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COCHAIN OPERATIONS AND HIGHER COHOMOLOGY OPERATIONS

By Stephan KLAUS

RESUME. Etendant un programme initié par Kristensen, cet article donne une construction algébrique des opérations de cohomologie d'ordre supérieur instables par des opérations de cochaîne simpliciale. Des pyramides d'opérations cocycle sont considérées, qui peuvent être utilisées pour une seconde construction des opérations de cohomologie d'ordre supérieur.

1. Introduction

In this paper we consider the relation between cohomology operations and simplicial cochain operations. This program was initialized by L. Kristensen in the case of (stable) primary, secondary and tertiary cohomology operations.

The method is strong enough that Kristensen obtained sum, product and evaluation formulas for secondary cohomology operations by skilful combinatorial computations with special cochain operations ([8], [9], [10]). As significant examples of applications we mention the independent proof for the Hopf invariant one theorem by the computation of Kristensen of Massey products in the Steenrod algebra [11], the examination of the β-family in stable homotopy by L. Smith using a secondary Hopf invariant ([16]) and the authors result on Brown-Kervaire invariants for Spin manifolds ([2], [3]).

Unfortunately, there are serious problems to construct cohomology operations of higher order by the method of Kristensen. The reason is that his construction of secondary and tertiary cohomology operations works similar to the definition of Massey products in the homology of a differential algebra. Due to the non-linear character of cochain operations, already in the tertiary case one has to introduce additional terms correcting the lack of linearity and it seems not to be known how to construct them in the general case of higher order operations ([12]).

In the second section we give a survey on the method of Kristensen of constructing primary and secondary cohomology operations for singular cohomology of topological spaces by cochain operations. We also demonstrate the problem of constructing tertiary cohomology operations by his method.

In the third section we give a modified construction of primary cohomology operations which works unstable and is rather quick, using only basic facts from simplicial topology. We obtain the stable representation result of Kristensen as a corollary.

In the fourth section we generalize our construction to the case of arbitrary unstable higher order cohomology operations. This also gives a description of the cohomology s-stage spaces in terms of cochain operations.

In the last section, we consider pyramids of cocycle operations, which also can be used to produce higher order cohomology operations by a glueing construction of the author [5] which works similar as a Massey product, but avoids the problems related to the non-linear character of cochain and cocyle operations.

As Kristensen we work in the category of simplicial sets whose homotopy category is equivalent to that of the category of topological spaces. The equivalence is given by the adjoint functors total simplicial set of a space and geometric realization of a simplicial set. The reason for working with simplicial sets is the representability of the cochain functor in that category. Let $C^k(-;\pi)$ denote the functor of normalized k-cochains on simplicial sets with coefficients in an abelian group π. It is essential to consider normalized cochains because then we have the following result of Eilenberg-MacLane ([15]):

$$C^k(X_\bullet;\pi) = mor(X_\bullet, L(\pi, k+1)_\bullet),$$

where mor denotes the set of simplicial maps and $L(\pi, k+1)_\bullet$ is the acyclic simplicial abelian group which corresponds to the acyclic chain complex

$$l(\pi, k+1)_* := (\ldots \to 0 \to \pi \xrightarrow{id} \pi \to 0 \to \ldots \to 0)$$

(non-zero in dimensions k and $k+1$) by Dold-Kan equivalence. The representing isomorphism is given by pulling back the fundamental cochain $\iota \in C^k(L(\pi, k+1)_\bullet; \pi)$. (The reason of the index shift in the common notation of $L(\pi, k+1)_\bullet$ is that this space is isomorphic to the simplicial path space of the Eilenberg-MacLane space $K(\pi, k+1)_\bullet$.)

We remark that the simplicial chain functor cannot be represented as $mor(L_\bullet, -)$ with some space $L_\bullet$, because the chains on a product $X_\bullet \times Y_\bullet$ are not given by the product of chains on $X_\bullet$ and on $Y_\bullet$. This is the reason that we prefer to work with cochain operations instead of chain operations.

In this paper, we consider the relation between cochain operations and higher cohomology operations, but we do not make explicit computations. We will consider the combinatorial structure of cochain operations more closely in [7], where we make explicit computations by relating cochain operations to coordinate arrangements over finite fields.

2. On Kristensen's theory of cochain operations

Let π, π' be abelian groups and $m, n \in \mathbb{N}$.

