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FREE EXTENSIONS OF DOUBLE CATEGORIES

by R.J.M. DAWSON, R.PARE and D.A. PRONK

RESUME. Dans cet article, les auteurs étudient les catégories doubles obtenues en ajoutant librement de nouvelles cellules ou flèches à une catégorie double existante. Ils discutent plus spécifiquement la décidabilité de l'égalité de cellules dans la nouvelle catégorie double.

Introduction

Extending a category **C** by a free arrow x (either an endomorphism or an arrow between two existing objects) is a very straightforward construction. The composable strings of the new category consist of composable strings from the original category, alternating with instances of the new element:

$$f_{00}\cdots f_{0m_0}xf_{10}\cdots f_{1m_1}x\cdots xf_{k0}\cdots f_{km_k}$$

where any of the strings $f_{i0}f_{im_i}$ may be empty, $\mathrm{cod}(f_{im_i}) = \mathrm{dom}(x)$ for $i < k$, and $\mathrm{dom}(f_{i0}) = \mathrm{cod}(x)$ for $i > 0$. (Of course, if x is not an endomorphism, this forces all but the first and last strings $f_{i0}f_{im_i}$ to be nonempty!)

Two of these strings may represent the same morphism of **C**[**x**]. Because the free morphisms x isolate the strings $f_{i0}f_{im_i}$ absolutely within the arrangement, there is a canonical form for the morphisms of **C**[**x**] in which each string has been composed to a single arrow:

$$f_0xf_1x\cdots xf_k\ .$$

We may, of course, repeat this construction, adding further free arrows. It is easily shown that the order in which they are added does not matter, even if some objects appear as domains or codomains of more than one arrow. We can conveniently represent such a system of arrows with shared objects as a (directed) graph G. The resulting category $\mathbf{C}[x_1, x_2, \ldots, x_n]$ can also be thought of as the free extension of $\mathbf{C}$ by the *category* $F(G)$ freely generated by G. This is actually the pushout of the $\mathbf{Cat}$ diagram

$$\mathbf{C} \longleftarrow \mathbf{D} \longrightarrow \mathbf{F(G)}$$

where $\mathbf{D}$ is the discrete category on the objects of G, and the maps are the obvious inclusions.

The free extension of a group H is slightly more complicated because the new element has an inverse; cancelling inverse pairs xx^{-1} or $x^{-1}x$ within a string may bring composable elements of H together; and this in turn may permit further cancellations. However, this process is strictly length-reducing, and there is still a canonical form

$$h_0 x^{\pm n_1} h_1 \cdots x^{\pm n_p} h_p$$

where only h_0 and h_p may be identities, $n_p > 0$, but p may equal 0.

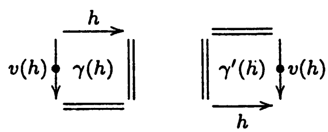
For the free extension $R[x]$ of a ring, we have the further complication of two operations, which can be used alternately to yield arbitrarily complicated expressions. However, the distributive law allows every such expression to be put in the canonical form of a polynomial over R. (If the ring is finite, the polynomial *functions* $R \to R$ form a proper quotient ring of $R[x]$ – for instance, over $\mathbb{Z}_2$, x and x^2 are different polynomials but the same function – but that will not concern us here.)

Consider, now, a double category - that is, a category object in the category $\mathbf{Cat}$ of categories. This may conveniently be thought of as a set of morphisms (more aptly described as 'cells' than 'arrows') forming (an objectless presentation of) a category under each of two different composition operations (which we may write as $\circ$ and $*$). These are linked by the middle-four interchange axiom, which plays a role slightly similar to that of the distributive law in a ring.

The cells which are the identities of the vertical composition operation themselves form a category under horizontal composition, and it is usual to call them the *horizontal arrows* of the double category; they of course correspond to the vertical domains and codomains of the cells.

The subcategory of *vertical arrows* is similarly defined, and these form horizontal domains and codomains of cells. We will find it useful to classify cells on the basis of which (if any) of their domains and codomains are identities in the appropriate arrow category.

If there is an isomorphism between the horizontal and vertical arrow categories, we call the double category *edge-symmetric.* Brown and Mosa introduce [3] the idea of a *connection* structure on such a double category (see also [5] and [19]), consisting of two sets of cells indexed by 'the' arrow category. For each pair of arrows $(h, v(h))$, of a horizontal arrow h and its corresponding vertical arrow $v(h)$, the cell $\gamma(h)$ has h as its vertical domain, $v(h)$ as its horizontal domain, and identities for the two codomains; $\gamma'(h)$ has identities as domains and h and $v(h)$ as codomains.



The indexing is natural in h, and $\gamma(h) \circ \gamma'(h) = e_h$, while $\gamma(h) * \gamma'(h) = i_{v(h)}$. They show that the category of 2-categories may be embedded in the category of double categories, so that the image is the symmetric double categories with connection.

In this paper, we will consider various free extensions of double categories, both by arrows and by cells. There is, of course, no difficulty in constructing the composable arrangements of the new double category. However, determining whether two such arrangements represent the same cell may be, under some circumstances, undecidable, contrary to what was the case for rings and categories. We will give some conditions under which there is undecidability, and other conditions under which canonical forms exist for the composable arrangements (which implies that the equivalence relation is decidable). These conditions will usually involve the existence or absence of of cells with three or four identity arrows as domains and codomains.

This work is motivated by the importance of some free or near-free constructions in higher dimensional category theory. One example of this is the free adjoint construction which we introduced in [8]; it was shown there that the 2-category obtained by freely adding adjoints to arrows of a category C was equivalent to a quotient 2-category of a certain 2-category of diagrams in C. It was shown in [9] that the equivalence relation is in general undecidable. The results in this paper reveal different aspects of this undecidability problem. We want to emphasize that although the work of this paper is presented in the context of double categories, many results also apply to 2-categories.

Another example of a near-free construction arises from the construction of a left adjoint to the embedding $\mathbf{2}-\mathbf{Cat} \hookrightarrow \mathbb{D}-\mathbf{Cat}$ (see, *eg*, Brown and Mosa [3]). A. and C. Ehresmann [10] gave an explicit construction for the left adjoint, together with a way to construct such functors for higher dimensional categories.

The resulting 2-category has arrows corresponding to both the horizontal and the vertical arrows of the original double category, and compositions of these; the category of horizontal arrows has in effect been freely extended by the category of vertical arrows. The distinction between the original horizontal and vertical composition has likewise been lost; this corresponds to the free addition of 'connection' cells. Thus, the study of free extensions sheds light on this left adjoint construction, a point that the authors intend to consider in future work.

Further interest in free extensions of double categories arises from the syntax of the category $\mathbb{D}-\mathbf{Cat}$ of double categories itself. Free extensions form a special kind of pushout in this category, and the work of this paper can be viewed as a first step in describing the properties of pushouts in it. Also, the construction of free extensions is a good way to create new examples of double categories.

1 Double Categories

Ehresmann [11] introduced the concept of a *double category*, defined to be a category object in the category **Cat** of categories, i.e. a diagram

$$D_2 \underset{\pi_1}{\overset{\pi_2}{\underset{m}{\rightrightarrows}}} D_1 \underset{d_1}{\overset{d_0}{\leftrightarrows}} D_0 \tag{1}$$

of categories and functors. A *double functor* is an internal functor between the category objects. We write $\mathbb{D}$-**Cat** for the category of double categories and double functors.

With this notation, we consider the objects of D_0 to be the *objects* of the double category and the arrows to be the *vertical arrows* in $\mathbf{D}$. We draw these arrows as $\longrightarrow\!\!\!\!\!\!\bullet\;\;$. We write $D_0 = (D_V, D_\bullet, *)$, i.e. the symbol $*$ will indicate vertical composition. Identity arrows in this category are denoted by I^v_A. We also write $D_1 = (D_\square, D_H, *)$. The objects of D_1 are considered to be the *horizontal arrows* of $\mathbf{D}$ drawn as $\longrightarrow$. These arrows receive their domain and codomain in $D_\bullet$ from the arrows d_0 and d_1 in diagram (1). We write $h_1 \circ h_2$ for $m(h_1, h_2)$. They can be composed using the arrow m from the diagram. We will indicate this horizontal composition by the symbol $\circ$. We will indicate the horizontal identity arrows by I^h_A. The arrows of D_1 are the *cells* of $\mathbf{D}$. The domain, codomain and composition in D_1 give their vertical domain, codomain and composition respectively. Note that their vertical domain and codomain are horizontal arrows. The arrows d_0 and d_1 in diagram (1) give their horizontal domain and codomain respectively (which consist of vertical arrows), and the arrow m gives their horizontal composition, which we will indicate by $\circ$ again. We use e_h and i_v for vertical and horizontal identity cells respectively, and draw them as:

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle v}\downarrow & i_v & \downarrow{\scriptstyle v} \\ A & = & A \end{array} \quad\text{or}\quad \begin{array}{ccc} A & = & A \\ {\scriptstyle v}\downarrow & = & \downarrow{\scriptstyle v} \\ A & = & A \end{array} \quad\text{and}\quad \begin{array}{ccc} B & \xrightarrow{h} & B \\ \| & e_h & \| \\ B & \xrightarrow[h]{} & B \end{array} \quad\text{or}\quad \begin{array}{ccc} B & \xrightarrow{h} & B \\ \| & \| & \| \\ B & \xrightarrow[h]{} & B. \end{array}$$

It is required that $e_{I^h_A} = i_{I^v_A}$ and we denote this cell by i_A.

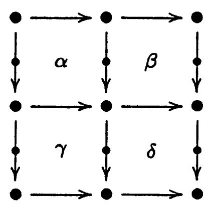
We draw a general cell in $\mathbf{D}$ as:

$$\begin{array}{ccc} A & \xrightarrow{h_0} & B \\ {\scriptstyle v_0}\downarrow & \alpha & \downarrow{\scriptstyle v_1} \\ C & \xrightarrow[h_1]{} & D \end{array}$$

where $A, B, C, D \in D_\bullet$, the $v_i = d^h_i(\alpha)$ are vertical arrows, the $h_i = d^v_i(\alpha)$ are horizontal arrows and α is a cell. Functoriality of the arrows in diagram (1) gives us:

1. The vertical composition of the horizontal domains (or codomains) of cells forms the horizontal domain (or codomain resp.) of the vertical composition of the cells. This justifies using the symbol $*$ also for the vertical composition of cells.

2. The middle four interchange law for a diagram of cells



which reads

$$(\alpha \circ \beta) * (\gamma \circ \delta) = (\alpha * \gamma) \circ (\beta * \delta)$$

Note that this only holds when both sides of the equality are well-defined.

3. Taking horizontal and vertical domains and codomains for a cell α commutes in the sense that

$$d_i^h d_j^v(\alpha) = d_j^v d_i^h(\alpha) \in D_\bullet,$$

which justifies the way the cell α above is drawn.

Finally note that the horizontal category $(D_H, D_\bullet, \circ)$ could also have been used to serve as D_0 in the description of $\mathbf{D}$, in which case we would have defined the vertical composition from the internal category structure in diagram (1).

Examples

1. Every category can be viewed as a double category by the inclusion functor $I_H\colon \mathbf{Cat} \to \mathbb{D}\text{-}\mathbf{Cat}$ where $I_H(M, O, \circ)$ is the double category with D_0 the discrete category with objects O, and D_1

is the discrete category with objects M. A diagram of the form $A \xrightarrow{f} B \xrightarrow{g} C$ in a category $\mathcal{C}$ is sent to a diagram of the form

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \| & & \| & & \| \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

in the double category $I_H(\mathcal{C})$. This is the free double category generated by a category in the sense that this functor is left adjoint to the forgetful functor $U: \mathbb{D} - \mathbf{Cat} \to \mathbf{Cat}$, which sends $\mathbf{D}$ to $(D_\Box, D_H, \circ)$.

