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ORDERED GROUPOIDS AND ETENDUES

by Mark V. LAWSON and Benjamin STEINBERG

Résumé

Kock et Moerdijk ont montré que chaque étendue est engendrée par un site dont tout morphisme est monic. Dans cet article nous donnons une caractérisation alternative des étendues en termes de groupoïdes ordonnés. Spécifiquement, nous définissons un site d'Ehresmann comme étant un groupoïde ordonné muni de ce que nous nommons une 'topologie d'Ehresmann' — c'est essentiellement une famille d'idéaux pour l'ordre, stable par conjugaison — et de cette façon nous pouvons définir la notion de faisceau sur un site d'Ehresmann. Notre résultat principal est que chaque étendue est équivalente à la catégorie des faisceaux sur un site d'Ehresmann approprié.

1 Introduction

Kock and Moerdijk proved that each étendue is generated by a site in which every morphism is monic. In this paper we provide an alternative characterisation of étendues in terms of ordered groupoids. Specifically, we define an Ehresmann site to be an ordered groupoid equipped with what we term an 'Ehresmann topology' — this is essentially a family of order ideals closed under conjugation — and in this way we are able to define the notion of a sheaf on an Ehresmann site. Our main result is that each étendue is equivalent to the category of sheaves on a suitable Ehresman site. In this section, we outline the background to this equivalence.

At an informal level, a topos can be regarded as a 'generalised space' [12], whereas an étendue is a topos $\mathcal{T}$ which is 'locally like a space'. More precisely, an étendue is a topos containing an object $E \in \mathcal{T}$ such that the unique map from E to the terminal object of $\mathcal{T}$ is an epimorphism and such that the slice topos $\mathcal{T}/E$ is equivalent to the category of sheaves on a locale [1]. Kock and Moerdijk [5] proved that every étendue can be presented by means of a site whose morphisms are monic, what we call a *left cancellative site* (see Section 2.3 for the full definition). Conversely, by Theorem 1.5 and Proposition 1.3 of [14], Rosenthal proved that the category of sheaves on a left cancellative site is an étendue.

This work highlights left cancellative categories and their presheaves. Such structures had been studied independently in inverse semigroup theory. In [6], Lausch generalised classical group cohomology to inverse semigroups. Subsequently, Loganathan [10] showed that this cohomology was the same as the cohomology of an associated category. Leech [9] studied this category in more detail and proved that it was left cancellative with extra structure. Now inverse semigroups can be regarded as special kinds of ordered groupoids, a result that goes back to Ehresmann [2], and has more recently been important in inverse semigroup theory [7]. It was therefore natural to generalise Leech's work from inverse semigroups to ordered groupoids. This was done by the first author [8] who obtained a correspondence between ordered groupoids and left cancellative categories. This work is summarised in Section 2.2. Loganathan's work on inverse semigroup cohomology was based on the fact that certain types of actions of an inverse semigroup corresponded to actions of the associated category. This result had been investigated by the second author, who had realised that inverse semigroups acting on presheaves of structures played an important role in inverse semigroup theory (for example, in [13]).

At this point, it was natural to ask if these two strands of work, topos theoretic on the one side, and ordered groupoid theoretic on the other, could be related. This is indeed the case and is the basis of our paper. Specifically, we define what we term an 'Ehresmann topology' on an ordered groupoid. An ordered groupoid equipped with an Ehresmann topology is called an 'Ehresmann site'. We prove two main results:

(1) Each left cancellative site can be constructed from some Ehresmann site (Theorem 3.10).

(2) Each étendue is equivalent to the category of sheaves defined on

some Ehresmann site (Theorem 4.5).

2 Background

In this section, we describe the background results needed to read this paper.

2.1 Ordered groupoids

We shall regard a category C as a a generalisation of a monoid: amongst the morphisms are the distinguished morphisms known as *identities* denoted by C_o. If $x \in C$ then $\mathbf{d}(x)$ is the unique identity such that $x\mathbf{d}(x) = x$ and $\mathbf{r}(x)$ is the unique identity such that $\mathbf{r}(x)x = x$. We usually write $\mathbf{d}(x) \xrightarrow{x} \mathbf{r}(x)$. The product xy is defined if and only if $\mathbf{d}(x) = \mathbf{r}(y)$. The *opposite category to* C is denoted C^{op}. A *left cancellative category* is one in which every morphism is monic. Dually a *right cancellative category* is one in which every morphism is epic. A category which is both left and right cancellative is termed *cancellative.* An element x of a category is said to be *invertible* or to be an *isomorphism* if there is an element y such that $yx = \mathbf{d}(x)$ and $xy = \mathbf{r}(x)$. If such an element y exists it is necessarily unique and is denoted by x^{-1}. If e and f are identities and $e \xrightarrow{x} f$ is an isomorphism then we say that e is *isomorphic to* f. A *groupoid* G is a category in which every element is invertible. The set of isomorphisms Iso(C) in a category C forms a groupoid. A subcategory D of C is said to be *full* if $D_o = C_o$[1]; it is said to be *dense* if each identity in C is isomorphic to one in D. For any undefined terms see [11].

Let $(P, \leq)$ be a poset. A subset $X \subseteq P$ is an *order ideal* if $y \leq x \in X$ implies that $y \in X$. For each $x \in P$ we denote by $x^{\downarrow} = \{y \in P\colon\ y \leq x\}$ the *principal order ideal* determined by x.

An *ordered groupoid* $(G, \leq)$ is a groupoid G equipped with a partial order $\leq$ satisfying the following axioms:

(OG1) If $x \leq y$ then $x^{-1} \leq y^{-1}$.

[1]The term *wide* is often used.

(OG2) If $x \leq y$ and $x' \leq y'$ and the products xx' and yy' are defined then $xx' \leq yy'$.

(OG3) If $e \in G_o$ is such that $e \leq \mathbf{d}(x)$ there exists a unique element $(x \,|\, e) \in G$ such that $(x \,|\, e) \leq x$ and $\mathbf{d}(x \,|\, e) = e$.

(OG3)* If $e \in G_o$ is such that $e \leq \mathbf{r}(x)$ there exists a unique element $(e \,|\, x) \in G$ such that $(e \,|\, x) \leq x$ and $\mathbf{r}(e \,|\, x) = e$.

It can proved that (OG3)* is a consequence of (OG1)–(OG3) (Proposition 4.1.3 of [7]).

An *ordered functor* between ordered groupoids is simply an order-preserving functor. An *ordered embedding* is an injective functor $\theta \colon G \to H$ such that $g \leq g' \Leftrightarrow \theta(g) \leq \theta(g')$.

Let G be an ordered groupoid. A connected component of the underlying groupoid of G is called a *$\mathcal{D}$-class* by analogy with inverse semigroup theory. If $g, h \in G$ then we write $g \leq_{\mathcal{J}} h$ if there exists $g' \in G$ such that $g \,\mathcal{D}\, g' \leq h$. If $e, f \in G_o$ then $e \leq_{\mathcal{J}} f$ means precisely that there is $x \in G$ such that $e \xrightarrow{x} e' \leq f$. In the case G comes from an inverse semigroup then this definition of $\leq_{\mathcal{J}}$ agrees with the usual one (see Proposition 3.2.8 of [7]).