Definition 1. An **unstable cochain operation** of type (π, m, π', n) is a natural transformation from the functor $C^m(-;\pi)$ to the functor $C^n(-;\pi')$. We do not assume any condition of linearity. Denote the set of these operations by $\mathcal{O}(\pi, m, \pi', n)$. For fixed π, a **cochain operation of degree** k (in the sense of Kristensen) is an element $a = (a_m)_{m\geq 0}$ in the set

$$\mathcal{O}^k := \prod_{m\geq 0} \mathcal{O}(\pi, m, \pi, m+k).$$

Cochain operations actually form a set by the representability result of Eilenberg and MacLane, which gives by the Yoneda lemma

$$\mathcal{O}(\pi, m, \pi', n) = mor(L(\pi, m+1)_\bullet, L(\pi', n+1)_\bullet) = C^n(L(\pi, m+1)_\bullet; \pi').$$

As $L(\pi, m+1)_n = 0$ for $n < m$, it follows $\mathcal{O}(\pi, m, \pi', n) = 0$ for $n < m$. By $L(\pi, m+1)_m = \pi$ with $0 \in \pi$ being the only degenerate simplex in dimension m, it follows $\mathcal{O}(\pi, m, \pi', m) = \{f : \pi \to \pi' \mid f(0) = 0\}$. In particular, this includes the $\mathbb{Z}$-linear maps $Hom(\pi, \pi')$ as a subset.

The coboundary homomorphism $d : C^m(-;\pi) \to C^{m+1}(-;\pi)$ gives a further example of a cochain operation

$$d \in \mathcal{O}(\pi, m, \pi, m+1).$$

All cochain operations $a \in \mathcal{O}(\pi, m, \pi', n)$ with $n > 0$ satisfy $a(0) = 0$, where 0 denotes the zero cochain on a simplicial set $X_\bullet$. This follows easily utilizing naturality for the projection $X_\bullet \to *_\bullet$ to a point. Addition in π' defines on $\mathcal{O}(\pi, m, \pi', n)$ the structure of an abelian group which corresponds to the 'pointwise' addition in the set of mappings $mor(L(\pi, m+1)_\bullet, L(\pi', n+1)_\bullet)$ induced from the simplicial abelian group $L(\pi', n+1)_\bullet$. Composition of cochain operations gives a map

$$\mathcal{O}(\pi_2, m_2, \pi_3, m_3) \times \mathcal{O}(\pi_1, m_1, \pi_2, m_2) \to \mathcal{O}(\pi_1, m_1, \pi_3, m_3)$$

which is linear in the left variable, *but non-linear in the right variable*, in general. In particular, $(a + a')b = ab + a'b$, but $a(b + b') \neq ab + ab'$ in general.

Definition 2. [8] For fixed π, Kristensen defined

$$\mathcal{O}^* := \prod_{k \geq 0} \mathcal{O}^k,$$

which is a graded abelian group (in fact, Kristensen mainly considered the case $\pi = \mathbb{Z}/p$). Composition $\mathcal{O}^k \times \mathcal{O}^l \to \mathcal{O}^{k+l}$ is defined by $(ab)_m := a_{m+l} b_m$, which again is linear only in the left variable. The coboundary homomorphisms for all m form an element in $\mathcal{O}^1$ which we also denote by d.

Kristensen defined a linear differential

$$\Delta : \mathcal{O}^k \to \mathcal{O}^{k+1},$$

$$\Delta a := da + (-1)^k ad.$$

In fact, Δ is a linear map as d is linear, and a differential because

$$\Delta\Delta a = d\big(da + (-1)^k ad\big) + (-1)^{k+1}\big(da + (-1)^k ad\big)d = 0$$

where we used that $a(0) = 0$. For $x \in C^m(X; \mathbb{Z}/p)$ with $dx = 0$ and $a \in \mathcal{O}^k$ with $\Delta a = 0$, we have $da(x) = ad(x) = a(0) = 0$. Thus a maps cocycles to cocycles. Now, the basic result of Kristensen is the following:

Theorem 1. [8] *Let $a \in \mathcal{O}^k$ with $\Delta a = 0$. Then for any cocycle $x \in Z^m(X;\mathbb{Z}/p)$, the cohomology class of $a(x) \in Z^{m+k}(X;\mathbb{Z}/p)$ depends only on the cohomology class of x. Hence the cochain operation a defines a cohomology operation, denoted by $[a]$. This cohomology operation is stable, and the canonical map $a \mapsto [a]$ gives an isomorphism $H(\mathcal{O}^*;\Delta) = A^*$ between the homology of $\mathcal{O}^*$ with respect to the differential Δ and the Steenrod algebra A^* mod p.*

By definition, we have $[ab] = [a][b]$ for two Δ-cycles in $\mathcal{O}^*$. Thus, the semi-linear composition in $\mathcal{O}^*$ corresponds to the bilinear composition (i.e., multiplication) in A^*. Kristensen gave various generalizations of his basic result to the case of several variables (for example, $A^* \oplus A^*$, [8]) and to the multi-linear case (for example, $A^* \otimes A^*$, [9]). The most general case is handled in [10]. The proofs of Kristensen are somewhat involved and use acyclic models and inductive computations using the explicit structure of the cohomology of Eilenberg-MacLane spaces known by the results of Serre and Cartan [18].