2. Another way to embed the category of categories into the category of double categories, is by taking the double category of commutative squares in the original category.

3. Analogously to the case for categories, we can embed the category of 2-categories in two ways into the category of double categories. Let $\mathcal{T}$ be a 2-category. In the first case cells in the new double category $I_H(\mathcal{T})$ are of the form:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \| & \alpha & \| \\ A & \xrightarrow[g]{} & B \end{array}$$

where f and g are 1-cells in $\mathcal{T}$ and $\alpha: f \Rightarrow g$ is a 2-cell in $\mathcal{T}$.

For the second case, cells of $\Box(\mathcal{T})$ are of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow\!\bullet & \alpha & \bullet\!\downarrow{\scriptstyle g} \\ C & \xrightarrow[k]{} & D \end{array}$$

where f, g, h and k are 1-cells and $\alpha: g \circ f \Rightarrow k \circ h$ is a 2-cell in $\mathcal{T}$.

Later in this paper we will need the following terminology for the two dimensional analogies of the notion of endomorphism in a double category.

Definition 1.1 (i) *We will call a cell which has only identity arrows as domains and codomains a* zero-sided *cell.*
(ii) *A cell which has identity arrows as at least three of its domains and codomains will be called* one-sided *(note that zero-sided cells are a special case of this). We may call a one-sided cell that is not zero-sided* horizontal *or* vertical *according to the direction of its non-identity arrow. Note that for a one- (or zero-)sided cell α all the objects $d_i d_j(\alpha)$ are the same. We will call this unique object the* base *of α.*
(iii) *A cell whose horizontal domain and codomain are identities will be called a* vertical *two-sided cell.* Horizontal *two-sided cells are defined analogously.*

It is part of the folklore that if α and β are both zero-sided cells based at A,

$$\alpha * \beta = (\alpha \circ I_A) * (I_A \circ \beta) = (\alpha * I_A) \circ (I_A * \beta) = \alpha \circ \beta = \beta * \alpha = \beta \circ \alpha \,.$$

Similar, but more limited, results hold for one-sided cells that have non-identity edges in different locations; for instance, if γ has a non-identity arrow as its vertical domain, and δ has one as its vertical codomain, then

$$\gamma \circ \delta = \gamma * \delta = \delta \circ \gamma.$$

2 Free Extensions by Cells

2.1 Free Extensions Given a double category $\mathbf{D}$, and a square of arrows

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array} \tag{2}$$

we may create another double category $\mathbf{D}[X]$ with the same objects and arrows as $\mathbf{D}$, but 2-cells consisting of the elements of $D_\Box$, the cell X, and those arising from these by composition. The new cells are equivalence classes of composable rectangular arrays such as that in Figure 1, under the axioms of double categories and composition in $\mathbf{D}$.

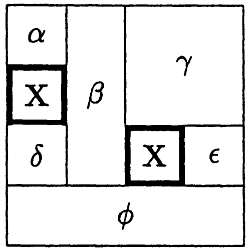


Figure 1: A composable rectangular array in $\mathbf{D}[X]$.

This double category is a free extension of $\mathbf{D}$, in the same sense that a polynomial ring $R[X]$ is a free extension of R:

Proposition 2.1 *For any double functor $F\colon \mathbf{D} \to \mathbf{E}$ and any cell α in $\mathbf{E}$ with boundary*

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle Fv}\downarrow & & \downarrow{\scriptstyle Fw} \\ FC & \xrightarrow[Fg]{} & FD \end{array} \tag{3}$$

there is a unique double functor $F_\alpha\colon \mathbf{D}[X] \to \mathbf{E}$, extending F, such that $F_\alpha(X) = \alpha$.

∎

Determining the structure of $\mathbf{D}[X]$, however, is somewhat more difficult than this analogy might lead us to assume. For any (commutative) ring R, there is a canonical form $\sum_0^n r_i X^i$ for $R[X]$ such that two polynomials over R are equal if and only if their canonical forms are the same. Similarly, if we extend a category C by a free arrow $x : a \to b$, the arrows of $C[x]$ can be put into the canonical form $f_0 x f_1 \cdots f_{n-1} x f_n$ where the f_i are arrows of C with $\mathrm{dom} f_i = b$ (except possibly when $i = 0$) and $\mathrm{cod} f_i = a$ (except possibly when $i = n$). In contrast, we shall see that equivalence of composable arrangements may be undecidable in $\mathbf{D}[X]$; it follows that there is no computable canonical form for elements of $\mathbf{D}[X]$.

2.2 Undecidability As observed above, the cells of a free extension $\mathbf{D}[X]$ are equivalence classes of composable rectangular arrangements involving X and the cells of $\mathbf{D}$. In this section we will show that this equivalence relation is, in general, undecidable.

If products in a group G (or indeed compositions in any category) are computable, one can easily solve the word problem for a free extension $G[X]$ since (as observed above) there is a normal form for $G[X]$. The normal form depends on the one-dimensional nature of composition in a category. The greater combinatorial complexity of the two-dimensional composition in a double category $\mathbf{D}$ allows the equivalence problem for composable arrangements in $\mathbf{D}[X]$ to be undecidable in some cases.

Theorem 2.2 *There exists a double category* $\mathbf{D}$ *for which the equivalence problem for composable arrangements can be solved (in a time linear in the length of the word), but such that, if a cell* X *with a certain boundary is freely added, the corresponding problem for* $\mathbf{D}[X]$ *is undecidable.*

Proof A *reversible rewrite system* consists of a finite alphabet $A = \{A_1 \ldots A_m\}$ and a set R of *rewrite rules* $\{R_1, \ldots, R_n\}$, each R_k consisting of a pair

$$(A_{j_{k1}} A_{j_{k2}} \ldots A_{j_{km_k}} \leftrightarrow A_{j'_{k1}} A_{j'_{k2}} \ldots A_{j'_{kn_k}})$$

of mutually substitutable words of A. The *word problem* for (A, R) consists of determining whether one specified word of A can be obtained from another by a sequence of "moves", each replacing a word of the form UVW by one of the form $UV'W$, where V and V' are (in either order) the two words of a rewrite rule.

The word problem for groups can be expressed as a special case of the word problem for reversible rewrite systems; so the latter is clearly undecidable in general. Given an arbitrary reversible rewrite system (A, R), we will construct a double category $\mathbf{D}_{(A,R)}$ such that the equivalence problem for composable arrangements in $\mathbf{D}_{(A,R)}$ can be solved by inspection of the (finite) list of rewrite rules, while the problem for $\mathbf{D}_{(A,R)}[X]$ is equivalent to the word problem for (A, R).

It will be useful to be able to assume that for any two words w_1 and w_2 and any single rewrite rule, one can determine in a finite amount of time, whether repeated use of this rule and its inverse could transform the first word into the second word. This seems to be a difficult problem for general rewrite systems; the problem arises when a substitution makes a partially-overlapping inverse substitution possible. However, any rewrite system is embedded, in an obvious fashion, in one in which this cannot happen, constructed as follows.

First, we extend the alphabet by two new symbols 'L' and 'R'. Every existing rule

$$(A_{j_{k1}} A_{j_{k2}} \ldots A_{j_{km_k}} \leftrightarrow A_{j'_{k1}} A_{j'_{k2}} \ldots A_{j'_{kn_k}})$$

is replaced by the rule

$$(LA_{j_{k1}} A_{j_{k2}} \ldots A_{j_{km_k}} R \leftrightarrow LA_{j'_{k1}} A_{j'_{k2}} \ldots A_{j'_{kn_k}} R),$$

and we add two new rules, $(L \leftrightarrow \emptyset)$ and $(R \leftrightarrow \emptyset)$. Any substitution of the old system can be performed in five steps in the new system by first inserting an L to mark the beginning of the string to be changed and an R to mark the end, applying the corresponding substitution from the new set, and then removing the two delimiters. The words of the old system are also words of the new system; and two of them are equivalent in the new system precisely when they were equivalent in the old system.

We show now that this system satisfies our requirements. Let w_1 and w_2 be any two words, and α a rewrite rule in this system. If α is the rule $(L \leftrightarrow \emptyset)$, one can check whether w_1 and w_2 are equivalent under this rule, by removing all occurrences of L in both words. w_1 and w_2 are equivalent under α if and only if the resulting words are the same. A similar argument can be used if α is the rule $(R \leftrightarrow \emptyset)$. If α is a replacement of an old rule, the number of new words one can make out of w_1 by repeated applications of α is bounded by 2^n where n is the number of matching LR-pairs in w_1. So one can determine in a finite amount of time whether w_2 is one of those words. We shall henceforth assume that the system (A, R) has this property.

Under this assumption, we construct the double category $\mathbf{D} = \mathbf{D}_{(A,R)}$. It has five objects, $\{a, b, c, d, e\}$, and horizontal arrows $f : b \to c$ and $f' : d \to e$. It has vertical arrows $u'' : a \to c$, $v : b \to d$, $v' : c \to e$, and $n + 1$ distinct vertical arrows $u, u_k : a \to b$ where the arrows u_k correspond to the rewrite rules.

The cells of $\mathbf{D}$ are generated by the following:

- A set of m cells $\{\alpha_i : i = 1 \ldots m\}$ corresponding to the letters of the alphabet. These all have vertical domain i_b and vertical codomain i_d and horizontal domain and codomain both equal to v.

- A set of n cells $\{\gamma_k : k = 1 \ldots n\}$, corresponding to the rewrite rules. The vertical domain of all of these cells is i_a, and their vertical codomain is i_b. The horizontal domain of each γ_k is u, and the horizontal codomain is u_k.

- A set of n cells $\{\delta_k : k = 1 \ldots n\}$, also corresponding to the rewrite rules. The vertical domain of all of these cells is i_a, and their vertical codomain is f. The horizontal domain of each γ_k is u_k, and the horizontal codomain is u''.

The fact that i_a is not a vertical codomain, nor i_d a vertical domain (except of course, trivially, of their own identity cells) restricts the possibilities for composition. In fact, the only composable arrangements in $\mathbf{D}$ are, up to middle-four interchange:

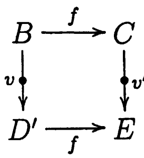
- $\gamma_k \circ \delta_k$; these are all equal and the composition will be denoted by β.

- $\alpha_{i_1} \circ \cdots \circ \alpha_{i_n}$; the compositions of these are free (that is, equal only if the arrangements are equal).

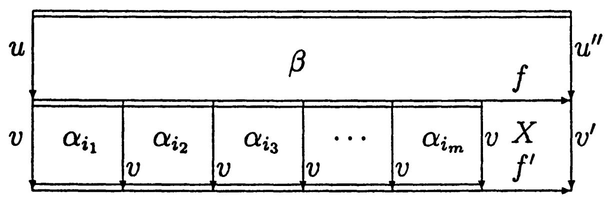
- $\gamma_k * (\alpha_{i_1} \circ \cdots \circ \alpha_{i_n})$; two such compositions, say $\gamma_k * (\alpha_{i_1} \circ \cdots \circ \alpha_{i_n})$ and $\gamma'_k * (\alpha_{i'_1} \circ \cdots \circ \alpha_{i'_{n'}})$, are defined to be equal if $k = k'$ and the string $A_{i_1} \cdots A_{i_n}$ can be converted to $A_{i'_1} \cdots A_{i'_{n'}}$ using only the rule R_k.

We note that the equality defined above is an equivalence relation, and closed under compositions; so no further equalities occur. Moreover, due to the properties that we have assumed for the rewrite system, equality can be checked in a time linear in the size of the arrangement.