We say that G has *maximal identities* if there is a function $G_o \to G_o$ denoted by $e \mapsto e^\circ$ such that the following two conditions hold:

(1) $e \leq e^\circ$.

(2) If $e \leq i^\circ, j^\circ$ then $i^\circ = j^\circ$.

It follows that the identities e° are maximal.

Let $x, y \in G$ and suppose that in the poset $(G_o, \leq)$, the greatest lower bound $e = \mathbf{d}(x) \wedge \mathbf{r}(y)$ exists. Then define $x \otimes y = (x \,|\, e)(e \,|\, y)$. The partial product $\otimes$, which extends the categorical product in G, is called the *pseudoproduct*. It can be proved that if $x \otimes (y \otimes z)$ and $(x \otimes y) \otimes z$ are both defined then they are equal (Lemma 4.1.6 of [7]).

The theory of ordered groupoids is due to Ehresmann [2]. What we call an 'ordered groupoid' is what Ehresmann termed a 'functorially ordered groupoid'. The terminology we use is now standard in inverse semigroup theory where ordered groupoids find useful applications [7].

2.2 Left cancellative categories and ordered groupoids

The aim of this section is to describe the relationship between left cancellative categories and ordered groupoids. The details may be found in [8]. In particular, we outline the proof that each left cancellative category is equivalent to one constructed from an ordered groupoid.

Let C be a left cancellative category. Put

$$U = \{(a,b) \in C \times C\colon \ \mathbf{d}(a) = \mathbf{d}(b)\}.$$

Define a relation $\sim$ on U as follows:

$$(a,b) \sim (a',b') \Leftrightarrow (a,b) = (a',b')u \text{ for some } u \in \mathrm{Iso}(C)$$

where $(a',b')u = (a'u, b'u)$. Then $\sim$ is an equivalence relation on U. We denote the equivalence class containing (a,b) by $[a,b]$, and the set of equivalence classes by $\mathbf{G}(C)$. Define

$$\mathbf{d}[a,b] = [b,b], \quad \mathbf{r}[a,b] = [a,a] \text{ and } [a,b]^{-1} = [b,a].$$

If $\mathbf{d}[a,b] = \mathbf{r}[c,d]$ define the partial product

$$[a,b][c,d] = [a,du]$$

where $u \in \mathrm{Iso}(C)$ is such that $b = cu$. Finally, define a relation $\leq$ on $\mathbf{G}(C)$ by

$$[a,b] \leq [c,d] \Leftrightarrow (a,b) = (c,d)p \text{ for some } p \in C,$$

where $(c,d)p = (cp,dp)$. Then $\mathbf{G}(C)$ is an ordered groupoid. If $\theta\colon C \to D$ is a functor between two left cancellative categories we may define an ordered functor $\mathbf{G}(\theta)\colon \mathbf{G}(C) \to \mathbf{G}(D)$ by $\mathbf{G}(\theta)([a,b]) = [\theta(a),\theta(b)]$. The proof of the following may be found in [8].

Theorem 2.1. *The construction* $\mathbf{G}$ *is a functor from the category of left cancellative categories and functors to the category of ordered groupoids with maximal identities and ordered functors preserving maximal identities.* □

Remark The identities of $\mathbf{G}(C)$ are $\{[a,a]\colon\ a \in C\}$ and the maximal identities are $\{[e,e]\colon\ e \in C_o\}$. For each $[a,a] \in \mathbf{G}(C)_o$, we have that $[a,a]^\circ = [\mathbf{r}(a), \mathbf{r}(a)]$ is the unique maximal identity above $[a,a]$. Furthermore, $[a,a]\,\mathcal{D}\,[\mathbf{d}(a), \mathbf{d}(a)]$.

Let G be an ordered groupoid. Put

$$\mathbf{C}(G) = \{(e,x) \in G_o \times G\colon\ \mathbf{r}(x) \leq e\}.$$

Define

$$\mathbf{d}(e,x) = (\mathbf{d}(x), \mathbf{d}(x)) \text{ and } \mathbf{r}(e,x) = (e,e)$$

and define a partial product on $\mathbf{C}(G)$ as follows: if $\mathbf{d}(e,x) = \mathbf{r}(f,y)$ then $(e,x)(f,y) = (e, x \otimes y)$, else it is undefined. Then $\mathbf{C}(G)$ is a category. If $\theta\colon\ G \to H$ is an ordered functor between two ordered groupoids, we may define a functor $\mathbf{C}(\theta)\colon\ \mathbf{C}(G) \to \mathbf{C}(H)$ by $\mathbf{C}(\theta)(e,x) = (\theta(e), \theta(x))$. The following is proved in [8].

Theorem 2.2. *The construction* $\mathbf{C}$ *is a functor from the category of ordered groupoids and ordered functors to the category of left cancellative categories and their functors.* □

The identities of $\mathbf{C}(G)$ are $\{(e,e)\colon\ e \in G_o\}$, and the groupoid G is isomorphic to the groupoid

$$\operatorname{Iso}(\mathbf{C}(G)) = \{(\mathbf{r}(x), x)\colon\ x \in G\}$$

under the map $x \mapsto (\mathbf{r}(x), x)$. Put

$$\operatorname{Mono}(\mathbf{C}(G)) = \{(e,f)\colon\ e,f \in G_o,\ f \leq e\}.$$

Then every element of $\mathbf{C}(G)$ can be uniquely factored as a product from

$$\operatorname{Mono}(\mathbf{C}(G))\operatorname{Iso}(\mathbf{C}(C)).$$

The following is proved in [8].

Theorem 2.3. *Let C be a left cancellative category. Then*

$$\iota\colon\ C \to \mathbf{CG}(C) \text{ defined by } \iota(c) = ([\mathbf{r}(c), \mathbf{r}(c)], [c, \mathbf{d}(c)])$$

is an injective functor which embeds C in $\mathbf{CG}(C)$ as a full dense subcategory. In particular, C and $\mathbf{CG}(C)$ are equivalent. □

The above theorem tells us that up to equivalence *every* left cancellative category can be constructed from an ordered groupoid.

2.3 Sites

We recall some standard definitions from topos theory [12].

Let C be a category and e an identity of C. A *sieve S on e in C* is a subset of C satisfying $S \subseteq eC$ and $SC \subseteq S$. Evidently if S is a sieve then in fact $S = SC$. Let S be a sieve on e in C, and let $f \xrightarrow{a} e$. Define

$$a^*S = \{b \in fC:\ ab \in S\}.$$

Then a^*S is a (possibly empty) sieve on f in C. If a is an isomorphism then $a^*S = a^{-1}S$.

A *Grothendieck topology* on C is a function J which assigns to each identity $e \in C$ a collection $J(e)$ of sieves satisfying the following three conditions:

(T1) $eC \in J(e)$.

(T2) If $f \xrightarrow{a} e$ and $S \in J(e)$ then $a^*S \in J(f)$.

(T3) If $S \in J(e)$ and R is any sieve on e such that $a^*R \in J(f)$ for all $f \xrightarrow{a} e \in S$ then $R \in J(e)$.