We will give a quick proof for the construction of unstable operations in the next section which gives Kristensen's basic result as a corollary.

The non-linearity of maps in $\mathcal{O}^*$ that is the reason for $\mathcal{O}^*$ not being an algebra with respect to composition cannot be avoided by restriction to linear maps in $\mathcal{O}^*$. With the exception of coefficient homomorphisms it is not possible to produce non-trivial cohomology operations from linear cochain operations, even in the case of stable (hence linear) cohomology operations (see [7]).

Now we come to Kristensen's representation of secondary cohomology operations by cochain operations. It is well-known that a stable secondary cohomology operation is associated to a relation between primary cohomology operations [1]. Let

$$\sum_{i=1}^{s} \alpha_i \beta_i = 0$$

be a relation of degree k in A^* and choose fixed representatives of the primary cohomology operations by cochain operations: $\alpha_i = [a_i]$ and

$\beta_i = [b_i]$ for all i. Then the cochain operation

$$r := \sum_{i=1}^{s} a_i b_i$$

in $\mathcal{O}^k$ satisfies $\Delta r = 0$ and $[r] = 0$. Hence there exists a cochain operation $R \in \mathcal{O}^{k-1}$ with $\Delta R = r$ which we also fix for the construction. Now, let $[x] \in H^m(X;\mathbb{Z}/p)$ be a cohomology class represented by a cocycle $x \in Z^m(X;\mathbb{Z}/p)$ such that $[x]$ lies in the kernel of all β_i. Choose cochains $y_i \in C^{m+|\beta_i|-1}(X;\mathbb{Z}/p)$ such that $dy_i = b_i(x)$ for all i. Then it is straightforward to check that the cochain

$$z := -R(x) + \sum_{i=1}^{s} a_i(y_i)$$

in $C^{m+k-1}(X;\mathbb{Z}/p)$ is a cocycle. We denote by $ker(\beta)$ the kernel of all β_i in $H^m(X;\mathbb{Z}/p)$ and by $coker(\alpha)$ the cokernel of the sum of all α_i into $H^{m+k-1}(X;\mathbb{Z}/p)$.

Theorem 2. [8] *The set of cohomology classes* $[z] \in H^{m+k-1}(X;\mathbb{Z}/p)$ *for all choices of* x *and* y_i *in the construction above only depends on the cohomology class of* x. *This defines a homomorphism*

$$\phi : ker(\beta) \to coker(\alpha)$$

$$[x] \mapsto \phi([x]) := \{[z]\}$$

that coincides with some secondary cohomology operation associated with the relation $\sum_{i=1}^{s} \alpha_i\beta_i = 0$. *Other choices of the cochain operation* R *for the relation give all secondary cohomology operations associated with the relation.*

In [17], it is shown how secondary operations in homotopy theory can be constructed from a 'relation of maps between pointed spaces' $A \to B \to C$ (i.e., the composition is null-homotopic) together with an explicit choice of zero-homotopy for the composition (see also [2], chapter 5). Thus the cochain operation R plays the role of a zero-homotopy for the composition of maps between Eilenberg-MacLane spaces which corresponds to the relation. We will come again to this point of view in the last section.

Kristensen not only handles the case of a stable secondary cohomology operation, but also of a 'metastable' operation, i.e. the composition $\sum_{i=1}^{s} \alpha_i\beta_i$ need not to vanish, but is an element in A^* of excess e. Then it is still possible to construct an unstable secondary cohomology operation in dimensions m smaller than e [8]. However, his method does not work for general unstable secondary cohomology operations.

Furthermore, there are problems with the construction of stable tertiary and higher order operations using $\mathcal{O}^*$ [12], [13]. The problem is that a definition as a higher Massey product in general does not work because of non-linearity of cochain operations:

