathbf{C}$. Let S denote a $\widehat{J}$-sieve on a family of components in $\mathbf{Y}$: S is engendered by a ψ-cover in $\mathbf{Y}$. By definable completeness we may refine this ψ-cover by a μ-cover, whose domain object is of course a family of components. The collective epimorphism associated with this μ-cover may not be in $\mathbf{C}$; however, we may apply Lemma 7.1, 2, and therefore refine the first μ-cover by a μ-cover whose associated collective epimorphism lies in $\mathbf{C}$. Hence, we may find a K-sieve that is contained in S, which shows that $\widehat{J}$ is contained in the upclosure of K. Therefore $\widehat{J}$ is contained in (hence equal to) the topology generated by K. Thus, the image topos of ρ coincides with $\mathscr{Y}$. In other words, the pure factor of ψ is a surjection.

Conversely, suppose that the pure factor of ψ is a surjection. Suppose we are given a ψ-cover in FC_ψ. We may choose the generating $\mathbf{C}$ such that the collective epimorphism associated with the given ψ-cover lies in $\mathbf{C}$. By definition of the $\mathbf{Y}$ associated with this $\mathbf{C}$, the ψ-cover lies in $\mathbf{Y}$. By Remark 8.2, for this $\mathbf{C}$ and $\mathbf{Y}$, $\widehat{J}$ equals the upclosure of K. Thus, the sieve engendered by the given ψ-cover contains a K-sieve. This K-sieve is engendered by a μ-cover associated with a collective epimorphism in $\mathbf{C}$. This μ-cover therefore refines the given ψ-cover. This shows that ψ is definably complete. □

Remark 8.4 Only the argument for necessity of the completeness condition (second paragraph above) depends on Johnstone's C3.3.14.

Example 8.5 We note some simple examples.

1. Pure surjections may be readily found. Any connected geometric morphism is a pure surjection. A simple example from topology of a pure surjection is a projection from a punctured plane to the real line.

2. It may strike the reader that definable completeness is like a compactness condition in the sense that an arbitrary cover can be

refined by a cover from a preferred class. This is indeed so, but definable completeness is broader than compactness since any locally connected topos $\mathscr{F} \longrightarrow \mathscr{S}$ is definably complete. This can be seen directly from the definition of definably complete, but observe that the pure factor is the connected, hence surjective, geometric morphism $\mathscr{F} \longrightarrow \mathscr{S}/f_!(1)$. This example is neither a spread nor pure.

3. The identity geometric morphism $\mathscr{E} \longrightarrow \mathscr{E}$ on any locally connected topos is a complete spread. Hence, it is definably complete.

9 Spreads revisited

Our purpose in this section is to prove Theorem 9.3. A spread is defined in [2] as a geometric morphism that has a definable family that generates at 1. It turns out that this is not equivalent to the following stronger notion.

Definition 9.1 A geometric morphism $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ over $\mathscr{S}$ is said to be *a spread* if ψ has a definable generating family.

Proposition 9.2 *Suppose that* $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ *has locally connected domain and bounded codomain. Then the following are equivalent:*

1. *ψ is a spread.*

2. *For any generating family chosen for $\mathscr{E}$, every object of $\mathscr{F}/f^*$ can be covered by an object in the image of the functor (8).*

3. *The pure factor τ of ψ is an inclusion.*

Proof. Assume that ψ is a spread. We refer to diagram (7). As always, $\mathbf{C}$ and $\mathbf{Y}$ denote the essentially small categories associated with a generating family for $\mathscr{E}$. An object $X \longrightarrow f^*A \cong \psi^*e^*A$ may be put in a diagram

$$\begin{array}{ccc} P & \twoheadrightarrow & X \\ \downarrow & & \downarrow \\ \psi^*E & \longrightarrow & \psi^*e^*A \end{array}$$

where $P \longrightarrow \psi^* E$ is definable. Now cover $E \longrightarrow e^* A$ by an object of $\mathbf{C}$, and consider the pullback below right.

$$\begin{array}{ccc} F & \xrightarrow{\;k\;} & E \\ \downarrow & & \downarrow \\ e^*B & \longrightarrow & e^*A \end{array} \qquad \begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & \lrcorner & \downarrow \\ \psi^*F & \xrightarrow{\;\psi^*k\;} & \psi^*E \end{array}$$

We have thus covered $X \longrightarrow f^*A$ by $Q \longrightarrow \psi^*F \longrightarrow f^*B$, where $Q \longrightarrow \psi^*F$ is definable, and $F \longrightarrow e^*B$ is an object of $\mathbf{C}$. But then the FC-coreflection of Q is an object of $\mathbf{Y}$ that covers $X \longrightarrow f^*A$. (Even if $Q \longrightarrow \psi^*F$ is not a subobject, by Remark 4.5 its FC-coreflection is a family of components.)

If every object of $\mathscr{F}/f^*$ can be covered by an object in the image of the functor (8), then ρ is an inclusion, so that τ is also.

Finally, suppose that τ and hence ρ is an inclusion. The essential geometric morphism associated with a discrete opfibration, in this case γ, is a spread. The composite of an inclusion followed by a spread is a spread, so that $\gamma \cdot \rho$ is a spread. Then ψ is a spread because any left factor of a spread is a spread. □

Theorem 8.3 and Proposition 9.2 give us the following.

Theorem 9.3 *A geometric morphism with locally connected domain and bounded codomain is a complete spread* (Def. 7.2) *iff it is definable complete and a spread* (Defs. 6.5 and 9.1).

Proof. A geometric morphism is a complete spread iff its pure factor is an equivalence iff the pure factor is an inclusion and a surjection iff the geometric morphism is a spread and definably complete. □

References

[1] M. Barr and R. Paré. Molecular toposes. *J. Pure Appl. Alg.*, 17:127–152, 1980.

[2] M. Bunge and J. Funk. Spreads and the symmetric topos. *J. Pure Appl. Alg.*, 113:1–38, 1996.

[3] M. Bunge, J. Funk, M. Jibladze, and T. Streicher. Distribution algebras and duality. *Advances in Mathematics*, 156:133–155, 2000.

[4] R. H. Fox. Covering spaces with singularities. In R. H. Fox et al., editors, *Algebraic Geometry and Topology: A Symposium in Honor of S. Lefschetz*, pages 243 257. Princeton University Press, Princeton, 1957.

[5] J. Funk. The display locale of a cosheaf. *Cahiers de Top. et Géom. Diff. Catégoriques*, 36(1):53–93, 1995.

[6] P. T. Johnstone. *Topos Theory.* Academic Press, Inc., London, 1977.

[7] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium.* Clarendon Press, Oxford, 2002.

[8] F. W. Lawvere. Intensive and extensive quantities. Notes for the lectures given at the workshop on Categorical Methods in Geometry, Aarhus, 1983.

[9] J.-L. Moens. *Caracterisation des topos de faisceaux sur un site interne à un topos.* PhD thesis, Univ. Louvain-la-Neuve, 1982.

[10] Thomas Streicher. Fibered categories à la Jean Benabou. Unpublished notes, 2003.

McGill University, Montreal, Canada.
University of Regina, Regina, Canada.
Georgian Academy of Sciences, Tbilisi, Georgia.
TU Darmstadt, Darmstadt, Germany.