A *site* is a pair (C, J) consisting of a category C equipped with a Grothendieck topology J. If C is left cancellative then we shall say that (C, J) is a *left cancellative site.*

Let C be a subcategory of C' and let J and J' be Grothendieck topologies on C and C' respectively. For each $e \in C_o$, there are two sets $J(e)$ and $J'(e)$. We write

$$C \cap J'(e) = \{C \cap A:\ A \in J'(e)\}.$$

If $J(e) = C \cap J'(e)$ for each $e \in C_o$, we shall say that the Grothendieck topology J' *extends* the Grothendieck topology J.

The proof of the following is straightforward.

Proposition 2.4. *Let C' be a category containing C as a full dense subcategory.*

(i) *Suppose that J is a Grothendieck topology on C. Then there is exactly one Grothendieck topology J' on C' which extends J.*

(ii) *Suppose that J' is a Grothendieck topology on C'. For each $e \in C_o$ define $J(e) = J'(e) \cap C$. Then J is a Grothendieck topology on C.*

□

We now recall the definition of a sheaf on a site. We refer the reader to [12] for sheaf theory background. Throughout this paper **Set** is the category of small sets. Let C be a category. Then a *presheaf* on C is a functor $F\colon C^{op} \to \mathbf{Set}$. The *category of presheaves*, denoted $\mathbf{Set}^{C^{op}}$, has presheaves as objects and natural transformations as morphisms. If C is a partially ordered set $(P, \leq)$ then a presheaf F over P can be equivalently defined as follows: for each $e \in P$ there is a set $F(e)$, and for each pair $f \geq e$ in P there is a function $\rho^f_e\colon F(f) \to F(e)$, called a *restriction map*, such that for each $e \in P$ we have that ρ^e_e is the identity on $F(e)$ and if $e \geq e' \geq e''$ then $\rho^{e'}_{e''}\rho^e_{e'} = \rho^e_{e''}$ [15].

A presheaf on C may be viewed as a *right action* of C on a suitable set in the following way. Let F be a presheaf on C. Put $\mathcal{A} = \bigsqcup_{e \in C_o} F(e)$, where the notation indicates that we are dealing with a disjoint union. Let $a \in F(e)$ and $f \xrightarrow{g} e$, and define

$$a \cdot g = F(g)(a).$$

One then verifies:

(A1) $a \cdot e = a$.

(A2) $a \cdot (gh) = (a \cdot g) \cdot h$ when $\mathbf{d}(g) = \mathbf{r}(h)$.

(A3) $a \cdot g \in F(f)$ and so $a \mapsto a \cdot g$ is a function from $F(e)$ to $F(f)$.

Right actions arising in this way may be axiomatised as follows [3]. Let C be a category, and let $\mathcal{A}$ be a set equipped with a function $\pi\colon \mathcal{A} \to C_o$ and put $\mathcal{A}(e) = \pi^{-1}(e)$. Let $\mathcal{A} \times C \to \mathcal{A}$ be a partial function where $(a, g) \mapsto a \cdot g$. We suppose that $\exists a \cdot g$ if and only if $\pi(a) = \mathbf{r}(g)$. In addition, the following three axioms hold:

(A1)′ $a \cdot \pi(a) = a$.

(A2)′ $a \cdot (gh) = (a \cdot g) \cdot h$ when $\mathbf{d}(g) = \mathbf{r}(h)$.

(A3)′ $\pi(a \cdot g) = \mathbf{d}(g)$.

Let (C, J) be a site, F a presheaf on C and let $S \in J(e)$. A *matching family for S* of elements of F is a function which assigns to each element $d \xrightarrow{g} e$ in S an element $a_g \in F(d)$ in such a way that

$$a_g \cdot h = a_{gh}$$

for all h in C such that $\mathbf{r}(h) = d$. This definition make sense, because S is a sieve and so $g \in S$ implies that $gh \in S$. An *amalgamation* of such a family is an element $a \in F(e)$ such that

$$a \cdot g = a_g$$

for each $g \in S$. The presheaf F is a *sheaf* if every matching family has a *unique* amalgamation. The category of sheaves on (C, J), denoted by $\mathbf{Sh}(C, J)$, is the full subcategory of $\mathbf{Set}^{C^{op}}$ whose objects are sheaves.

The following result provides the essential link between our work and topos theory. The proof follows from the Comparison Lemma of [5].

Proposition 2.5. *Let C' be a category containing C as a full dense subcategory. Let J be a Grothendieck topology on C, and let J' be the unique extension of J to a Grothendieck topology on C' constructed according to Proposition 2.4. Then the category of sheaves on the site (C, J) is equivalent to the category of sheaves on (C', J').*

□

3 Ehresmann topologies on ordered groupoids

In Section 2.2, we described the correspondence between left cancellative categories and ordered groupoids. In this section, we investigate what happens when the left cancellative category in question is equipped with a Grothendieck topology. The main result is Theorem 3.10.

3.1 Ehresmann sites

Let G be an ordered groupoid. An *Ehresmann topology* on G is an assignment of order ideals $T(e)$ of $e^{\downarrow}$ for each identity e in G satisfying the following three axioms:

(ET1) $e^{\downarrow} \in T(e)$ for each identity e.

(ET2) Let e and f be identities such that $f \leq_{\mathcal{J}} e$. Then for each $x \in G$ such that $f \xrightarrow{x} e' \leq e$ we have that $x^{-1} \otimes A \otimes x \in T(f)$ for each $A \in T(e)$.

(ET3) Let e be an identity, let $A \in T(e)$ and let $B \subseteq e^{\downarrow}$ be an order ideal. Suppose that for each $f \xrightarrow{x} e' \leq e$ where $e' \in A$ we have that $x^{-1} \otimes B \otimes x \in T(f)$. Then $B \in T(e)$.

The following definition and result is technical but useful. Let G be an ordered groupoid with maximal identities in which each $\mathcal{D}$-class contains maximal identities (cf. Theorem 2.1). A *pre-Ehresmann topology* on G is defined in exactly the same way as an Ehresmann topology *except* that the word 'identity' is replaced everywhere by the phrase 'maximal identity'. The three axioms we obtain in this way are labelled (PET1), (PET2) and (PET3) respectively.

The following result shows that on ordered groupoids with maximal identities and in which each $\mathcal{D}$-class contains a maximal identity, there is a bijection between Ehresmann topologies and pre-Ehresmann topologies. We omit the routine proof.

Proposition 3.1. *Let G be an ordered groupoid with maximal identities in which each $\mathcal{D}$-class contains maximal identities.*

(i) *Let T' be a pre-Ehresmann topology on G. For each identity $e \in G_o$ define*

$$T(e) = \{x^{-1} \otimes A \otimes x : \quad \textit{where } e \xrightarrow{x} i^{\circ} \textit{ and } A \in T'(i^{\circ})\}.$$

Then T defines an Ehresmann topology on G. In addition, for each maximal identity $e \in G_o$, we have that $T(e) = T'(e)$.

(ii) *Let S be an Ehresmann topology on G. For each maximal identity $e \in G_o$, define $T'(e) = S(e)$. Then T' is a pre-Ehresmann topology on G. Let T be the Ehresmann topology on G defined by applying the method of* (i) *above to T'. Then $T = S$.* □

3.2 From left cancellative sites to Ehresmann sites and back

Let C be a left cancellative category. We show first that there is a bijection between Grothendieck topologies on C and Ehresmann topologies on $\mathbf{G}(C)$ (Theorem 3.5). The proof of the following is routine and omitted.