Let α_1, α_2 and α_3 be stable primary cohomology operations which we represent by cochain operations a_1, a_2 and a_3 with $\Delta a_i = 0$. Assume that

$$\alpha_1\alpha_2 = 0 \quad \text{and} \quad \alpha_2\alpha_3 = 0.$$

Hence there are cochain operations R_{12} and R_{23} with

$$\Delta R_{12} = a_1a_2 \quad \text{and} \quad \Delta R_{23} = a_2a_3.$$

For a definition as a Massey product in the usual sense, we need to assume that the cochain operation

$$r := R_{12}a_3 - a_1R_{23}$$

represents a vanishing cohomology operation. Then we would get a further cochain operation R with $\Delta R = r$ which is necessary for the definition of the tertiary cohomology operation associated to α_1, α_2 and α_3. But, unfortunately, it holds

$$\begin{aligned}\Delta r &= (dR_{12}a_3 \pm R_{12}a_3d) - (da_1R_{23} \pm a_1R_{23}d) \\ &= a_1a_2a_3 \pm a_1(\pm R_{23}d \pm a_2a_3) \pm a_1R_{23}d,\end{aligned}$$

which in general does not vanish by the non-linearity of a_1 (here, we are lazy with the signs in Δ in order to simplify the notation). In [12] and [13], L. Kristensen and I. Madsen compensate this effect for tertiary operations by tricky modifications using cochain operations measuring the lack of linearity. Concerning higher order operations, we cite the authors from [13], p.145: "Massey products of length 4 and 5 can be defined in a similar fashion. The lack of distributivity in $\mathcal{O}^*$ makes the defining formulas rather involved. Because of this complication we are not able to define Massey products of arbitrary length."

In the fourth section, our modified method allows us to construct unstable higher cohomology operations of any order.

3. Unstable cochain operations and primary cohomology operations

For our modification of Kristensen's method we consider unstable cochain operations $\mathcal{O}^n_m := \mathcal{O}(\pi, m, \pi', n)$ instead of $\mathcal{O}^*$. Left and right composition with the coboundary homomorphism d define differentials

$$d^L : \mathcal{O}^n_m \to \mathcal{O}^{n+1}_m \quad \text{and} \quad d^R : \mathcal{O}^n_m \to \mathcal{O}^n_{m-1}.$$

Thus the Kristensen differential Δ is given by $d^L \pm d^R$. As d^L is just the differential in the cochain complex $C^*(L(\pi, m)_\bullet; \pi')$, it holds that $ker(d^L) = im(d^L)$ because $L(\pi, m)_\bullet$ is contractible.

We recall [18] that unstable primary cohomology operations of type (π, m, π', n) can be identified with the cohomology group

$$A^n_m := H^n(K(\pi, m)_\bullet; \pi')$$

of the Eilenberg-MacLane space $K(\pi, m)_\bullet$. Here, the connection between $L(\pi, m)_\bullet$ and $K(\pi, m)_\bullet$ (standard simplicial model) is given as follows [15]: The differential $d : L(\pi, m)_\bullet \to L(\pi, m + 1)_\bullet$ is a homomorphism of simplicial abelian groups with $ker(d) = K(\pi, m - 1)_\bullet$ and also $im(d) = K(\pi, m)_\bullet$. The space $L(\pi, m)_\bullet$ can be identified with the simplicial path space $PK(\pi, m + 1)_\bullet$ and d with the projection map p followed by inclusion. The simplicial loop space Ω of a Kan simplicial set respects the standard models of Eilenberg-MacLane spaces, i.e. $\Omega K(R, m)_\bullet = K(R, m - 1)_\bullet$ and $\Omega L(R, m)_\bullet = L(R, m - 1)_\bullet$.

Theorem 3. *Any unstable primary cohomology operation α of type (π, m, π', n) is represented by a cochain operation $a \in \mathcal{O}^n_m$ such that*

$$dad = 0.$$

This gives an isomorphism

$$A^n_m = \frac{\{a \in \mathcal{O}^n_m \mid dad = 0\}}{\{a \in \mathcal{O}^n_m \mid da = 0\} + \{a \in \mathcal{O}^n_m \mid ad = 0\}} =$$

$$= \frac{ker(d^L d^R)}{ker(d^L) + ker(d^R)}.$$

Proof. In order to simplify notation, we set $K^m_\bullet := K(\pi, m)_\bullet$ and $L^m_\bullet := L(\pi, m+1)_\bullet$. We drop the coefficients π' in the notation of cochains, cocycles and coboundaries on a space. Now, restriction to the subspace $i : K^m_\bullet \subset L^m_\bullet$ gives an epimorphism of cochain complexes

$$i^* : C^*(L^m_\bullet) \to C^*(K^m_\bullet)$$

and we first have to determine $(i^*)^{-1} Z^n(K^m_\bullet)$ as a subset of $\mathcal{O}^n_m = C^n(L^m_\bullet)$. As $ip = d : L^{m-1}_\bullet \to L^m_\bullet$, and

$$p^* : C^*(K^m_\bullet) \to C^*(L^{m-1}_\bullet)$$

is injective, it follows $(i^*)^{-1} Z^n(K^m_\bullet) = \{a \mid dd^*a = 0\} = ker(d^L d^R)$. At last we have to determine $(i^*)^{-1} B^n(K^m_\bullet)$ as a subset of $\mathcal{O}^n_m$. This is given by $(i^*)^{-1}(d^L C^{n-1}(K^m_\bullet)) = d^L C^{n-1}(L^m_\bullet) + ker(i^*)$, but $im(d^L) = ker(d^L)$ on $L^m_\bullet$ and $ker(i^*) = ker(d^*) = ker(d^R)$. □