Suppose that we introduce a further cell X into $\mathbf{D}$, with boundary



and which composes freely. We consider the following diagram in $\mathbf{D}[\mathbf{X}]$:



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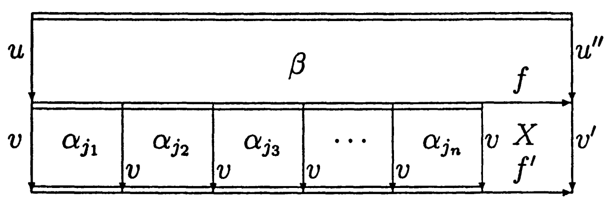
and ask whether it is equivalent in $\mathbf{D}[\mathbf{X}]$ to some other diagram



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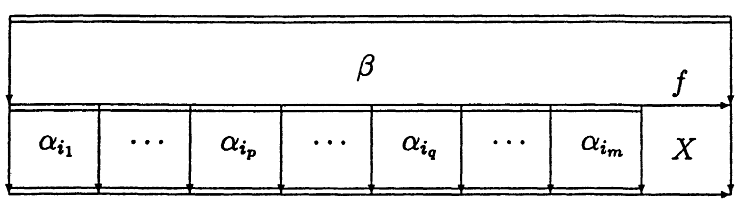
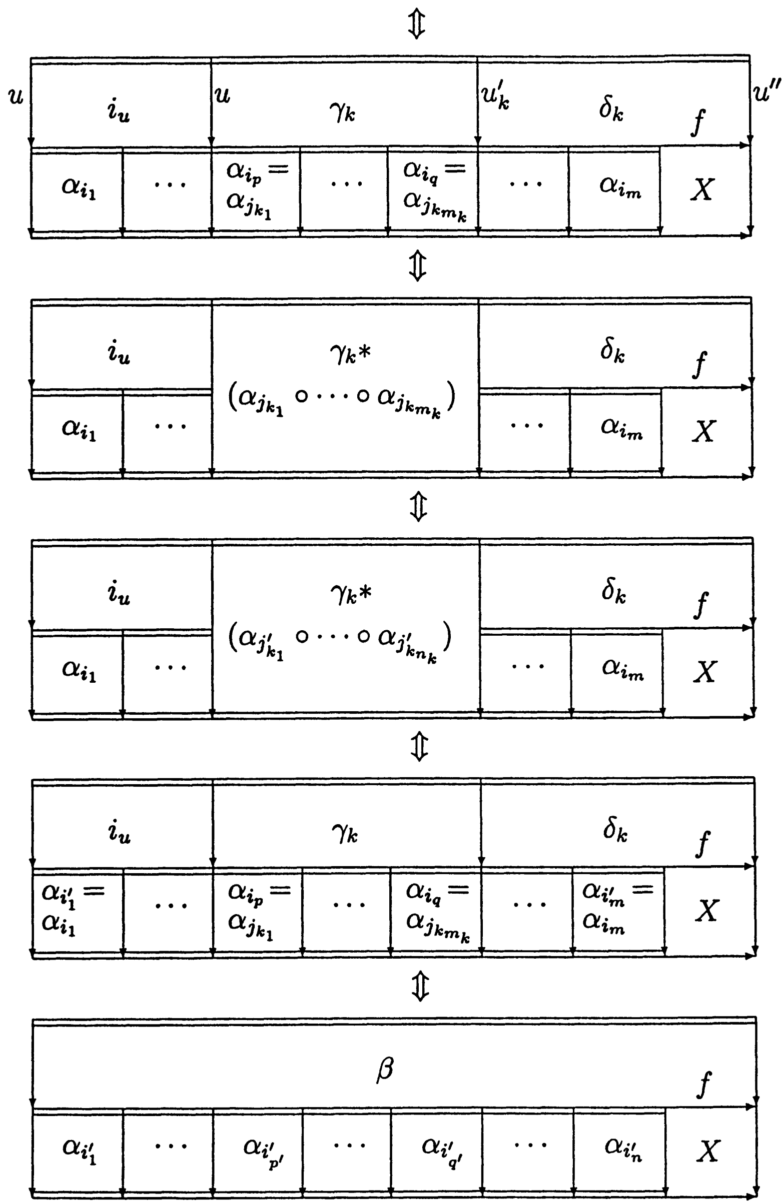
In general, answering this question is equivalent to the word problem for the rewrite system. It is not hard to see that the nontrivial equations in $\mathbf{D}[X]$ are generated by single rewrite rules R_k as follows.

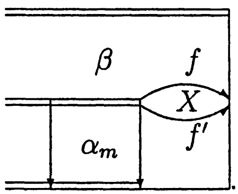




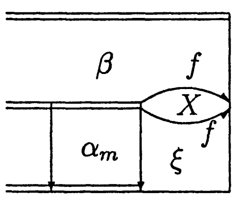
It follows that if $A_{i'1} \cdots A_{i'n}$ can be obtained from $A_{i1} \cdots A_{im}$ by a sequence of such substitutions, the corresponding compositions in $\mathbf{D}[x]$

are equal. However, while the individual substitution rules can be determined from equations in **D** in linear time, no composition in **D** "witnesses" any combination of these substitutions using more than one rule. Therefore, we conclude that, even though the equivalence problem for composable arrangements in **D** is decidable, the corresponding problem in $\mathbf{D}[X]$ is undecidable. ∎

Note The construction clearly works when the extension element has two adjacent edges that are not identities. Moreover, the construction may be adapted to a free extension by an element which has one edge that is not an identity, or two different and nonadjacent non-identity edges – note that even if the edge f' below is an identity, β can still take part in no composition that does not involve X.



Indeed, if the free element has two equal and nonadjacent non-identity edges, we can still adapt the construction: consider the double category suggested by the following arrangement



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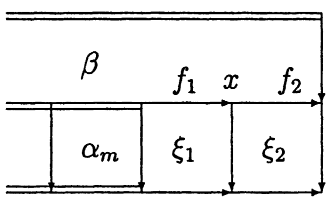
in which ξ composes freely with the α_i but all compositions in **D** involving ξ and any δ_k (and possibly other cells) are equal. Once more, it may be verified that composable arrangements in **D** either contain ξ in which case equivalence is trivial, or cannot contain cells composable to β in which case equivalence involves at most one rewrite rule. However, in $\mathbf{D}[X]$, equality of certain arrangements similar to that shown is undecidable.

The most difficult case is that in which both domains and both codomains of the new cell are identities. In this case the double category $\mathbf{D}$ which we construct contains two cells ξ_1 and ξ_2 as shown below, whose vertical domains f_1 and f_2 are not identity arrows, and such that $\xi_1 \circ \xi_2$ acts in the same way that ξ acted in the previous construction. The boundaries of the free cell X are identity arrows on the object x, which is the domain of f_2.



(4)

As before, the inclusion of the cell X at the point indicated prevents the wholesale identification of arrangements of this type.

We conclude:

Proposition 2.3 *If we replace any subset of the arrows in (2) by identities and optionally specify the horizontal (or vertical) arrows to be equal, there exists a double category* $\mathbf{D}$ *such that the equivalence problem for the free extension of* $\mathbf{D}$ *by a cell of that shape is undecidable.* ∎

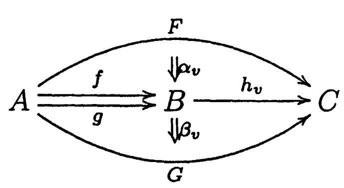
The following proposition shows that the absence of a cell in $\mathbf{D}$ with the same boundary as X is not an essential feature of the construction in the proof of Theorem 2.2. This proposition may be relevant in finding naturally arising double categories for which free extensions give rise to undecidability problems. Ordinarily, we do not concern ourselves too much with the presentations of double categories, and it is tempting to suppose that we can find the composition of any two cells. Even when the double category has countably many cells, however, this may require infinitely many separate pieces of information, enough for instance to list the halting behaviour of every state of a universal Turing machine.

In such a case, the assumption that we can know the composition of any two cells would imply mathematical omniscience. To avoid this, we are only considering double categories that are given by *finite* definitions of some sort or another.

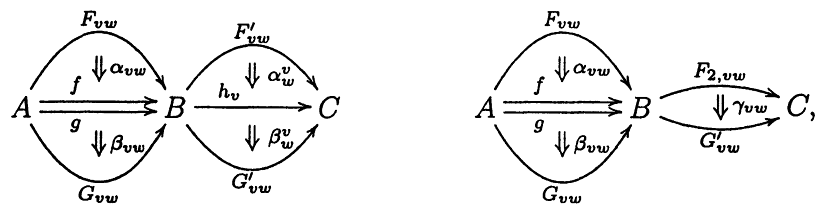
Proposition 2.4 *Let* $\mathbf{D}$ *be a double category which may be freely extended by a cell* X*, such that the equivalence of arrangements of cells of* $\mathbf{D}[X]$ *is undecidable; and let* $F\colon \mathbf{D} \to \mathbf{C}$ *be a faithful functor, the action of which on the elements of* $\mathbf{D}$ *is computable. Then* $\mathbf{C}$ *has a free extension* $\mathbf{C}[Y]$ *in which the equivalence of arrangements of cells is undecidable.*

Proof Suppose that this is not the case. Extend $\mathbf{C}$ freely by a a cell Y whose boundaries are the images of those of X under F. Then F induces a functor $\overline{F}\colon \mathbf{D}[X] \to \mathbf{C}[Y]$. Two arrangements $\mathcal{A}$ and $\mathcal{A}'$ of $\mathbf{D}[X]$ are equivalent if and only if $F\mathcal{A}$ and $F\mathcal{A}'$ are equivalent. This contradicts our assumption that the equivalence problem is undecidable for $\mathbf{D}[X]$. ∎

Observation The argument above cannot be used to show that freely adding a 2-cell to a 2-category may lead to undecidability for the equivalence relation for the new 2-cells. However, the following argument shows that this may lead to undecidability by showing how one may simulate the connectivity problem for an infinite bipartite graph in this situation. (This problem is known to be undecidable, see [7].) Let $\mathbf{G}$ be a bipartite graph for which connectivity is undecidable. Suppose that $\mathcal{A}$ is a 2-category and the cell $X\colon f \Rightarrow g\colon A \rightrightarrows B$ is to be added freely. Let $\mathcal{A}$ contain two arrows $F\colon A \to C$ and $G\colon A \to C$ and a triple $((\alpha_v, h_v, \beta_v)$ for every vertex v in G, where α_v and β_v are 2-cells as in the following diagram:



For each edge vw in G, let there be the following cells and arrows



satisfying

$$\begin{array}{ll} F_{vw} \circ F'_{vw} = F, & G_{vw} \circ G'_{vw} = G, \\ \alpha_{vw} \circ \alpha^v_w = \alpha_v, & \beta_{vw} \circ \beta^v_w = \beta_v, \\ \alpha^v_w * \beta^v_w = \gamma_{vw}, & \gamma_{vw} = \gamma_{wv}, \\ F_{vw} = F_{wv}, & G_{vw} = G_{wv}, \\ F'_{vw} = F'_{wv}, & G'_{vw} = G'_{wv}. \end{array}$$

The details of this example are left to the reader.

3 Free Extensions by Arrows

In this section we will consider, not only the free extension of a double category by one arrow, but also the more general case in which we add multiple arrows, possibly with non-trivial compositions relating them. Let $\mathbf{D}$ be a double category, and let $\mathcal{X}$ be a category with the same objects as $\mathbf{D}$. We will construct the double category $\mathbf{D}[\mathcal{X}]$ that is the pushout of the following diagram in $\mathbb{D}$–$\mathbf{Cat}$

$$\begin{array}{ccc} \mathrm{Disc}(\mathbf{D}) & \longrightarrow & I_H(\mathcal{X}) \\ \downarrow & & \\ \mathbf{D} & & \end{array}$$

where $\mathrm{Disc}(\mathbf{D})$ is the discrete double category (that is, with only identity arrows and cells) on the objects of $\mathbf{D}$, and I_H is as described in Section

1. That is, we extend the horizontal arrows of **D** by $\mathcal{X}$, and we extend the cells of **D** by the identity cells on arrows of $\mathcal{X}$, *i.e.* cells of the form

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ \| & i_x & \| \\ A & \xrightarrow[x]{} & B. \end{array}$$

Note that the cells from **D** only compose with these new cells along identity arrows

A case of particular interest is that in which $\mathcal{X}$ is the free category on a graph G. It may be seen that extending **D** by $\mathcal{X}$ is equivalent to extending **D** by, in turn, an arrow corresponding to each edge of G.