Lemma 3.2. *Let C be a left cancellative category and e an identity in C.*

(i) *Let S be a sieve on e in C. Put*

$$[S] = \{[a, a] \colon \ a \in S\}.$$

Then $[S]$ is an order ideal of $[e, e]^{\downarrow}$.

(ii) *Let A be an order ideal in $[e, e]^{\downarrow}$. Put*

$$\|A\| = \{b \in C \colon \ [b, b] \in A\}.$$

Then $\|A\|$ is a sieve on e in C.

The operations $S \mapsto [S]$ and $A \mapsto \|A\|$ are mutually inverse. It follows that there is a bijection between the set of sieves on e in C and the set of order ideals of $[e, e]^{\downarrow}$ in $\mathbf{G}(C)$.

□

The following two lemmas are of a technical character.

Lemma 3.3. *Let C be a left cancellative category. Let e and f be identities in C. For each $f \xrightarrow{a} e$, we have that*

$$[f, f] \xrightarrow{[a, f]} [a, a] \leq [e, e].$$

Conversely, if

$$[f,f] \stackrel{[x,y]}{\to} [b,b] \leq [e,e]$$

then there is $f \stackrel{a}{\to} e$ *such that* $[x,y] = [a,f]$.

Proof. Let $f \stackrel{a}{\to} e$ in C. Then $\mathbf{d}(a) = f$ and so $[a,f]$ is a well-defined element of $\mathbf{G}(C)$. Now $(a,a) = (e,e)a$, and so $[a,a] \leq [e,e]$. It follows that

$$[f,f] \stackrel{[a,f]}{\to} [a,a] \leq [e,e]$$

as required.

Let

$$[f,f] \stackrel{[x,y]}{\to} [b,b] \leq [e,e].$$

Then $\mathbf{d}[x,y] = [f,f]$ and $\mathbf{r}[x,y] = [b,b]$. Thus $y = fu$ and $x = bv$ for some invertible elements $u,v \in C$. Now $[b,b] \leq [e,e]$ implies that $b = eb$, and so $\mathbf{r}(b) = e$. It follows that $[x,y] = [bv,fu] = [bvu^{-1},f]$. Put $a = bvu^{-1}$. Then $f \stackrel{a}{\to} e$, as required. □

Lemma 3.4. *Let* $f \stackrel{a}{\to} e$ *in* C *and let* S *be a sieve on* e *in* C. *Then*

$$[a^*S] = [a,f]^{-1} \otimes [S] \otimes [a,f].$$

Proof. Let $[x,x] \in [a^*S]$. Then $x \in a^*S$ and so $ax \in S$. Thus $[ax,ax] \in [S]$. We now calculate $[f,a] \otimes [ax,ax] \otimes [a,f]$. First $[f,a] \otimes [ax,ax] = [x,ax]$ and $[x,ax] \otimes [a,f] = [x,x]$. It follows that

$$[x,x] \in [a,f]^{-1} \otimes [S] \otimes [a,f].$$

Conversely, let

$$[y,y] \in [a,f]^{-1} \otimes [S] \otimes [a,f].$$

Then $[y,y] = [f,a] \otimes [x,x] \otimes [a,f]$. Observe first that because the pseudoproduct exists, we can assume without loss of generality that $[x,x] \leq [a,a]$. Thus $x = ap$ for some $p \in C$. It follows that $[y,y] = [p,p]$. Hence $y = pu$ for some invertible element $u \in C$. We therefore have that $ay = apu = xu \in S$, since S is a sieve. Thus $y \in a^*S$, as required. □

Theorem 3.5. *Let* C *be a left cancellative category, and let* $\mathbf{G} = \mathbf{G}(C)$ *be the ordered groupoid associated with* C.

(i) *Let J be a Grothendieck topology on C. For each maximal identity $[e,e] \in \mathbf{G}$ define*

$$T'([e,e]) = \{[S] \colon \ S \in J(e)\}.$$

Then T' is a pre-Ehresmann topology on $\mathbf{G}$.

(ii) *Let T' be a pre-Ehresmann topology on $\mathbf{G}$. For each identity $e \in C$ define*

$$J(e) = \{\|A\| \colon \ A \in T'([e,e])\}.$$

Then J is a Grothendieck topology on C.

There is a bijective correspondence between Grothendieck topologies on C and Ehresmann topologies on $\mathbf{G}(C)$.

Proof. (i) If e is an identity in C then $[e,e]$ is a maximal identity in $\mathbf{G}$. Observe that $[e,e] = [f,f]$ if and only if $e = f$. Thus distinct identities in C give rise to distinct maximal identities in $\mathbf{G}$.

By Lemma 3.2, the set $T'([e,e])$ is a collection of order ideals of $[e,e]^{\downarrow}$. We show that the three axioms for a pre-Ehresmann topology hold.

(PET1) holds. By assumption, J satisfies (T1) and so $eC \in T(e)$. It is easy to check that $[eC] = [e,e]^{\downarrow}$. Thus $[e,e]^{\downarrow} \in T'([e,e])$.

(PET2) holds. Let $[f,f]$ and $[e,e]$ be maximal identities such that $[f,f] \leq_{\mathcal{J}} [e,e]$. Let

$$[f,f] \xrightarrow{[x,y]} [b,b] \leq [e,e],$$

and let $A \in T'([e,e])$. By Lemma 3.3, there exists $f \xrightarrow{a} e$ such that $[x,y] = [a,f]$. By definition, there exists a sieve $S \in J(e)$ such that $A = [S]$. By (T2), $a^*S \in J(f)$ and so by definition $[a^*S] \in T'([f,f])$. But by Lemma 3.4, we have that

$$[a^*S] = [a,f]^{-1} \otimes [S] \otimes [a,f];$$

thus

$$[x,y]^{-1} \otimes A \otimes [x,y] \in T'([f,f])$$

as required.

(PET3) holds. Let $[e,e]$ be a maximal identity, let $A \in T'([e,e])$ and let $B \subseteq [e,e]^{\downarrow}$ be an arbitrary order ideal. Assume that for each $[f,f] \overset{[x,y]}{\to} [b,b] \leq [e,e]$ where $[b,b] \in A$ we have that $[x,y]^{-1} \otimes B \otimes [x,y] \in T'([f,f])$. We shall prove that $B \in T'([e,e])$. By definition, there is a sieve $S \in J(e)$ such that $[S] = A$. By Lemma 3.2, $R = ||B||$ is a sieve on e such that $[R] = B$. Let $f \overset{a}{\to} e \in S$ be arbitrary. By Lemma 3.3, we have that

$$[f,f] \overset{[a,f]}{\to} [a,a] \leq [e,e]$$

where $[a,a] \in A$. Thus by assumption,

$$[a,f]^{-1} \otimes B \otimes [a,f] \in T'([f,f]).$$

But by Lemma 3.4, $[a,f]^{-1} \otimes B \otimes [a,f] = [a^*R]$. Hence $[a^*R] \in T'([f,f])$ and so $a^*R \in J(f)$. Since (T3) holds, $R \in J(e)$ and so $B \in T'([e,e])$, as required.

(ii) By Lemma 3.2, the elements of $J(e)$ are sieves. We show that the three axioms for a Grothendieck topology hold.

(T1) holds. Observe that $||[e,e]^{\downarrow}|| = eC$. The result follows by (PET1).