We recall that the **cohomology suspension** $\alpha' \in A^{n-1}_{m-1}$ of a cohomology operation $\alpha \in A^n_m$ is defined by the action of α in the suspension of any space, i.e. by the following commutative diagram:

$$\begin{array}{ccc} H^m(\Sigma X; \pi) & \xrightarrow{\alpha} & H^n(\Sigma X; \pi') \\ \sigma \uparrow & & \sigma \uparrow \\ H^{m-1}(X; \pi) & \xrightarrow{\alpha'} & H^{n-1}(X; \pi') \end{array}$$

where σ denotes the suspension isomorphism. As the loop functor is adjoint to the suspension functor, it follows that α' is given by $\Omega\alpha$ where we consider α as a map between Eilenberg-MacLane spaces.

Theorem 4. *For $a \in \mathcal{O}^n_m$, the condition $dad = 0$ is equivalent to the existence of $a' \in \mathcal{O}^{n-1}_{m-1}$ with $ad = da'$. Denote by $\alpha \in A^n_m$ the cohomology operation represented by a, then a' represents the cohomology suspension $\Omega\alpha \in A^{n-1}_{m-1}$ of α.*

Proof. By the exactness of d^L, the condition $d^L(ad) = dad = 0$ is equivalent to the existence of a' with $ad = d^L(a') = da'$. As $da'd = add = 0$, the cochain operation a' also induces a cohomology operation. Let $a|_K$ be the restriction of a to $K(\pi, m)_\bullet$. As a maps cocycles to cocycles, $a|_K$ takes values in $K(\pi', n)_\bullet \subset L(\pi', n+1)_\bullet$. Now we apply the path functor P to the map $a|_K$, which gives a map $\bar{a} := P(a|_K)$ from $PK(\pi, m)_\bullet = L(\pi, m)_\bullet$ to $PK(\pi', n)_\bullet = L(\pi', n)_\bullet$. Hence $\bar{a} \in \mathcal{O}_{m-1}^{n-1}$, and it holds $d\bar{a} = ad$ by definition as d is just the projection map of the path space followed by inclusion. Restriction of $\bar{a}$ to $\Omega K(\pi, m)_\bullet = K(\pi, m-1)_\bullet$ clearly represents the homotopy class $\Omega\alpha$, i.e. the cochain operation $\bar{a}$ represents the cohomology suspension. Now, we have $d(\bar{a} - a') = ad - ad = 0$, and by the exactness of d^L, it follows $\bar{a} - a' = da''$ with some $a'' \in \mathcal{O}_{m-1}^{n-2}$. Thus also a' represents the cohomology suspension. □

As a corollary we obtain Kristensen's basic result:

Corollary 1. *There is an isomorphism*

$$A^k = \frac{\{a \in \mathcal{O}^k \mid \Delta a = 0\}}{\Delta \mathcal{O}^{k-1}}.$$

Proof. As a stable cohomology operation $\alpha \in A^k$ can be arbitrarily desuspended, it can be represented in each dimension m by $a_m \in \mathcal{O}_m^{m+k}$ such that $da_m = a_{m+1}d$ for all m. In fact, using the construction $a \in \mathcal{O}_m^{m+k} \mapsto \bar{a} \in \mathcal{O}_{m-1}^{m+k-1}$ in the proof of the theorem above, we get such a system $(a_m)_{m \geq 0}$ by taking any element in the inverse limit

$$\lim_{\bar{a} \longleftarrow a} \{a_m \mid [a_m] = \alpha_m\}.$$

With

$$a := \begin{cases} (a_m)_{m \geq 0} & \text{for } k \text{ odd} \\ ((-1)^m a_m)_{m \geq 0} & \text{for } k \text{ even} \end{cases}$$

this is equivalent to $\Delta(a) = 0$. If there is $b = (b_m)_{m \geq 0} \in \mathcal{O}^{k-1}$ with $a = \Delta(b)$, we have $a_m = db_m + (-1)^{k-1} b_{m+1} d$ and thus $[a_m] = 0 \in A_m^{m+k}$ for all m as the left summand is in $ker(d^L)$ and the right summand is in $ker(d^R)$. Conversely, if $[a_m] = 0 \in A_m^{m+k}$ for all m, we want to prove that there exists $b \in \mathcal{O}^{k-1}$ with $a = \Delta b$. As $a = du + v$ with $du \in ker(d^L) = im(d^L)$ and $v \in ker(d^R)$, we get from $0 = \Delta(a) = da + (-1)^k ad = dv + (-1)^k dud$ that $v + (-1)^k ud = dx$ for some x. Application of d^R shows that $dxd = 0$. Thus

$$a = du + (-1)^{k-1} ud + dx = \Delta(u) + dx,$$

and it remains to prove existence of y with $dx = \Delta(y)$. We will prove this using the following two statements:

1. For $x^{(\leq m)} = (x_0, x_1, ..., x_m, 0, 0, ...)$ with $dx_i d = 0$ for all i, there exists $y^{(\leq m)} = (y_0, y_1, ..., y_m, 0, 0, ...)$ with $dx^{(\leq m)} = \Delta(y^{(\leq m)})$.
2. For $x^{(m)} = (0, ..., 0, x_m, 0, 0, ...)$ with $dx_m d = 0$ and m in the stable range (i.e., $m > k$), there exists $y^{(m)} = (0, ..., 0, y_m, y_{m+1}, ...)$ with $dx^{(m)} = \Delta(y^{(m)})$.

By both statements, the decomposition $x = x^{(\leq m)} + x^{(m+1)} + x^{(m+2)} + x^{(m+3)} + ...$ with some m in the stable range yields a solution $y := y^{(\leq m)} + y^{(m+1)} + y^{(m+2)} + y^{(m+3)} + ...$ of $dx = \Delta(y)$. Here, the infinite sums in $\mathcal{O}^* = \prod_l \mathcal{O}_l^{l+*}$ make sense as there are only finitely many summands in any degree l.

Proof of the first statement: We start with $y_m := x_m$ and look for y_i satisfying $(*_i) : dx_i = dy_i + (-1)^{k-1} y_{i+1} d$ for $i = m-1, ..., 0$. We remark that $dy_i d = 0$ by $(*_i)$. Asuming that we already have found y_i for $i = m, ..., m-j+1$, there exists a solution y_{m-j} of $(*_{m-j})$ as $d(dx_{m-j} - (-1)^{k-1} y_{m-j+1} d) = 0$ and by the exactness of d^L.

Proof of the second statement: As we are in the stable range, we already have proved that there is some cochain operation z_m in the class $[x_m] \in A_m^{m+k}$ that has infinitely many desuspensions. Thus, by changing some signs for k odd, there are $z_{m+1}, z_{m+2}, ...$ with $dz_i = (-1)^k z_{i+1} d$ for all $i \geq m$. By $[x_m] = [z_m]$, we have $x_m = z_m + u + v$ with $u \in ker(d^L)$ and $v \in ker(d^R)$. Now we set $y_m := v$ and $y_i := -z_i$ for $i > m$, which gives

$$\Delta(y^{(m)}) = dy^{(m)} + (-1)^{k-1} y^{(m)} d =$$

$$(0, ..., 0, vd, dv - (-1)^{k-1} z_{m+1} d, -dz_{m+1} - (-1)^{k-1} z_{m+2} d, ...) = dx^{(m)}.$$

□

4. Unstable cochain operations and higher cohomology operations

In order to work with the multi-variable case which we need for higher order operations, we introduce a category of unstable cochain operations (compare also with [6]). For a graded abelian group $\pi =$

$\oplus_{n\geq 0}\pi_n$ we define generalized Eilenberg-MacLane spaces

$$L\pi_\bullet := \prod_{n\geq 0} L(\pi_n, n+1)_\bullet \quad \text{and} \quad K\pi_\bullet := \prod_{n\geq 0} K(\pi_n, n)_\bullet.$$

The representation result of Eilenberg-MacLane gives

$$C(X_\bullet;\pi) := \prod_{n\geq 0} C^n(X_\bullet;\pi_n) = mor(X_\bullet, L\pi_\bullet),$$

$$H(X_\bullet;\pi) := \prod_{n\geq 0} H^n(X_\bullet;\pi_n) = [X_\bullet, K\pi_\bullet].$$

We will need some notation for manipulating coefficient groups: For a graded abelian group π, $\Omega\pi$ means shifting down π by one step (i.e., $\Omega\pi_n := \pi_{n+1}$) and $\pi^1 \oplus \pi^2$ denotes the direct sum (i.e., $(\pi^1 \oplus \pi^2)_n = \pi^1_n \oplus \pi^2_n$).