The components of $\mathbf{D}[\mathcal{X}]$ can be described as follows. The class of horizontal arrows is generated by the horizontal arrows of **D** and the arrows of $\mathcal{X}$. So without loss of generality we may assume that elements of $\mathbf{D}[\mathcal{X}]$ are of the form

$$A_0 \xrightarrow{h_0} B_1 \xrightarrow{x_1} A_1 \xrightarrow{h_1} B_2 \quad \cdots \quad B_n \xrightarrow{x_n} A_n \xrightarrow{h_n} B_{n+1}$$

with h_i in D_H and x_i in $\mathcal{X}$. The cells of $\mathbf{D}[\mathcal{X}]$ are generated by the cells in **D** and the (vertical) identity cells i_x for the new arrows. These identity cells can only compose with cells of **D** along vertical identity arrows, giving composable arrangements of the form

$$\begin{array}{ccccccccccccc} A_0 & \xrightarrow{h_0} & B_1 & \xrightarrow{x_1} & A_1 & \xrightarrow{h_1} & B_2 & \xrightarrow{x_2} & A_2 & & A_n & \xrightarrow{h_n} & B_{n+1} \\ {\scriptstyle w}\downarrow & \alpha_0 & \| & i_{x_1} & \| & \alpha_1 & \| & i_{x_2} & \| & \cdots & \| & \alpha_n & \downarrow{\scriptstyle w'} \\ A_0' & \xrightarrow[h_1']{} & B_1 & \xrightarrow[x_1]{} & A_1 & \xrightarrow[h_2']{} & B_2 & \xrightarrow[x_2]{} & A_2 & & A_n & \xrightarrow[h_n']{} & B_{n+1}'. \end{array} \tag{5}$$

The structure of $\mathbf{D}[\mathcal{X}]$ therefore depends heavily on the nature of the cells in **D** with one or more identity arrows as horizontal domain or codomain. We will say that two composable arrangements are equivalent when they compose to the same cells in $\mathbf{D}[\mathcal{X}]$.

Suppose that $D_\Box$ contains no horizontal one-sided cells except for the identities on objects. Then none of the arrows h_i or h_i' in (5) are identities. If also none of the x_i are identities, we call such an arrangement an *h-arrangement* and the resulting cell in $\mathbf{D}[\mathcal{X}]$ an *h-cell*; these exhaust the new cells in $\mathbf{D}[\mathcal{X}]$.

Proposition 3.1 *If the double category* $\mathbf{D}$ *has no horizontal one-sided cells, then each cell* α *in* $\mathbf{D}[\mathcal{X}]$ *is a cell of* $\mathbf{D}$ *or an h-cell.*

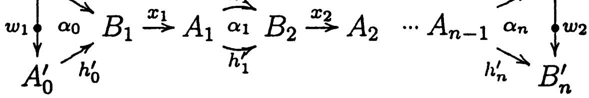
Proof We will show this by induction on the number of cells involved in the composable arrangement. If there is only one cell α, it is either a cell in $\mathbf{D}$, or a vertical identity of an arrow of $\mathcal{X}$ (a special case of an h-cell). Otherwise, without loss of generality, we may assume the last composition to be vertical, that is, $\alpha = \alpha' * \alpha''$. There are four cases to be considered, depending on the types of α' and α''. We will use the following lemma:

Lemma 3.2 *The domains and codomains of h-cells are not arrows of* $\mathbf{D}$.

Let such a domain or codomain be $h_0 \circ x_1 \circ \cdots \circ x_n \circ h_n$. The (non-identity) factors x_i of such an arrow are separated from each other by (non-identity) factors of the form h_j and therefore cannot cancel. ∎

From this lemma it follows that if one of α' and α'' is an h-cell, then so is the other. If $\operatorname{cod}\alpha' = \operatorname{dom}\alpha''$, they must have the same length and their component cells can be composed vertically, to yield another h-cell. ∎

Note The h-arrangements will also be depicted as:



Lemma 3.3 *Two h-arrangements in* $D[\mathcal{X}]_\square$ *are equivalent if and only if all their corresponding components are equal.*

Proof It suffices to show that no composable arrangement can be composed in two different ways to yield distinct cells of this form. We will prove this by induction on the number of cells in the arrangement.

Assume the arrangement $\mathcal{A}$ contains only one cell. Clearly this cell is of the form i_x, where $x: B \longrightarrow A$ is an arrow in $\mathcal{X}$; and the unique h-cell to which $\mathcal{A}$ may be composed is $i_{I_B} \circ i_x \circ i_{I_A}$.

Suppose that $\mathcal{A}$ consists of more than one cell, and that every arrangement smaller than $\mathcal{A}$ which composes to an h-cell does so uniquely. At least one of the following three situations occurs:

1. $\mathcal{A} = \mathcal{A}' \circ \mathcal{A}''$, where $\mathcal{A}'$ and $\mathcal{A}''$ both compose to h-cells;
2. $\mathcal{A} = \mathcal{A}' \circ \mathcal{A}''$, where one of $\mathcal{A}'$ and $\mathcal{A}''$ composes to a cell in $\mathbf{D}$ (and hence contains only cells of $\mathbf{D}$) and the other one composes to an h-cell;
3. $\mathcal{A} = \mathcal{A}' * \mathcal{A}''$ (which implies that $\mathcal{A}'$ and $\mathcal{A}''$ compose to h-cells).

If $\mathcal{A}$ may be factored in more than one way, Theorem 1.2 of [4] applied to the double category of arrangements guarantees that the composition is independent of the choice. Combining the same theorem (applied to $\mathbf{D}$) with the inductive hypothesis, we see that in each case $\mathcal{A}'$ and $\mathcal{A}''$ compose uniquely, from which the result follows. ∎

It follows from Lemma 3.2 that h-cells are never equal to cells in $D_\square$. So every cell in $\mathbf{D}[\mathcal{X}]$ has a unique representation as either an h-cell or a cell in $D_\square$ and we can consider this as a normal form for the cells in $\mathbf{D}[\mathcal{X}]$. In particular this shows:

Theorem 3.4 *Let $\mathbf{D}$ be a double category that does not contain any non-identity zero- or one-sided cells. If the equivalence of composable arrangements in $\mathbf{D}$ is decidable, then so is the corresponding problem in $\mathbf{D}[\mathcal{X}]$.* ∎

Remark This result is also applicable to 2-categories without non-trivial cells with the identity as domain or codomain.

4 Horizontal Arrangements

In this section we consider the case where **D** contains no non-identity zero-sided cells, but contains (without loss of generality) horizontal one-sided cells.

4.1 Split Horizontal Cells One can compose the horizontal one sided cells with cells of the form i_x to obtain composable arrangements of the forms

$$\begin{array}{ccccccccccc}
C_1 & \xrightarrow{b_1} & C_1 & \xrightarrow{x_1} & C_2 & \cdots & C_{m-1} & \xrightarrow{x_{m-1}} & C_m & \xrightarrow{b_m} & C_m \\
\| & \beta_1 & \| & i_{x_1} & \| & & \| & i_{x_{m-1}} & \| & \beta_m & \| \\
C_1 & = & C_1 & \xrightarrow[x_1]{} & C_2 & \cdots & C_{m-1} & \xrightarrow[x_{m-1}]{} & C_m & = & C_m
\end{array} \qquad (6)$$

and

$$\begin{array}{ccccccccccc}
C'_1 & = & C'_1 & \xrightarrow{x'_1} & C'_2 & \cdots & C'_{n-1} & \xrightarrow{x'_{n-1}} & C'_n & = & C'_n \\
\| & \gamma_1 & \| & i_{x'_1} & \| & & \| & i_{x'_{n-1}} & \| & \gamma_n & \| \\
C'_1 & \xrightarrow[c_1]{} & C'_1 & \xrightarrow[x'_1]{} & C'_2 & \cdots & C'_{n-1} & \xrightarrow[x'_{n-1}]{} & C'_n & \xrightarrow[c_n]{} & C'_n
\end{array} \qquad (7)$$

We will call these *uh-arrangements* and *lh-arrangements* respectively, and their compositions *uh-cells* and *lh-cells*. We represent these arrangements as

$$\begin{array}{ccc}
 & \overset{p}{\rightsquigarrow} & \\
\| & \beta & \| \\
 & \underset{x}{\longrightarrow} &
\end{array} \quad \text{and} \quad \begin{array}{ccc}
 & \overset{x'}{\longrightarrow} & \\
\| & \gamma & \| \\
 & \underset{q}{\rightsquigarrow} &
\end{array} \qquad (8)$$

where the cells are

$$\beta = \beta_1 \circ i_{x_1} \circ \cdots \circ i_{x_{m-1}} \circ \beta_m \text{ and } \gamma = \gamma_1 \circ i_{x'_1} \circ \cdots \circ i_{x'_{n-1}} \circ \gamma_n,$$

the (straight) arrows are

$$x = x_1 \circ x_2 \circ \cdots \circ x_{m-1}, \text{ and } x' = x'_1 \circ x'_2 \circ \cdots \circ x'_{n-1},$$

and the squiggly arrows are

$$p = b_1 \circ x_1 \circ b_2 \circ \cdots \circ x_{m-1} \circ b_m, \text{ and } q = c_1 \circ x'_1 \circ c_2 \circ \cdots \circ x'_{n-1} \circ c_n.$$

Note that

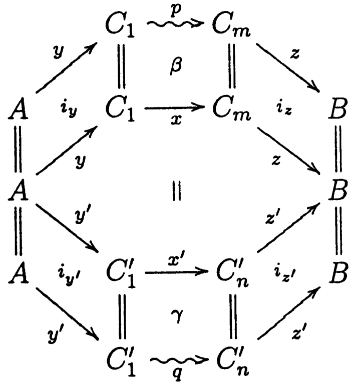
$$\text{if } p = q, \text{ then } x = x'. \tag{9}$$

If $y: A \longrightarrow C_1$, $y': A \longrightarrow C'_1$, $z: C_m \longrightarrow B$, and $z': C'_n \longrightarrow B$ are arrows in $\mathcal{X}$, such that $y \circ x \circ z = y' \circ x' \circ z'$, the cells (6) and (7), and the identities on the arrows y, y', z, and z' can be composed as

$$(i_y \circ \beta * i_z) * (i_{y'} \circ \gamma \circ i_{z'}),$$

or as a diagram,

(10)



We call these *split horizontal* arrangements and split horizontal cells, or *sh-arrangements* and *sh-cells* for short.