(T2) holds. Let $f \overset{a}{\to} e$ and $S \in J(e)$. By definition and Lemma 3.2, $[S] = A \in T'([e,e])$, and by Lemma 3.3, $[f,f] \overset{[a,f]}{\to} [a,a] \leq [e,e]$. Thus by (PET2), we have that

$$[a,f]^{-1} \otimes A \otimes [a,f] \in T'([f,f]).$$

Thus by definition, $||[a,f]^{-1} \otimes A \otimes [a,f]|| \in J(f)$. But by Lemma 3.4, we have that $[a,f]^{-1} \otimes A \otimes [a,f] = [a^*S]$. Thus by Lemma 3.2, it follows that $a^*S \in J(f)$, as required.

(T3) holds. Let $S \in J(e)$ and let R be any sieve on e such that $a^*R \in J(f)$ for all $f \overset{a}{\to} e \in S$. By definition and Lemma 3.2, $||S|| = A \in T'([e,e])$ and $\{R\} = B$ an order ideal in $[e,e]^{\downarrow}$. Let $[f,f] \overset{[x,y]}{\to} [b,b] \leq [e,e]$ be such that $[b,b] \in A$. By Lemma 3.3, there exists $f \overset{a}{\to} e$ such that $[x,y] = [a,f]$. Now $[b,b] = [x,x] = [a,a]$ and $[b,b] \in A$. It follows by Lemma 3.2, that $a \in S$. By assumption, $a^*R \in J(f)$. Thus $[a^*R] \in T'([f,f])$ and so by Lemma 3.4, we have that $[x,y]^{-1} \otimes B \otimes [x,y] \in$

$T'([f,f])$. It follows by (PET3), that $B \in T'([e,e])$ and so $R \in J(e)$, as required.

Our final claim is immediate from (i) and (ii) above, Lemma 3.2 and Proposition 3.1 □

Let G be an ordered groupoid. We show next that there is a bijection between Ehresmann topologies on G and Grothendieck topologies on $\mathbf{C}(G)$ (Theorem 3.8). The proof of the following is routine and omitted.

Lemma 3.6. *Let G be an ordered groupoid and e an identity in G.*

(i) *Let A be an order ideal of $e^{\downarrow}$ in G. Put*

$$A^{\flat} = (\{e\} \times A)\mathbf{C}.$$

Then $A^{\flat}$ is a sieve on (e,e) in $\mathbf{C}$.

(ii) *Let S be a sieve on (e,e) in $\mathbf{C}$. Put*

$$S^{\sharp} = \{f :\ (e,f) \in S\}.$$

Then $S^{\sharp}$ is an order ideal in $e^{\downarrow}$.

The operations $A \mapsto A^{\flat}$ and $S \mapsto S^{\sharp}$ are mutually inverse. It follows that there is a bijection between the set of order ideals of $e^{\downarrow}$ in G and the set of sieves on (e,e) in $\mathbf{C}(G)$.

□

The following lemma is of a technical nature.

Lemma 3.7. *Let G be an ordered groupoid and $\mathbf{C} = \mathbf{C}(G)$. Let A be an order ideal in $e^{\downarrow}$ and let $(f,f) \stackrel{(e,x)}{\rightarrow} (e,e)$. Then*

$$x^{-1} \otimes A \otimes x = ((e,x)^{*}A^{\flat})^{\sharp}.$$

Proof. It is straightforward to check that $x^{-1} \otimes A \otimes x$ is an order ideal of $f^{\downarrow}$.

Let $S = A^{\flat}$. We now prove that

$$(x^{-1} \otimes A \otimes x)^{\flat} = (e,x)^{*}S.$$

The result stated in the lemma is then obtained easily by applying Lemma 3.6. Let $(f,y) \in (x^{-1} \otimes A \otimes x)^{\flat}$. Then by definition, $(f,y) = (f,i)(i,y)$ for some $i \in x^{-1} \otimes A \otimes x$. It follows that there is $j \in A$ such that $i = x^{-1} \otimes j \otimes x$. It is easy to check that $x \otimes i = j \otimes x$. Thus

$$(e,x)(f,y) = (e,x)(f,i)(i,y) = (e, j \otimes x)(i,y) = (e,j)(j, j \otimes x)(i,y).$$

But $(e,j) \in A^{\flat}$. Thus $(e,x)(f,y) \in A^{\flat}$ and so

$$(x^{-1} \otimes A \otimes x)^{\flat} \subseteq (e,x)^* S,$$

as required.

To prove the reverse inclusion, let $(f,y) \in (e,x)^* S$. Thus by definition $(e,x)(f,y) \in A^{\flat}$, and so

$$(e,x)(f,y) = (e,i)(i,z)$$

for some $i \in A$. It is easy to check that this implies that $y = x^{-1} \otimes i \otimes e \otimes z = x^{-1} \otimes i \otimes x \otimes x^{-1} \otimes z$. Thus

$$(f,y) = (f, x^{-1} \otimes i \otimes x)(x^{-1} \otimes i \otimes x, x^{-1} \otimes z)$$

where $(f, x^{-1} \otimes i \otimes x) \in (x^{-1} \otimes A \otimes x)^{\flat}$ and so $(f,y) \in (x^{-1} \otimes A \otimes x)^{\flat}$ as required. □

Theorem 3.8. *Let G be an ordered groupoid. Then there is a bijective correspondence between Ehresmann topologies on G and Grothendieck topologies on $\mathbf{C}(G)$.*

Proof. Let T be an Ehresmann topology on G. For each $(e,e) \in \mathbf{C}$ define

$$J((e,e)) = \{A^{\flat} : \ A \in T(e)\}.$$

By Lemma 3.6, $J(e,e)$ is a collection of sieves on (e,e) in $\mathbf{C}$. We show that J is a Grothendieck topology on $\mathbf{C}$.

(T1) holds: By (ET1), we have that $e^{\downarrow} \in T(e)$. Thus $(e^{\downarrow})^{\flat} \in J((e,e))$. By definition, $(e^{\downarrow})^{\flat} = (e \times (e^{\downarrow}))\mathbf{C}$. Let $(e,x) \in (e,e)\mathbf{C}$. Then $(e,x) = (e, \mathbf{r}(x))(\mathbf{r}(x), x)$. But $\mathbf{r}(x) \in e^{\downarrow}$ and so $(e,x) \in (e^{\downarrow})^{\flat}$. Thus $(e^{\downarrow})^{\flat} = (e,e)\mathbf{C}$, and the result follows.

(T2) holds: Let $(f,f) \overset{(e,x)}{\to} (e,e)$ and $S \in J((e,e))$. We prove that $(e,x)^*S \in J((f,f))$. By definition, $S = A^\flat$ for some $A \in T(e)$. Also $f \overset{x}{\to} \mathbf{r}(x) \leq e$ Thus by (ET2), we have that $x^{-1} \otimes A \otimes x \in T(f)$. By Lemma 3.7 and Lemma 3.6,

$$(x^{-1} \otimes A \otimes x)^\flat = (e,x)^*S.$$

This proves the claim.