Definition 3. Let $\mathcal{O}$ be the category with objects the graded abelian groups π and the set of morphisms from π to π' given by

$$\mathcal{O}(\pi,\pi') := mor(L\pi_\bullet, L\pi'_\bullet) = C(L\pi_\bullet;\pi').$$

We call $\mathcal{O}$ the **category of unstable cochain operations**.

One can define in the same way a category A of primary unstable cohomology operations and generalize our representation result to this case. More general, we consider unstable higher order cohomology operations, now. We recall the well-known definition which we transfer to the simplicial category:

Definition 4. An **unstable higher order cohomology operation** ϕ (briefly **UHCO**) is a tower of simplicial sets

$$\begin{array}{ccc} E^s_\bullet & \xrightarrow{f_s} & K\pi^s_\bullet \\ \downarrow & & \\ \vdots & & \\ E^2_\bullet & \xrightarrow{f_2} & K\pi^2_\bullet \\ \downarrow & & \\ K\pi^0_\bullet = E^1_\bullet & \xrightarrow{f_1} & K\pi^1_\bullet\,. \end{array}$$

where $\pi^0, \pi^1, \dots, \pi^s$ are graded abelian groups and $E^{i+1}_\bullet$ is the pullback of the path fibration $L\Omega\pi^i_\bullet = PK\pi^i_\bullet \to K\pi^i_\bullet$ over the map $f_i : E^i_\bullet \to K\pi^i_\bullet$:

$$E^{i+1}_\bullet := \{(x, y) \in E^i_\bullet \times L\Omega\pi^i_\bullet \mid f_i(x) = p(y)\}.$$

The **order** of ϕ is $s - 1$, i.e. one less than the height of the tower. We call $I(\phi) := \pi^0$ the **input coefficients** and $O(\phi) := \pi^s$ the **output coefficients** of ϕ. The **composition** $\phi \circ \psi$ of UHCOs ϕ and ψ with $O(\psi) = I(\phi)$ is defined by successively pulling back the fibrations $E^{i+1}_\bullet \to E^i_\bullet$ of ϕ over the top horizontal map of ψ. This again gives an UHCO of order equal to the sum of orders of ϕ and ψ.

As in the stable case [6], UHCOs up to isomorphism form a large category with respect to composition ('large' means that the morphisms between two objects do not form a set but rather a class):

Definition 5. Let B be the category with objects the graded abelian groups π and the morphisms from π to π' given by the class $B(\pi, \pi')$ of UHCOs with input coefficients equal to π and output coefficients equal to π'. We call B the **category of UHCOs**.

Theorem 5. *Let ϕ be an UHCO of order s as above which is defined on a subset of $H(X_\bullet; \pi^0)$ and takes values in $H(X_\bullet; \pi^s)$ (modulo indeterminacy, i.e. values are subsets of $H(X_\bullet; \pi^s)$). Then ϕ can be represented by a system of cochain operations $a_1, a_2, \dots a_s$ and $a'_1, a'_2, \dots, a'_s$ such that $a_i \in \mathcal{O}(\rho^i, \pi^i)$ and $a'_i \in \mathcal{O}(\Omega\rho^i, \Omega\pi^i)$ with*

$$\rho^i := \pi^0 \oplus \Omega(\pi^1 \oplus \pi^2 \oplus \dots \oplus \pi^{i-1}).$$

In particular, $\rho^1 = \pi^0$. The cochain operations have to satisfy the following system $(*)$ of equations for $1 \leq i \leq s$:

$$(*) : \quad a_i\big(dx_1,\ dx_2 + a'_1(x_1),\ dx_3 + a'_2(x_1, x_2),\ dx_4 + a'_3(x_1, x_2, x_3), \dots$$

$$\dots, dx_i + a'_{i-1}(x_1, x_2, \dots, x_{i-1})\big) \ = \ da'_i(x_1, x_2, ..., x_i).$$

Here, $x_1 \in C(X_\bullet; \Omega\pi^0)$ and $x_i \in C(X_\bullet; \Omega^2\pi^i)$ for $i > 1$ denote variables (arbitrary cochains on any space $X_\bullet$). Conversely, any system of cochain operations $a_1, a_2, \dots a_s$ and $a'_1, a'_2, \dots, a'_s$ satisfying these equations defines an UHCO. Evaluation of ϕ on a cohomology class