Theorem 4.1 *Let* $\mathbf{D}$ *be a double category without non-identity zero-sided cells and* $\mathcal{X}$ *any category. The split horizontal cells of* $\mathbf{D}[\mathcal{X}]$ *form a 2-category, with horizontal and vertical composition given by pasting.*

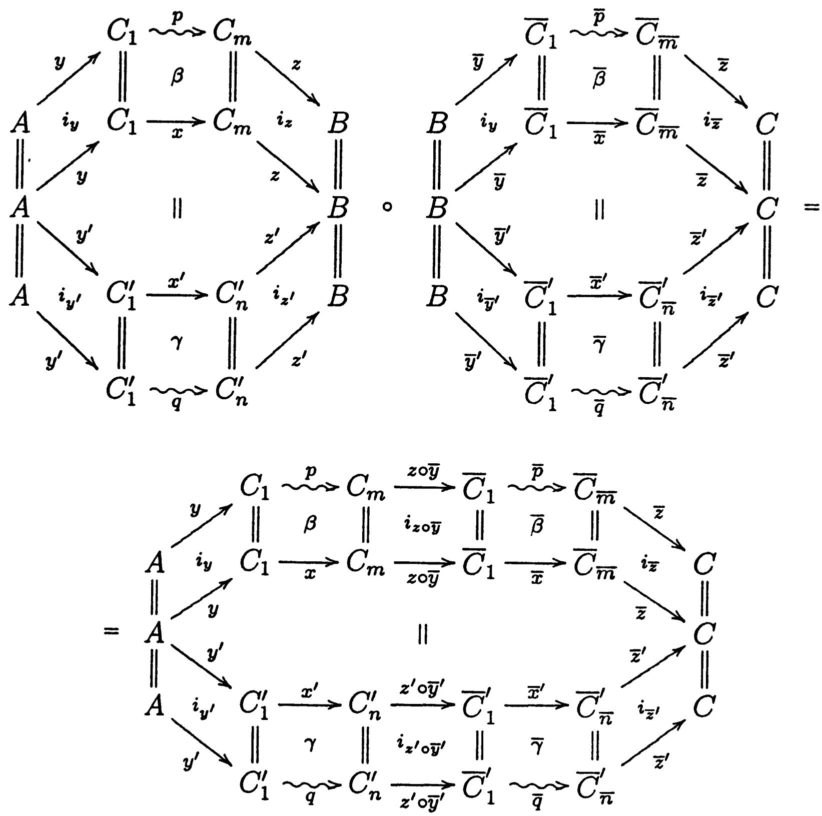
Proof It is clear that the horizontal domains and codomains of these cells are identities. It remains to be shown that they are closed under horizontal and vertical composition.

By middle four interchange for pastings, the horizontal composition of two split horizontal cells is easily seen to be a split horizontal cell again:

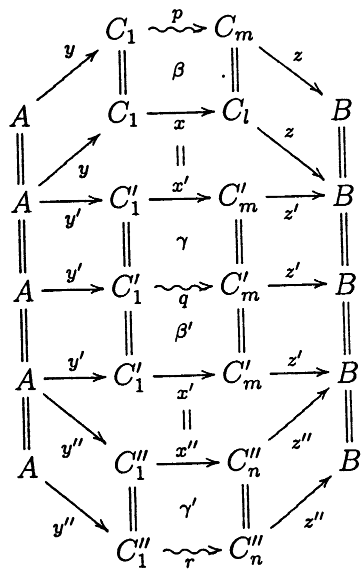
$$[(i_y \circ \beta \circ i_z) * (i_{y'} \circ \gamma \circ i_{z'})] \circ [(i_{\overline{y}} \circ \overline{\beta} \circ i_{\overline{z}}) * (i_{\overline{y}'} \circ \overline{\gamma} \circ i_{\overline{z}'})] =$$

$$(i_y \circ (\beta \circ i_{z * \overline{y}} \circ \overline{\beta}) \circ i_{\overline{z}}) * (i_{y'} \circ (\gamma \circ i_{z' \circ \overline{y}'} \circ \overline{\gamma}) \circ i_{\overline{z}'})$$

In a diagram this can be seen as:



When we vertically compose two split horizontal cells, they paste as follows:



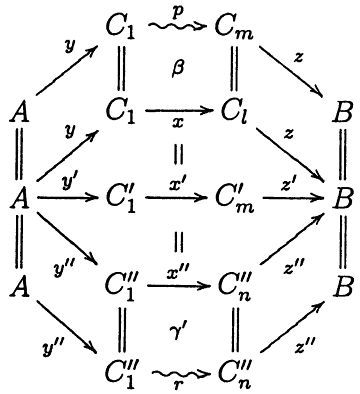
(11)

Note that the vertical domain of γ and the vertical codomain of β' are both equal to $x' = x'_1 \circ \cdots \circ x'_m$; this follows from (9).

Expanding $\gamma * \beta'$ yields

$$
\begin{array}{ccccccccccc}
C'_1 & = & C'_1 & \xrightarrow{x'_1} & C'_2 & = & C'_2 & \cdots & C'_m & = & C'_m \\
\| & \gamma_1 & \| & i_{x'_1} & \| & \gamma_2 & \| & \cdots & \| & \gamma_m & \| \\
C'_1 & \xrightarrow{h'_1} & C'_1 & \xrightarrow{x'_1} & C'_2 & \xrightarrow{h'_2} & C'_2 & \cdots & C'_m & \xrightarrow{h'_m} & C'_m \\
\| & \beta'_1 & \| & i_{x'_1} & \| & \beta'_2 & \| & \cdots & \| & \beta'_m & \| \\
C'_1 & = & C'_1 & \xrightarrow[x'_1]{} & C'_2 & = & C'_2 & \cdots & C'_m & = & C'_m
\end{array}
$$

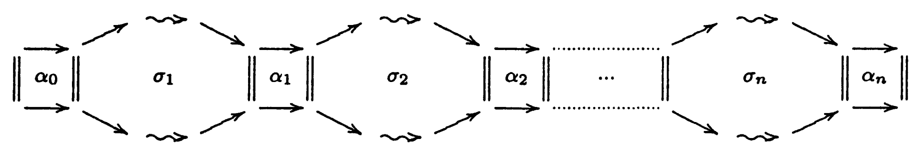
As the cells $\gamma_i * \beta'_i$ are zero-sided and hence by assumption are identities, this simplifies to $i_{x'_1 \circ \cdots \circ x'_n} = i_{x'}$, and diagram (11) becomes



which is clearly split horizontal.

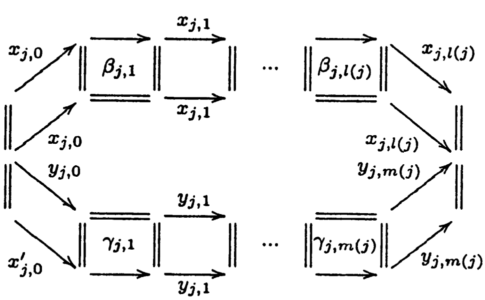
The middle four interchange law for cells in this 2-category follows from the corresponding law for pasting diagrams. ∎

4.2 H-Arrangements Composing split horizontal arrangements σ_i horizontally with horizontal cells α_i of **D** yields arrangements of the form



(12)

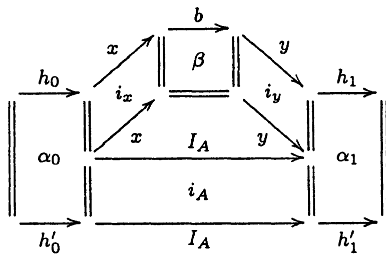
where σ_j equals



(13)

We call (12) an *H-arrangement* if none of the arrows $x_{j,k}$ or $y_{j,k}$ is an identity arrow and none of the α-cells, with the possible exception of α_0 and α_n, is one-sided. The resulting cell in $\mathrm{D}[\mathcal{X}]$ is an *H-cell.*

Note that if the category $\mathcal{X}$ has a pair of arrows $A \xrightarrow{x} B \xrightarrow{y} A$ with $x \circ y = I_A$, the following diagram gives a composable arrangement which is *not* an H-arrangement:



(14)

Theorem 4.2 *Let* D *be a double category without non-identity zero-sided cells, and* $\mathcal{X}$ *any category. The H-cells in* $\mathrm{D}[\mathcal{X}]$ *form a 2-category, with horizontal and vertical composition given by pasting.*

Proof Consider two H-arrangements

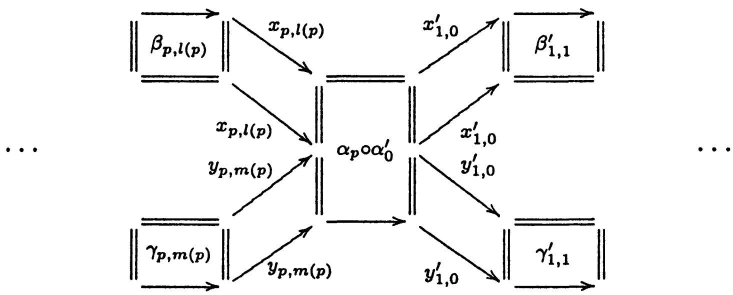
$$\mathcal{H} = \alpha_0 \circ \sigma_1 \circ \cdots \alpha_p \text{ and } \mathcal{H}' = \alpha_0' \circ \sigma_1' \circ \cdots \alpha_q'.$$

Suppose that the horizontal codomain of α_p is the horizontal domain of α_0', so that $\mathcal{H}$ and $\mathcal{H}'$ can be composed horizontally. If neither the vertical domain or codomain of $\alpha_l \circ \alpha_0'$ is an identity arrow, the horizontal composition

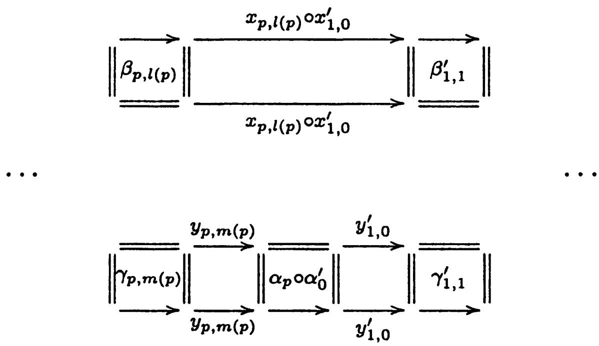
$$\mathcal{H} \circ \mathcal{H}' = \alpha_0 \circ \sigma_1 \circ \cdots \circ \sigma_p \circ (\alpha_p \circ \alpha_0') \circ \sigma_1' \circ \cdots \circ \sigma_q' \circ \alpha_q'$$

is *ipso facto* an H-arrangement. If (without loss of generality) the vertical domain of $\alpha_p \circ \alpha_0'$ is an identity arrow, the part of the arrangement

$\mathcal{H} \circ \mathcal{H}'$ around the cell $\alpha_p \circ \alpha'_0$ is of the form



This can be rewritten as



This is an H-arrangement unless $x_{p,l(p)} \circ x'_{1,0} = I$. In that case, compose $\beta_{p,l(p)}$ and $\beta'_{1,1}$ horizontally to obtain an H-arrangement. This completes the proof that H-arrangements are closed under horizontal composition.

Now assume that $\mathcal{H}$ and $\mathcal{H}'$ above can be composed vertically. We prove by induction on $l + m$ (the number of two-sided horizontal cells in these H-arrangements) that

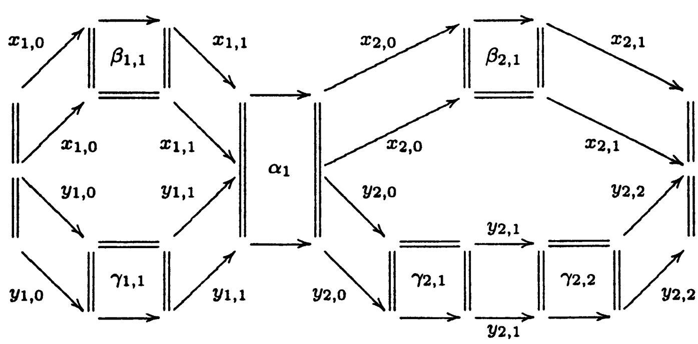
$$(\alpha_0 \circ \sigma_1 \circ \cdots \alpha_p) * (\alpha'_0 \circ \sigma'_1 \circ \cdots \alpha'_q) \tag{15}$$

is an H-cell. If $l = m = 1$, σ_1 and σ'_1 can be composed vertically since there are no identity arrows from $\mathcal{X}$ in their domains or codomains. It follows by middle-four interchange that $\mathcal{H} * \mathcal{H}' = (\alpha_0 * \alpha'_0) \circ (\sigma_1 * \sigma'_1) \circ (\alpha_1 * \alpha'_1)$. It follows from Theorem 4.1 that $\sigma_1 * \sigma'_1$ is an sh-arrangement and it is obvious that it has no identity arrows from $\mathcal{X}$ in its domain or codomain, since these are the same as from the original σ_1 and σ'_1 respectively. So (15) is an H-cell.