(T3) holds: let $S \in J((e,e))$ and let R be any sieve on (e,e) such that $(e,a)^*R \in J((f,f))$ for all $(f,f) \overset{(e,a)}{\to} (e,e) \in S$. We shall prove that $R \in J((e,e))$. Put $A = S^\sharp \in T(e)$ and $B = R^\sharp \subseteq e^\downarrow$ and let $f \overset{x}{\to} e' \leq e$ where $e' \in A$. It follows that $(e,x) \in S$. By assumption, $(e,x)^*R \in J((f,f))$ and so $((e,x)^*R)^\sharp \in T(f)$. But then by Lemma 3.7 we have that $x^{-1} \otimes B \otimes x \in T(f)$. It follows by (ET3), that $B \in T(e)$ and so $R \in J((e,e))$. Thus given an Ehresmann topology T on G, we have constructed a Grothendieck topology J on $\mathbf{C}$.

Let J be a Grothendieck topology on $\mathbf{C}$. For each $e \in G_o$ define

$$T(e) = \{S^\sharp \colon \ S \in J((e,e))\}.$$

We prove that T is an Ehresmann topology on G.

(ET1) By (T1), we have that $S = (e,e)\mathbf{C} \in J((e,e))$. By definition $S^\sharp \in T(e)$, and $S^\sharp = \{f \in G_o \colon \ (e,f) \in S\}$. But this is precisely $e^\downarrow$ as required.

(ET2) Let $f \overset{x}{\to} e' \leq e$ and $A \in T(e)$. By definition, $A = S^\sharp$ where $S \in J((e,e))$, and $(f,f) \overset{(e,x)}{\to} (e,e)$. Thus by (T2), we have that $(e,x)^*S \in J((f,f))$. So by definition $((e,x)^*S)^\sharp \in T(f)$. However, $((e,x)^*S)^\sharp = x^{-1} \otimes A \otimes x$ by Lemma 3.7. Thus $x^{-1} \otimes A \otimes x \in T(f)$ as required.

(ET3) Let $A \in T(e)$ and let $B \subseteq e^\downarrow$ be an arbitrary order ideal. Suppose that for each $f \overset{x}{\to} e' \leq e$ where $e' \in A$ we have that $x^{-1} \otimes B \otimes x \in T(f)$. We shall prove that $B \in T(e)$. Put $S = A^\flat \in J((e,e))$ and $R = B^\flat$ a sieve on (e,e). Let $(f,f) \overset{(e,x)}{\to} (e,e)$ be arbitrary such that $(e,x) \in S$. Then $((e,x)^*R)^\sharp = x^{-1} \otimes B \otimes x$ by Lemma 3.7, and $x^{-1} \otimes B \otimes x \in T(f)$ by assumption. Thus $(e,x)^*R \in J((f,f))$. By (T3), we have that $R \in J((e,e))$. Hence $B \in T(e)$.

The fact that Ehresmann topologies on G are in bijective correspondence with Grothendieck topologies on $\mathbf{C}(G)$ now follows from Lemma 3.6. □

We are now ready to return to étendues. Let (C, J) be a left cancellative site. By Theorem 2.3, there is an injective functor

$$\iota\colon\ C \to \mathbf{CG}(C) = C'$$

which is full, faithful and dense. By Theorem 3.5, J induces an Ehresmann topology T on $\mathbf{G}(C)$, and by Theorem 3.8, T induces a Grothendieck topology J' on C'.

Lemma 3.9. *With the above notation, we have $\iota(J(e)) = J'(\iota(e)) \cap \iota(C)$ for every $e \in C_o$*

Proof. By Theorem 3.8, we have that $S \in J'(\iota(e))$ if and only if $S = B^\flat$ where $B \in T([e,e])$. By Theorem 3.5, we have that $B \in T([e,e])$ if and only if $B = [A]$ where $A \in J(e)$. Thus $S = [A]^\flat$. We prove that $\iota(A) = [A]^\flat \cap \iota(C)$, which will prove the lemma. Let $a \in A$. Then $\iota(a) = ([e,e],[a,\mathbf{d}(a)])$. But

$$([e,e],[a,\mathbf{d}(a)]) = ([e,e],[a,a])([a,a],[a,\mathbf{d}(a)]),$$

where $([e,e],[a,a]) \in \{[e,e]\} \times [A]$. Thus $\iota(a) \in [A]^\flat \cap \iota(C)$.

A typical element of $[A]^\flat \cap \iota(C)$ has the form $([e,e],[ap,x]) = \iota(b)$, for some $b \in C$. But then there is an invertible $u \in C$ such that $ap = bu$ and so $b = apu^{-1}$ which implies $b \in A$, as required. □

It follows from the above lemma and Proposition 2.4, that if we identify C and $\iota(C)$, then J' is the unique extension of J to C'. But then by Proposition 2.5, the category of sheaves on (C, J) is equivalent to the category of sheaves on (C', J'). The main theorem of this section is the following.

Theorem 3.10. *Every étendue is presented by a site constructed from an ordered groupoid equipped with an Ehresmann topology.* □

4 Sheaves

In the previous section, we described a correspondence between sites on left cancellative categories and Ehresmann sites. We now turn to the correspondence between sheaves on a site and sheaves on an Ehresmann site. The main theorem of the whole paper is Theorem 4.5

4.1 Sheaves on Ehresmann sites

Let G be an ordered groupoid and let F be a presheaf over over $(G_o, \leq)$. Recall that there are functions ρ^f_e from $F(f)$ to $F(e)$ whenever $f \geq e$. Let $X = \sqcup_{e \in G_o} F(e)$ and let $\pi\colon\ X \to G_o$ be the function which maps elements of $F(e)$ to e. Suppose in addition that there is a right action of the groupoid G on the set X (defined via π) which satisfies the following compatibility condition (CC): if $g \leq h$ where $f' \overset{g}{\leftarrow} e'$ and $f \overset{h}{\leftarrow} e$ then for each $a \in F(f)$ we have that

$$\rho^e_{e'}(a \cdot h) = \rho^f_{f'}(a) \cdot g.$$

We say in this case that the *ordered groupoid* G *acts on* **Set** *on the right.* We shall also say that G *acts on the right on the presheaf* F *with values in* **Set**. Observe that if $a \in F(e)$ and $f \overset{g}{\to} e$ then the map $a \mapsto a \cdot g$ from $F(e)$ to $F(f)$ is a bijection.

In the case of categories, right actions correspond to presheaves. We now give the 'presheaf definition' corresponding to our 'right action definition' above.

First we need some notation. Let $\mathbf{C}(G_o)$ be the category whose morphisms are the ordered pairs (e, f) such that $e \geq f$. It will be useful to regard $\mathbf{C}(G_o)$ as a category with objects G_o so that $(e, f)\colon\ f \to e$.[2]

Let G be an ordered groupoid. A *presheaf* on the ordered groupoid G with values in **Set** is a pair of functors (F_1, F_2) where $F_1\colon\ G^{op} \to$ **Set** and $F_2\colon\ \mathbf{C}(G_o)^{op} \to$ **Set**, where G^{op} is the opposite category of the

[2]Note that $\mathbf{C}(G_o)$ is the just usual category associated to the partially ordered set G_o [11] and so the construction $\mathbf{C}(G)$ can be viewed as a generalisation of this standard construction. Also observe that $\mathbf{C}(G_o)$ is exactly the subcategory of $\mathbf{C}(G)$ denoted Mono($\mathbf{C}(G)$) in Section 2.3.

groupoid G, such that $F_1(e) = F_2(e)$ for each $e \in G_o$, and such that the following *compatibility condition* (4.1) holds: suppose that $e \xrightarrow{h} f$ and $e' \xrightarrow{g} f'$ and $g \leq h$. Then

$$\begin{array}{ccc} F_1(f) & \xrightarrow{F_1(h)} & F_1(e) \\ {\scriptstyle F_2(f,f')}\downarrow & & \downarrow{\scriptstyle F_2(e,e')} \\ F_1(f') & \xrightarrow[F_1(g)]{} & F_1(e') \end{array} \tag{4.1}$$

commutes. Presheaves are equivalent to right actions; this becomes apparent when we observe that when $e \geq f$ the function $F_2(e,f)$ is precisely ρ^e_f. Our usual notation will be to write F instead of F_1. We normally write $F\colon\ (G,\leq)^{op} \to \mathbf{Set}$ to denote the presheaf (or right action) with values in **Set**.

Now let G be an ordered groupoid equipped with an Ehresmann topology T, and let F be a presheaf on G. A *matching family* $\{a_i\}_{i\in A}$ for an order ideal $A \in T(e)$ is a choice of $a_i \in F(i)$ for each $i \in A$ such that

$$\text{if } j \leq i \in A \text{ then } \rho^i_j(a_i) = a_j.$$

An *amalgamation* of such a family is an element $a \in F(e)$ such that

$$\rho^e_i(a) = a_i \text{ for all } i \in A.$$

The presheaf F is a *sheaf* if every matching family has a unique amalgamation.

The *category of presheaves of the ordered groupoid* G *in* **Set**, denoted $\mathbf{Set}^{(G,\leq)^{op}}$, is a subcategory of $\mathbf{Set}^{G^{op}} \times \mathbf{Set}^{\mathbf{C}(G_o)^{op}}$. The *objects* are the pairs (F_1,F_2) with $F_1(e) = F_2(e)$ for each $e \in G_o$ which satisfy the compatibility condition (CC). The *morphisms* of $\mathbf{Set}^{(G,\leq)^{op}}$ consist of those morphisms in $\mathbf{Set}^{G^{op}} \times \mathbf{Set}^{\mathbf{C}(G_o)^{op}}$ of the form

$$(\eta,\epsilon)\colon\ (F_1,F_2) \to (H_1,H_2)$$

such that, for each $e \in G_o$, the components

$$\eta_e\colon\ F_1(e) \to H_1(e) \text{ and } \epsilon_e\colon\ F_2(e) \to H_2(e)$$

are equal (recall: $F_1(e) = F_2(e)$ and $H_1(e) = H_2(e)$). The morphisms are therefore the natural transformations of the form (η, η). Equivalently, the family $\eta = \{\eta_e\}_{e \in G_o}$ is simultaneously a natural transformation from F_1 to H_1 and from F_2 to H_2. We normally call a morphism in $\mathbf{Set}^{(G,\leq)^{op}}$ a *natural transformation* and just denote it by $\eta\colon F \to H$. There is a component arrow $\eta_e\colon F(e) \to H(e)$ for each $e \in G_o$.

The category of sheaves on (G, T), denoted $\mathbf{Sh}(G, T)$, is the full subcategory of $\mathbf{Set}^{(G,\leq)^{op}}$ whose objects are sheaves.

4.2 Ehresmann sheaves to Grothendieck sheaves and back

Let G be an ordered groupoid equipped with an Ehresmann topology T. Our goal is to prove that $\mathbf{Sh}(G, T)$ is isomorphic to $\mathbf{Sh}(\mathbf{C}(G), J)$ where J is the Grothendieck topology constructed in Section 3. We may then deduce, by Theorem 3.10, that every étendue is equivalent to the category of sheaves on an Ehresmann site.

We show first (Theorem 4.3) that right actions of an ordered groupoid G correspond exactly to right actions of $\mathbf{C}(G)$; that is $\mathbf{Set}^{(G,\leq)^{op}} \cong \mathbf{Set}^{\mathbf{C}(G)^{op}}$. We then show that Ehresmann sheaves in the former correspond exactly to Grothendieck sheaves in the latter (Theorem 4.4).

The category $\mathbf{C}(G)$ has identities the elements of the form (e, e) where $e \in G_o$. It will sometimes be convenient below to regard $\mathbf{C}(G)$ as a category whose objects are G_o in line with the way categories are usually regarded rather than the 'generalised monoid approach' which we have used in most of this paper.

From Theorem 2.2, the groupoid G is isomorphic to the groupoid of isomorphisms of $\mathbf{C}(G)$ by a function $\iota\colon G \to \mathrm{Iso}(\mathbf{C}(G))$ defined by $g \mapsto (\mathbf{r}(g), g)$. The category $\mathbf{C}(G_o)$ is the category with objects G_o and morphisms $(e, f)\colon f \to e$ whenever $e \geq f$; this category is essentially $\mathrm{Mono}(\mathbf{C}(G))$ from Theorem 2.2. It follows that a functor defined on $\mathbf{C}(G)^{op}$ restricts to give two functors defined on $\mathrm{Iso}(\mathbf{C}(G))^{op}$ and $\mathrm{Mono}(\mathbf{C}(G))^{op}$ respectively. This provides the motivation for the following definition.

We define a function $\psi\colon \mathbf{Set}^{\mathbf{C}(G)^{op}} \to \mathbf{Set}^{(G,\leq)^{op}}$ as follows. If F is an object then

$$\psi(F) = (F \circ \iota, F \mid_{\mathbf{C}(G_o)}),$$

where for convenience, we set

$$\psi(F) = F^{\sharp}\colon\ (G, \leq)^{op} \to \mathbf{Set}.$$

Note that:

- $F^{\sharp}(e) = F(e)$ for $e \in G_o$
- $\rho^e_f = F((e,f))$ for $e \geq f$
- $F^{\sharp}(x) = F((\mathbf{r}(x), x))$

If η is a morphism, where $\eta\colon\ F \to H$ is a natural transformation, then define

$$\psi(\eta) = (\eta, \eta).$$

It is trivial to see that η is simultaneously a natural transformation from $F \circ \iota$ to $H \circ \iota$ and from $F \mid_{\mathbf{C}(G_o)}$ to $H \mid_{\mathbf{C}(G_o)}$ since we are just restricting F and H to full subcategories of $\mathbf{C}(G)$. Hence $\psi(\eta) = \eta\colon\ F^{\sharp} \to H^{\sharp}$ is well defined. It is routine to check the following.

Proposition 4.1. *The function ψ is a functor.* □

By Theorem 2.2, each element of $\mathbf{C}(G)$ can be uniquely factored into the product of an element of $\mathbf{C}(G_o)$ and an element of $\mathrm{Iso}(\mathbf{C}(G))$: namely,

$$(e, g) = (e, \mathbf{r}(g))(\mathbf{r}(g), g).$$

But $\mathbf{C}(G_o)$ and $\mathrm{Iso}(\mathbf{C}(G))$ are the foundations for defining presheaves on G. This observation lies behind the following definition.