Suppose that the vertical composition of any two H-cells that contain fewer α-cells than $\mathcal{H}$ and $\mathcal{H}'$ is an H-cell. Since $\mathcal{H}$ and $\mathcal{H}'$ are vertically composable, we have $\mathrm{cod}\,(\mathcal{H}) = \mathrm{dom}\,(\mathcal{H}')$. Moreover, by the definition of H-arrangement, each has the same representation as a composition $h_0 \circ y_1 \cdots \circ y_n \circ h_n$, where each of the h_i is the vertical codomain of either a horizontal one-sided cell γ or a horizontal two-sided cell α. It is also the vertical domain of either a horizontal two-sided cell α' or a horizontal one-sided cell β'. So each α matches up with either an α' or a β'.

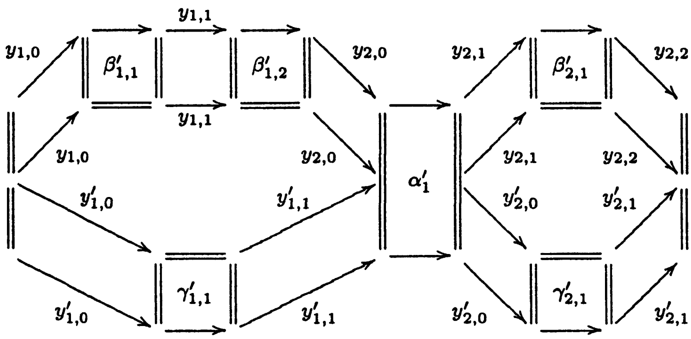
If there is an index j with $1 < j < p$, such that α_j matches up with a cell α'_k, let I_A be the horizontal codomain of α_j. In this case we can rewrite the composition (15) as the horizontal composition of $(\alpha_0 \circ \sigma_1 \circ \cdots \circ \sigma_j \circ \alpha_j) * (\alpha'_0 \sigma'_1 \circ \cdots \circ \sigma'_k \circ \alpha'_k)$ and $(i_A \circ \sigma_{j+1} \circ \alpha_{j+1} \circ \cdots \circ \sigma_p \circ \alpha_p) * (i_A \circ \sigma'_{k+1} \circ \alpha'_{k+1} \circ \cdots \circ \sigma'_q \circ \alpha'_q)$. By the induction hypothesis, each of these cells is an H-cell; this implies by the first part of this proof that (15) is an H-cell.

The only case left is when every α in $\mathcal{H}$ matches up with a β in $\mathcal{H}'$, and consequently every α in $\mathcal{H}'$ matches up with a γ in $\mathcal{H}$. For example, the vertical composition of

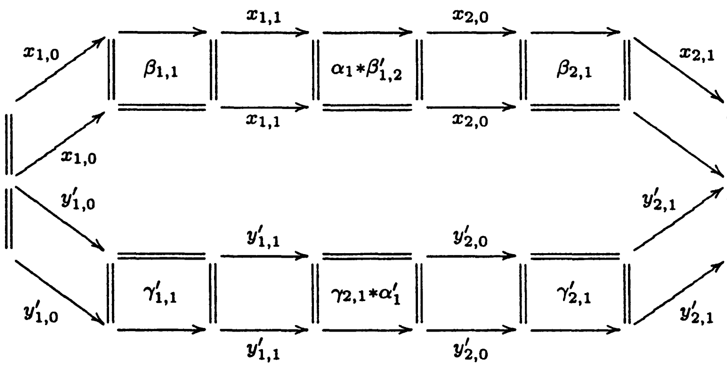


(16)

and

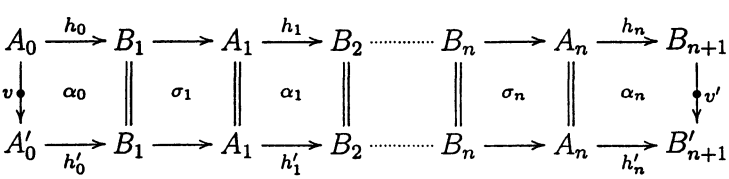


can be represented by the H-arrangement



In general, the vertical composition will be of the form $(\alpha_0 * \alpha'_0) \circ \sigma \circ (\alpha_p * \alpha'_q)$, where σ is an sh-arrangement. ∎

4.3 Augmented H-Cells It is obvious that not all cells in the double category $\mathbf{D}[\mathcal{X}]$ are H-cells. We need to extend the class of H-arrangements by allowing horizontal composition with three sided cells from $\mathbf{D}$. We call the resulting arrangement an *augmented H-arrangement.*



(17)

This composes to an *augmented H-cell* in $\mathbf{D}[\mathcal{X}]$. If the category $\mathcal{X}$ contains non-trivial factorizations of identity arrows, in general, $\mathbf{D}[\mathcal{X}]$ will contain cells which are not augmented H-cells nor cells from $\mathbf{D}$ (cf. diagram 14)). However, we will show in the next section that if $\mathcal{X}$ does not contain such factorizations, every cell in $\mathbf{D}[\mathcal{X}]$ is either an augmented H-cell or a cell from $\mathbf{D}$.

5 Decidability Results

In this section we want to discuss some conditions on the category $\mathcal{X}$ that will ensure that the equivalence problem for arrangements in $\mathbf{D}[\mathcal{X}]$ is decidable. One would like to be able to decide whether two augmented H-arrangements are equivalent. The equivalence relation on these arrangements is in general non-trivial; for instance, in (16) α_1 may factor as a vertical composition of two one-sided cells, possibly even in more than one way. If horizontal cells factor at most uniquely into pairs of one-sided cells, we have (rather uninterestingly) decidability. However, if two one-sided cells can be composed and then factored into a different pair, we will need conditions to limit the propagation of this equivalence. For example, if in (16) we have

$$x_{2,0} = y_{2,0} = y_{2,0} \circ y_{2,1}, \quad x_{2,1} = y_{2,1} \circ y_{2,2} = y_{2,2}, \tag{18}$$

the cell $\beta_{2,1}$ can be composed with either $\gamma_{2,1}$ or $\gamma_{2,2}$, and then subsequently refactored, perhaps differently. In Section 6 we will show how this may lead to undecidability. The equations (18) form one example of what we will call an Escher factorization (named in honour of the creator of such works as "*Waterfall*" and "*Ascending and Descending*"). (The reader may recall that Ehresmann introduced factorizations similar to these in the proof of Proposition 52 in [12].)

Definition An *Escher factorization* is a quadruple (f, g_1, g_2, h) of arrows

$$A \xrightarrow{f} B \xrightarrow{g_1} B' \xrightarrow{g_2} B \xrightarrow{h} C \tag{19}$$

such that $f = fg_1g_2$, $g_1g_2h = h$, where at least one of the g_i is not an identity. We call the object B the *base* of the Escher factorization. Let

$g = g_1 g_2$. If $g \neq I$ we call the Escher factorization *proper.*

Escher factorizations are never found (for instance) in a partial order, or a category that is generated freely by a graph. On the other hand, any category with initial and terminal objects and some non-identity endomorphism has an Escher factorization, as does any category with a nontrivial idempotent arrow.

Lemma 5.1 *Any nontrivial factorization of an identity arrow gives rise to an Escher factorization.*

Proof Let $I = kl$ be a nontrivial factorization (*i.e.*, neither factor equals I). This gives rise to an Escher factorization (I, k, l, I). ∎

Note that the example above with an idempotent arrow shows that not every Escher factorization arises this way.

In this section we show that the absence of Escher factorizations in $\mathcal{X}$ is sufficient to ensure that the equivalence of composable arrangements in $\mathbf{D}[\mathcal{X}]$ is decidable, even when $\mathbf{D}$ contains non-identity one- or zero-sided cells. (It will be shown later (Theorem 5.5) that this condition is also, in a certain sense, necessary.)

Theorem 5.2 *Let* $\mathbf{D}$ *be a double category without non-identity zero-sided cells and* $\mathcal{X}$ *a category without nontrivial factorizations of identities; then each cell in* $\mathbf{D}[\mathcal{X}]$ *is either a cell in* $D_\Box$ *or an augmented H-cell.*

Proof Let $\mathcal{A}$ be an arrangement of cells in $D[\mathcal{X}]_\Box$. If $\mathcal{A}$ contains only cells in $D_\Box$, it is obvious that it composes to a cell in $D_\Box$. We will prove by induction on the number of cells in $\mathcal{A}$ that if $\mathcal{A}$ contains a cell of the form i_x, then the composition of $\mathcal{A}$ can be written as an augmented H-cell.

If $\mathcal{A}$ contains only one cell, it has to be of the form i_x. This is a special case of an sh-cell, so $\mathcal{A}$ composes trivially to an augmented H-cell. Let $\mathcal{A}$ contain more than one cell, and assume that every arrangement with fewer cells than $\mathcal{A}$ containing cells of the form i_x composes to an augmented H-cell. The arrangement $\mathcal{A}$ must be a composition of smaller arrangements, either

$$\mathcal{A} = \mathcal{A}_1 \circ \mathcal{A}_2, \text{ or } \mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2.$$

In the first case, by the induction hypothesis, $\mathcal{A}_1$ and $\mathcal{A}_2$ compose to cells that are either augmented H-cells or in $D_\square$; it is obvious that in this case $\mathcal{A}$ composes to an augmented H-cell.

In the second case, both $\mathcal{A}_1$ and $\mathcal{A}_2$ compose to augmented H-cells, say $\mathcal{A}_1 = \alpha_0 \circ \sigma_1 \circ \alpha_1 \circ \cdots \circ \sigma_l \circ \alpha_l$ and $\mathcal{A}_2 = \alpha'_0 \circ \sigma'_1 \circ \alpha'_1 \circ \cdots \circ \sigma'_m \circ \alpha'_m$. The arrangements $\mathcal{A}_1 = \sigma_1 \circ \alpha_1 \circ \cdots \circ \sigma_l$ and $\mathcal{A}_2 = \sigma'_1 \circ \alpha'_1 \circ \cdots \circ \sigma'_m$ are H-cells. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are vertically composable, the cells $\mathcal{A}'_1$ and $\mathcal{A}'_2$ are also vertically composable, and

$$\mathcal{A}_1 * \mathcal{A}_2 = (\alpha_0 * \alpha'_0) \circ (\mathcal{A}'_1 * \mathcal{A}'_2) \circ (\alpha_{l+1} * \alpha'_{m+1}).$$

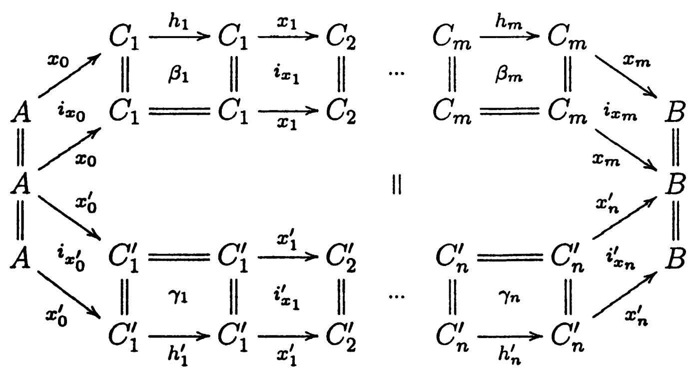
It follows from Theorem 4.2 that $\mathcal{A}'_1 * \mathcal{A}'_2$ is an H-cell; therefore $\mathcal{A}_1 * \mathcal{A}_2$ is an augmented H-cell. ∎

Call an arrangement $\alpha_0 \circ \sigma_1 \circ \alpha_1 \circ \cdots \circ \sigma_n \alpha_n$ an *expanded* H-arrangement if none of the α_i with $i = 1, \ldots, n-1$ can be factored as a vertical composition of two one-sided cells.