We define a function

$$\tau\colon\ \mathbf{Set}^{(G,\leq)^{op}} \to \mathbf{Set}^{\mathbf{C}(G)^{op}}$$

as follows. If F is an object, that is $F\colon\ (G, \leq)^{op} \to \mathbf{Set}$, then define

$$\tau(F) = F^{\flat}\colon\ \mathbf{C}(G)^{op} \to \mathbf{Set}$$

by

$$F^{\flat}((e,x)) = F(x)\rho^{e}_{\mathbf{r}(x)}.$$

If η is a morphism, where $\eta\colon F \to H$ is a natural transformation in $\mathbf{Set}^{(G,\leq)^{op}}$, then define

$$\tau(\eta) = \eta\colon F^{\flat} \to H^{\flat}.$$

The proof of the following is routine.

Proposition 4.2. *The function τ is a functor.* □

It is easy to check that ψ and τ are inverses of each other. We therefore have the following theorem which is a generalisation of a result first formulated by Loganathan [10].

Theorem 4.3. $\mathbf{Set}^{(G,\leq)^{op}} \cong \mathbf{Set}^{\mathbf{C}(G)^{op}}$ □

Suppose now (G,T) is an Ehresmann site and $(\mathbf{C}(G),J)$ is the corresponding site constructed according to Theorem 3.8. Presheaves on $(G,\leq)$ and $\mathbf{C}(G)$ coincide. To prove $\mathbf{Sh}(G,Y) \cong \mathbf{Sh}(\mathbf{C}(G),J)$, it suffices to show that ψ and τ, defined before Propositions 4.1 and 4.2 respectively, take sheaves to sheaves. This is proved in the following theorem.

Theorem 4.4. *Let (G,T) be an Ehresmann site and let $(\mathbf{C}(G),J)$ be the associated site constructed according to Theorem 3.8. Then* $\mathbf{Sh}(G,T) \cong \mathbf{Sh}(\mathbf{C}(G),J)$.

Proof. We show first that ψ maps sheaves to sheaves. Let

$$F \in \mathbf{Sh}(\mathbf{C}(G),J).$$

We show $F^{\sharp} \in \mathbf{Sh}(G,T)$.

Suppose $\{a_f\}_{f\in A}$ is a matching family for $A \in T(e)$. We construct a matching family for $A^{\flat}$. Suppose $(e,x) \in A^{\flat}$. Then

$$(e,x) = (e,f)(f,y)$$

with $f \in A$. But $(e,f)(f,y) = (e,y)$ so $y = x$. Hence $f \geq \mathbf{r}(x)$ so $\mathbf{r}(x) \in A$. Define

$$a_{(e,x)} = a_{\mathbf{r}(x)} \cdot x \tag{4.2}$$

Note that $a_{\mathbf{r}(x)} \cdot x \in F^\sharp(\mathbf{d}(x)) = F(\mathbf{d}((e,x))$. We show $\{a_{(e,x)}\}_{(e,f)\in A^\flat}$ is a matching family for $A^\flat$. Suppose $(e,x) \in A^\flat$ and $(e,x)(f,y)$ is defined in $\mathbf{C}(G)$; note that $f = \mathbf{d}(x)$. Then

$$(e,x)(f,y) = (e,(x \mid \mathbf{r}(y))y).$$

Let $g = \mathbf{r}((x \mid \mathbf{r}(y))y)$. Then

$$a_{(e,(x \mid \mathbf{r}(y))y)} = a_g \cdot (x \mid \mathbf{r}(y))y.$$

Since $\mathbf{r}(x) \geq g$, $a_g = \rho_g^{\mathbf{r}(x)}(a_{\mathbf{r}(x)})$ so

$$a_g \cdot (x \mid \mathbf{r}(y))y = \rho_g^{\mathbf{r}(x)}(a_{\mathbf{r}(x)}) \cdot (x \mid \mathbf{r}(y))y.$$

But

$$\rho_g^{\mathbf{r}(x)}(a_{\mathbf{r}(x)}) \cdot (x \mid \mathbf{r}(y))y = \rho_{\mathbf{r}(y)}^f(a_{\mathbf{r}(x)} \cdot x)y \tag{4.3}$$

by (CC). On the other hand

$$a_{(e,x)} \cdot (f,y) = (a_{\mathbf{r}(x)} \cdot x) \cdot (f,\mathbf{r}(y))(\mathbf{r}(y),y) = \rho_{\mathbf{r}(y)}^f(a_{\mathbf{r}(x)} \cdot x)y \tag{4.4}$$

Putting together (4.3) and (4.4), we see that $\{a_{(e,x)}\}_{(e,x)\in A^\flat}$ is a matching family for $A^\flat$. Hence there exists a unique amalgamation $a \in F(e) = F^\sharp(e)$.

We now show a is an amalgamation for $\{a_f\}_{f\in A}$. Suppose $f \in A$. Then $(e,f) \in A^\flat$ and

$$\rho_f^e(a) = a \cdot (e,f) = a_{(e,f)} = a_f \cdot f = a_f.$$

We conclude a is an amalgamation. Suppose b is another amalgamation for $\{a_f\}_{f\in A}$, we show b is an amalgamation for $\{a_{(e,x)}\}$. Indeed

$$b \cdot (e,x) = a \cdot (e,\mathbf{r}(x))(\mathbf{r}(x),x) = \rho_{\mathbf{r}(x)}^e(b) \cdot x = a_{\mathbf{r}(x)} \cdot x = a_{(e,x)}.$$

We conclude that $b = a$ and that $F^\sharp$ is a sheaf.

We now prove that τ takes sheaves to sheaves. Let $F \in \mathbf{Sh}(G,T)$. We show $F^\flat$ is a sheaf.

Suppose $\{a_{(e,x)}\}_{(e,f)\in S}$ is a matching family for $S \in J(e)$. We construct a matching family for $S^\sharp$. Let $f \in S^\sharp$. Then $(e,f) \in S$. Define $a_f = a_{(e,f)}$. Note that $a_{(e,f)} \in F^\flat(f) = F(f)$. Suppose $f \geq g$. Then

$$\rho^f_g(a_f) = a_{(e,f)} \cdot (f,g) = a_{(e,g)} = a_g,$$

so $\{a_f\}_{f\in S^\sharp}$ is a matching family for $S^\sharp$. Hence it has a unique amalgamation $a \in F(e) = F^\flat(e)$.

We show a is an amalgamation for $\{a_{(e,x)}\}_{(e,x)\in S}$. Indeed if $(e,x) \in S$, then $(e,x)(x,x^{-1}) = (e,\mathbf{r}(x)) \in S$ and so $\mathbf{r}(x) \in S^\sharp$. Thus

$$a \cdot (e,x) = \rho^e_{\mathbf{r}(x)}(a) \cdot x = a_{\mathbf{r}(x)} \cdot x = a_{(e,\mathbf{r}(x))} \cdot (\mathbf{r}(x),x) = a_{(e,x)}$$

and so a is an amalgamation. Suppose $b \in F^\flat(e)$ is another amalgamation for $\{a_{(e,x)}\}_{(e,f)\in S}$. We show b is an amalgamation for $\{a_f\}_{f\in S^\sharp}$. Indeed, if $f \in S^\sharp$ then $(e,f) \in S$. Hence

$$\rho^e_f(b) = b \cdot (e,f) = a_{(e,f)} = a_f,$$

as desired. □

The main theorem of this section and the whole paper now follows from Theorems 3.10 and 4.4.

Theorem 4.5. *Every étendue is equivalent to the topos of sheaves on an Ehresmann site.* □

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