Proposition 5.3 *Every H-arrangement is equivalent to an expanded H-arrangement.*

Proof If the H-arrangement is not expanded, find a factorization for each horizontal two-sided α-cell as a vertical composition of two three sided cells $\beta * \gamma$ and use β and γ in the place of α and 'disconnect' them. ∎

In a split horizontal arrangement



the cells β_i and γ_j are called *partners* if $x_0 \circ \cdots \circ x_{i-1} = x'_0 \circ \cdots \circ x'_{j-1}$ and $x_i \circ \cdots \circ x_m = x'_i \circ \cdots \circ x'_n$. (Note that a cell may possibly have more than one partner or no partners at all.)

Lemma 5.4 *If $\mathcal{X}$ has no Escher factorization, then each β-cell in a split horizontal arrangement in $\mathbf{D}[\mathcal{X}]$ has maximally one partner.*

Proof Suppose that β_i has both γ_j and γ_k as partners and that $j < k$. Let $f = x_0 \circ \cdots \circ x_{i-1}$, $g = x'_j \circ \cdots \circ x'_{k-1}$ and $h = x_i \circ \cdots \circ x_m$. The fact that β_i and γ_j are partners implies that $f = x'_0 \circ \cdots \circ x'_{j-1}$ and $h = x'_j \circ \cdots \circ x'_n$. Now it follows from the fact that β_i and γ_k are partners that $f = fg$ and $gh = h$. Moreover, $x_j \neq I$, so we have an Escher factorization in $\mathcal{X}$, contradiction. We conclude that β_i cannot have more than one partner. ∎

Theorem 5.5 *If a category $\mathcal{X}$ has no Escher factorizations, and the double category $\mathbf{D}$ has no zero-sided cells except for identities, then the equivalence of augmented H-arrangements in $\mathbf{D}[\mathcal{X}]$ is decidable.*

Proof We show that under the conditions of the theorem two expanded augmented H-arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$ only compose to the same cells when they satisfy the following requirements

1. $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same domain and codomain, say

$$\mathrm{dom}(\mathcal{H}_1) = \mathrm{dom}(\mathcal{H}_2) = h_0 \circ x_1 \circ h_1 \circ \cdots x_n \circ h_n$$

   and

$$\mathrm{cod}(\mathcal{H}_1) = \mathrm{cod}(\mathcal{H}_2) = h'_0 \circ x'_1 \circ h'_1 \circ \cdots x'_m \circ h'_m;$$

2. $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same shape, *i.e.* h_i is the domain of an α (resp. β) cell in $\mathcal{H}_1$ if and only if it is the domain of an α (resp. β) cell in $\mathcal{H}_2$ and analogously for the h'_i;
3. Corresponding three-sided and two-sided cells from $\mathbf{D}$ (*i.e.*, the α-cells) in $\mathcal{H}_1$ and $\mathcal{H}_2$ are equal;
4. Corresponding one-sided cells β_i in $\mathcal{H}_1$ and β'_i in $\mathcal{H}_2$ are equal or they have partners γ_j and γ_j' respectively and $\beta_i * \gamma_j = \beta'_i * \gamma'_j$.

Note that since partners are unique if they exist, this makes the equivalence relation on augmented H-arrangements decidable.

To prove our claim we represent the expanded extended H-arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$ by diagrams as in Figure 2 to emphasize that these are a composable arrangement of cells.

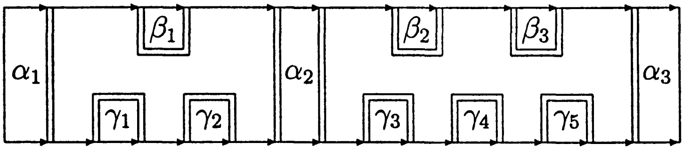


Figure 2: A composable arrangement in $\mathbf{D}[X]$.

The unmarked regions are filled with identity cells for arrows in $\mathcal{X}$. An arrangement as in Figure 2 can be changed into an equivalent arrangement by any of the following operations:

- Factoring cells from $\mathbf{D}$;
- Applying generalized associativity for double categories;
- Composing cells from $\mathbf{D}$ in $\mathbf{D}$.

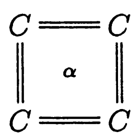
If two or more cells of $\mathbf{D}$ are factorized vertically along horizontal arrows which are not identity arrows, and the factors are recomposed, the pairing must (due to the absence of Escher factorizations) preserve left-to-right order. Thus, the first factor of one cannot then be composed with the second factor of another, (even if the horizontal arrows are equal) as this would leave some remaining horizontal arrows unpaired in the interior of the arrangement. The β- and γ-cells don't have non-trivial factorizations along identity arrows, since the category $\mathbf{D}$ does not contain any non-trivial zero-sided cells. By assumption, the arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$ are expanded, so the α-cells cannot be factorized. So the only thing one can do is to compose β- and γ-cells which are partners and possibly factor them in a different way. Since the fact that $\mathcal{X}$ does not contain Escher factorizations implies that partners are unique, this implies the last three conditions above. ∎

Corollary 5.6 *Let* D *be a double category without non-identity zero-sided cells and* $\mathcal{X}$ *a category without Escher factorizations. If the equivalence problem for composable arrangements in* D *is decidable, then so is the corresponding problem in* $\mathbf{D}[\mathcal{X}]$.

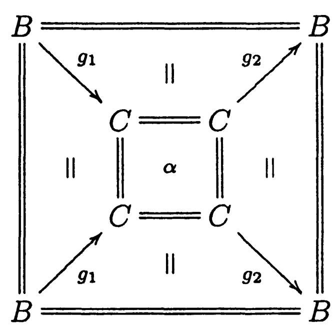
Remark It is obvious that this result also holds when D is a 2-category.

6 Undecidability Results

Let $\mathcal{X}$ be a category with a one-sided inverse pair of arrows $g_1: B \to C$, $g_2: C \to B$, with $g_1 \circ g_2 = I_B$. Let D be a double category with the same objects as $\mathcal{X}$ and a non-identity zero-sided cell with base C:



Then we have the following composable arrangement in $\mathbf{D}[\mathcal{X}]$:



(20)

Assume furthermore that D contains a set of cells as in diagram (4) of Section 2.2, where $x = B$. The zero-sided cell (20) composes freely with these cells. There are therefore composable arrangements in $\mathbf{D}[\mathcal{X}]$, containing (20) for which equivalence is undecidable.

Theorem 6.1 *For any category* $\mathcal{X}$ *containing a non-trivial factorization of an identity arrow there exists a double category* D *such that the* *equivalence of composable arrangements in* $\mathbf{D}[\mathcal{X}]$ *is undecidable, whereas the corresponding problem in* $\mathbf{D}$ *is decidable.*

The factorization of the identity arrow used above is an (improper) Escher factorization; a proper Escher factorization will not work. In the remainder of this section we present an alternative construction, based on the *abacus* model for universal computation, using a proper Escher factorization.

Abacuses, first introduced by Lambek [14] and (as "register machines") by Minsky [15], are a class of abstract models for computation, similar to Turing machines. Like Turing machines, they have finitely many program states; however, instead of having infinitely many registers each with finite capacity, they have finitely many registers, each capable of storing an arbitrary natural number. It is known [15] that there exists a 2-register abacus which is universal, and hence has undecidable halting problem. Because it has a fixed, finite, number of components, an abacus may be modeled more easily than a Turing machine or general rewrite system by structures that have, in some sense, constant size.

6.1 Definition of an abacus For the convenience of the reader we recall (cf. [9]):

Definition 6.2 *An (n-register)* abacus *consists of*
(i) *A finite set of* states S;
(ii) *Variables* $X_1, \cdots, X_n$ *in* $\mathbb{N}$ *which are considered as the contents of the registers;*
(iii) *A function Inst:* $S \to \{\mathtt{INCX}, \mathtt{INCY}, \mathtt{DECX}, \mathtt{DECY}, \mathtt{HALT}\}$;
(iv) *A* starting state $s_0 \in S$;
(v) *Transition functions* $\sigma\colon S \to S$ *and* $\sigma'\colon S \to S$.

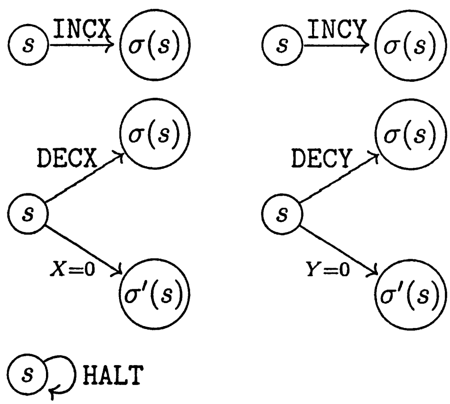
In this paper we will only be interested in the notion of a 2-register abacus, so from now on we will take $n = 2$ in the definition above, and use X and Y as variables for the registers. The behaviour of the abacus is a function $S \times \mathbb{N} \times \mathbb{N} \to S \times \mathbb{N} \times \mathbb{N}$

$$\mathfrak{S} = (s, X, Y) \overset{\Sigma}{\mapsto} \mathfrak{S}' = (s', X', Y')$$

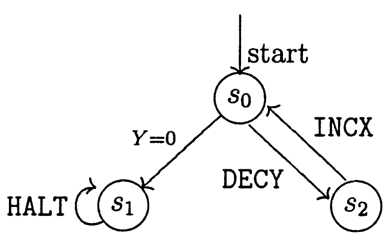
defined as follows

$$(s',X',Y') = \begin{cases} (\sigma(s), X+1, Y) & \text{if } \mathrm{Inst}(s) = \mathtt{INCX} \\ (\sigma(s), X, Y+1) & \text{if } \mathrm{Inst}(s) = \mathtt{INCY} \\ (\sigma(s), X-1, Y) & \text{if } \mathrm{Inst}(s) = \mathtt{DECX} \text{ and } X > 0 \\ (\sigma'(s), X, Y) & \text{if } \mathrm{Inst}(s) = \mathtt{DECX} \text{ and } X = 0 \\ (\sigma(s), X, Y-1) & \text{if } \mathrm{Inst}(s) = \mathtt{DECY} \text{ and } Y > 0 \\ (\sigma'(s), X, Y) & \text{if } \mathrm{Inst}(s) = \mathtt{DECY} \text{ and } Y = 0 \\ (s, X, Y) & \text{if } \mathrm{Inst}(s) = \mathtt{HALT}. \end{cases}$$

We can represent this by a graph whose nodes are the elements of S and whose edges are of the form



So the σ and σ' permit branching at nodes s with $\mathrm{Inst}(s) \in \{\mathtt{DECX}, \mathtt{DECY}\}$. Here is a simple example which adds the contents of the X-register and the Y-register and puts the sum in the X-register and 0 in the Y-register.



If one starts with $X = 4$ and $Y = 3$, this abacus would go through the following states: $(s_0, 4, 3) \mapsto (s_2, 4, 2) \mapsto (s_0, 5, 2) \mapsto (s_2, 5, 1) \mapsto (s_0, 6, 1) \mapsto (s_2, 6, 0) \mapsto (s_0, 7, 0) \mapsto (s_1, 7, 0)$ HALT.

6.2 The double category $\mathbf{D}_{\mathbb{A}}$ Let $\mathcal{X}$ be a category containing an Escher factorization $A \xrightarrow{f} B \xrightarrow{g} B \xrightarrow{h} C$, as in (19), and $\mathbb{A}$ a 2-register abacus. With the notation as above, the double category $\mathbf{D}_{\mathbb{A}}$ is defined as follows. The objects of $\mathbf{D}_{\mathbb{A}}$ are the objects of $\mathcal{X}$. Let B be the base of the Escher factorization in $\mathcal{X}$. The horizontal arrows of $\mathbf{D}_{\mathbb{A}}$ are generated by

$$\{c, r, x, y \colon B \to B\}.$$

If the Escher factorization is not proper, i.e. $g = I$, its composites $g_1 \colon B \to D$ and $g_2 \colon D \to B$ are not identities. In this case, the double category $\mathbf{D}_{\mathbb{A}}$ also contains an arrow $d \colon D \to D$

The cells of $\mathbf{D}_{\mathbb{A}}$ are generated by:

1. For each $n \in \mathbb{N}$, two cells

$$\begin{array}{ccc} B & \xrightarrow{x} & B \\ \| & X(n) & \| \\ B & = & B \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \xrightarrow{y} & B \\ \| & Y(n) & \| \\ B & = & B \end{array}$$

2. For each program control state s, a cell

$$\begin{array}{ccc} B & \xrightarrow{c} & B \\ \| & C(s) & \| \\ B & = & B \end{array}$$

3. For each instruction $\rho \in \{\mathtt{INCX}, \mathtt{INCY}\}$, two cells

$$\begin{array}{ccc} B & = & B \\ \| & \Theta(\rho,0) & \| \\ B & \xrightarrow[r]{} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} B & = & B \\ \| & \Theta(\rho,1) & \| \\ B & \xrightarrow[r]{} & B \end{array}$$

4. For each instruction $\rho \in \{\mathtt{DECX}, \mathtt{DECY}\}$, three cells

$$\begin{array}{ccc} B & = & B \\ \| & \Theta(\rho,0) & \| \\ B & \xrightarrow[r]{} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} B & = & B \\ \| & \Theta(\rho,1) & \| \\ B & \xrightarrow[r]{} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} B & = & B \\ \| & \Theta(\rho,2) & \| \\ B & \xrightarrow[r]{} & B \end{array}$$

5. A cell

$$\begin{array}{ccc} B & = & B \\ \| & H & \| \\ B & \xrightarrow[r]{} & B \end{array}$$

6. If the Escher factorization is not proper, a cell

$$\begin{array}{ccc} D & \xrightarrow{d} & D \\ \| & \delta & \| \\ D & = & D \end{array}$$

7. Cells

$$\begin{array}{ccc} B & \xrightarrow{k} & B \\ \| & \omega^k_{k'} & \| \\ B & \xrightarrow[k']{} & B \end{array}$$

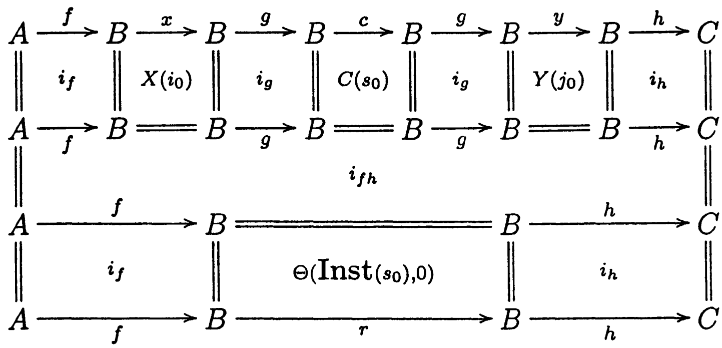
for $k' \in \{I_B, r\}$ and any arrow k which factors through both c and either x or y.

The arrows compose freely. The cells compose freely with the following exceptions. Any composable arrangement with domain k and codomain k' as in (7) above, involving both a cell of the form $C(s)$ and a cell of the form $X(i)$ or $Y(i)$, composes to $\omega^k_{k'}$. We also have the following specific identities:

$$\begin{aligned}
C(s) \circ \Theta(\rho, 1) &= C(\sigma(s)) \circ \Theta(\mathbf{Inst}(\sigma(s)), 0) \\
C(s) \circ \Theta(\rho, 2) &= C(\sigma'(s)) \circ \Theta(\mathbf{Inst}(\sigma'(s)), 0) \\
X(i) \circ \Theta(\mathtt{INCX}, 0) &= X(i+1) \circ \Theta(\mathtt{INCX}, 1) \\
X(0) \circ \Theta(\mathtt{DECX}, 0) &= X(0) \circ \Theta(\mathtt{DECX}, 2) \\
X(i) \circ \Theta(\mathtt{DECX}, 0) &= X(i-1) \circ \Theta(\mathtt{DECX}, 1) \text{ if } i > 0 \\
Y(i) \circ \Theta(\mathtt{INCY}, 0) &= Y(i+1) \circ \Theta(\mathtt{INCY}, 1) \\
Y(0) \circ \Theta(\mathtt{DECY}, 0) &= Y(0) \circ \Theta(\mathtt{DECY}, 2) \\
Y(i) \circ \Theta(\mathtt{DECY}, 0) &= Y(i-1) \circ \Theta(\mathtt{DECY}, 1) \text{ if } i > 0
\end{aligned}$$

It is straightforward to verify that equality in $(D_{\mathbb{A}})_{\square}$ is decidable in linear time.

We will now show that given any initial state of the abacus there are composable arrangements in $\mathbf{D}_{\mathbb{A}}[\mathcal{X}]$ which are equivalent if and only if the abacus halts from a certain initial state. Let (s_0, i_0, j_0) be the initial state of the abacus $\mathbb{A}$. If the Escher factorization in $\mathcal{X}$ is proper, consider the following composable arrangement in $\mathbf{D}_{\mathbb{A}}[\mathcal{X}]$:



The upper edge of the central identity cell i_{fh} may be factored as fh in three formally different ways, *viz*, $f(ggh)$, $(fg)(gh)$, and $(fgg)h$. Thus, such an arrangement may always be composed as

$$(i_f * i_f) \circ (X(m) * \Theta) \circ ((i_g \circ C(s) \circ i_g \circ Y(n) \circ i_h) * i_h),$$

$$((i_f \circ X(m) \circ i_g) * i_f) \circ (C(s) * \Theta) \circ (i_g \circ Y(n) \circ i_h) * i_h), \text{ or}$$

$$((i_f \circ X(m) \circ i_g \circ C(s) \circ i_g) * i_f) \circ (Y(n) * \Theta) \circ (i_h * i_h).$$

In each case, it may be possible to refactor the central term in a different way, using the identities above. We will call a refactorization that replaces the left side of one of these identities by the right side "forward", and one that replaces a right side by a left side "backward".

Examining the identities, we see that after a forward refactorization involving an X cell, we have only three possibilities.

- If we compose the resulting Θ cell with a Y cell, the composite has no other factorizations.

- If we compose the Θ cell with the X cell again, this is backward, and the inverse of the previous refactorization.

- If we compose the Θ cell with the C cell, there is a unique refactorization, which is again forward.

The possibilities after a forward refactorization that involves a Y cell are analogous. After a forward refactorization involving a C cell, there may be multiple backwards options, but there will be a unique forward refactorization, always involving an X cell or a Y cell.

Refactorizations involving C cells correspond to changes of state in the abacus, and those involving X or Y cells correspond to register operations. A pair of forward refactorizations, changing first an X or Y cell and then a C cell, corresponds to one step in the operation of the abacus. Backward refactorizations correspond to backward steps in a reversible abacus; the halting problem for such a machine is known to be equivalent to that for the corresponding non-reversible machine (see the proof of Theorem 1 in [9]). Determining whether there exist integers m and n and a control state s such that the composition of this arrangement is equal to the composition of the arrangement (21) is equivalent to the halting problem for the abacus $\mathbb{A}$.

$$
\begin{array}{ccccccccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{x} & B & \xrightarrow{g} & B & \xrightarrow{c} & B & \xrightarrow{g} & B & \xrightarrow{y} & B & \xrightarrow{h} & C \\
\| & i_f & \| & X(m) & \| & i_g & \| & C(s) & \| & i_g & \| & Y(n) & \| & i_h & \| \\
A & \xrightarrow{f} & B & = & B & \xrightarrow[g]{} & B & = & B & \xrightarrow[g]{} & B & = & B & \xrightarrow[h]{} & C \\
\| & & & & & & & i_{fh} & & & & & & & \| \\
A & & \xrightarrow{f} & & B & & & = & & & B & & \xrightarrow{h} & & B \\
\| & & i_f & & \| & & & H & & & \| & & i_h & & \| \\
A & & \xrightarrow[f]{} & & B & & & \xrightarrow[r]{} & & & B & & \xrightarrow[h]{} & & C
\end{array}
\tag{21}
$$

If the Escher factorization is not proper, consider the following composable arrangement in $\mathbf{D}_{\mathbb{A}}[\mathcal{X}]$:

$$
\begin{array}{c}
A \xrightarrow{f} B \xrightarrow{x} B \xrightarrow{g_1} D \xrightarrow{d} D \xrightarrow{g_2} B \xrightarrow{c} B \xrightarrow{g_1} D \xrightarrow{d} D \xrightarrow{g_2} B \xrightarrow{y} B \xrightarrow{h} C \\
\Vert \; i_f \; \Vert \; X(i_0) \; \Vert \; i_{g_1} \; \Vert \; \delta \; \Vert \; i_{g_2} \; \Vert \; C(s_0) \; \Vert \; i_{g_1} \; \Vert \; \delta \; \Vert \; i_{g_2} \; \Vert \; Y(j_0) \; \Vert \; i_h \; \Vert \\
A \xrightarrow{f} B = B \xrightarrow[g_1]{} D = D \xrightarrow[g_2]{} B = B \xrightarrow[g_1]{} D = D \xrightarrow[g_2]{} B = B \xrightarrow[h]{} C \\
\Vert \qquad\qquad i_{fh} \qquad\qquad \Vert \\
A \xrightarrow{\quad f \quad} B = = = = B \xrightarrow{\quad h \quad} C \\
\Vert \quad i_f \quad \Vert \quad \Theta(\mathbf{Inst}(s_0),0) \quad \Vert \quad i_h \quad \Vert \\
A \xrightarrow[f]{\qquad} B \xrightarrow[r]{\qquad\qquad} B \xrightarrow[h]{\qquad} C
\end{array}
$$

Determining whether there exist integers m and n and a control state s such that the composition of this arrangement is equal to the composition of the arrangement (22) is equivalent to the halting problem for the abacus $\mathbb{A}$.

$$
\begin{array}{c}
A \xrightarrow{f} B \xrightarrow{x} B \xrightarrow{g_1} D \xrightarrow{d} D \xrightarrow{g_2} B \xrightarrow{c} B \xrightarrow{g_1} D \xrightarrow{d} D \xrightarrow{g_2} B \xrightarrow{y} B \xrightarrow{h} C \\
\Vert \; i_f \; \Vert \; X(m) \; \Vert \; i_{g_1} \; \Vert \; \delta \; \Vert \; i_{g_2} \; \Vert \; C(s) \; \Vert \; i_{g_1} \; \Vert \; \delta \; \Vert \; i_{g_2} \; \Vert \; Y(n) \; \Vert \; i_h \; \Vert \\
A \xrightarrow{f} B = B \xrightarrow[g_1]{} D = D \xrightarrow[g_2]{} B = B \xrightarrow[g_1]{} D = D \xrightarrow[g_2]{} B = B \xrightarrow[h]{} C \\
\Vert \qquad\qquad i_{fh} \qquad\qquad \Vert \\
A \xrightarrow{\quad f \quad} B = = = = B \xrightarrow{\quad h \quad} C \\
\Vert \qquad\quad \Vert \quad H \quad \Vert \quad i_h \quad \Vert \\
A \xrightarrow[f]{\qquad} B \xrightarrow[r]{\qquad\qquad} B \xrightarrow[h]{\qquad} C
\end{array} \tag{22}
$$

We conclude:

Theorem 6.3 *For every category $\mathcal{X}$ containing an Escher factorization there is a double category* $\mathbf{D}$ *such that the equivalence of composable arrangements in* $\mathbf{D}[\mathcal{X}]$ *is undecidable.*

Remark Since all the cells in the constructions of this section have identity cells as horizontal domains and codomains, it is clear that this result is also applicable for $\mathbf{D}$ a 2-category. Note that this is not true for the result of Theorem 6.1. So for 2-categories only the proper Escher factorizations can cause undecidability.

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