$\xi \in H(X_\bullet;\pi^0)$ with some representing cocycle $x_1 \in Z(X_\bullet;\pi^0)$ is given by the set of cocycles $z \in Z(X_\bullet;\pi^s)$ modulo coboundaries with

$$z = a_s(x_1, x_2, \dots, x_s)$$

where the cochains $x_i \in C(X_\bullet;\Omega\pi^i)$ have to satisfy the system $(**)$ of equations for $1 \le i \le s-1$

$$(**): \quad a_i(x_1, x_2, \dots, x_i) = dx_{i+1}.$$

Proof. From $E^1_\bullet = K\pi^0_\bullet \subset L\pi^0_\bullet$ and $E^i_\bullet \subset E^{i-1}_\bullet \times L\Omega\pi^{i-1}_\bullet$, we see that

$$E^i_\bullet \subset L\pi^0_\bullet \times L\Omega\pi^1_\bullet \times L\Omega\pi^2_\bullet \times \cdots L\Omega\pi^{i-1}_\bullet = L\rho^i_\bullet.$$

With respect to this product decomposition, the elements of $L\rho^i_\bullet$ and $E^i_\bullet$ are of the form $(x_1, x_2, \dots, x_i)$. Denoting the inclusion by $j_i : E^i_\bullet \subset L\rho^i_\bullet$, we get a surjection of cochain groups

$$j_i^* : \mathcal{O}(\rho^i, \pi^i) = C(L\rho^i_\bullet;\pi^i) \longrightarrow C(E^i_\bullet;\pi^i).$$

The right hand cochain group contains the element $f_i \in mor(E^i_\bullet, K\pi^i_\bullet) = Z(E^i_\bullet;\pi^i)$, and we will consider an inverse image $a_i \in \mathcal{O}(\rho^i, \pi^i)$ in the left hand cochain group. Now we prove the following two statements (1) and (2) by induction on $i = 1, 2, \dots, s$:

(1) : There are cochain operations $a_1, \dots, a_{i-1}$ and $a'_1, \dots, a'_{i-1}$ such that the sequence of cochain operations

$$L\Omega\rho^i_\bullet \xrightarrow{d'_{(i)}} L\rho^i_\bullet \xrightarrow{d_{(i)}} L\Omega^{-1}\rho^i_\bullet$$

with

$$d_{(i)}(x_1, \dots, x_i) := \big(dx_1, dx_2 - a_1(x_1), \dots, dx_i - a_{i-1}(x_1, \dots, x_{i-1})\big),$$

$$d'_{(i)}(x_1, \dots, x_i) := \big(dx_1, dx_2 + a'_1(x_1), \dots, dx_i + a'_{i-1}(x_1, \dots, x_{i-1})\big),$$

satisfies

$$E^i_\bullet = ker(d_{(i)}) = im(d'_{(i)}).$$

Here, $ker(d_{(i)})$ denotes the simplicial subset consisting of $x \in L\rho^i_\bullet$ with $d_{(i)}(x) = 0$ and $im(d'_{(i)})$ denotes the simplicial subset consisting of $d'_{(i)}(x) \in L\rho^i_\bullet$ for all $x \in L\Omega\rho^i_\bullet$.

(2) : A cochain operation $a_i : L\rho^i_\bullet \to L\pi^i_\bullet$ restricts to some map $f_i : E^i_\bullet \to K\pi^i_\bullet$ if and only if

$$da_i d'_{(i)} = 0,$$

which is equivalent to the existence of some cochain operation $a'_i : L\Omega\rho^i_\bullet \to L\Omega\pi^i_\bullet$ such that

$$(*) : \quad a_i d'_{(i)} = da'_i.$$

For $i = 1$, statement (1) is just the exactness of

$$L\Omega\rho^1_\bullet \xrightarrow{d} L\rho^1_\bullet \xrightarrow{d} L\Omega^{-1}\rho^1_\bullet$$

and $E^1_\bullet = K\rho^1_\bullet = im(d) = ker(d)$ holds true. Statement (2) is proved in our representation result for primary cohomology operations.

Now we assume both statements (1) and (2) for some $1 \leq i < s$ and will prove it for $i + 1$:

By definition, we have

$$\begin{aligned} E^{i+1}_\bullet &= \{(x, x_{i+1}) \in E^i_\bullet \times L\Omega\pi^i_\bullet \mid f_i(x) = p(x_{i+1})\} \\ &= \{(x_1, \dots, x_i, x_{i+1}) \in L\rho^i_\bullet \times L\Omega\pi^i_\bullet \mid \\ &\quad (x_1, \dots, x_i) \in E^i_\bullet \text{ and } a_i(x_1, \dots, x_i) = dx_{i+1}\}, \end{aligned}$$

where the last equality follows by extending f_i to a_i. This can be written as $dx_{i+1} - a_i(x_1, \dots, x_i)$ and by $(1)_i$, and it follows $E^{i+1}_\bullet = ker(d_{(i+1)})$. By the second part of $(1)_i$, we know that

$$E^i_\bullet = im(d'_{(i)}),$$

$$(x_1, \dots, x_i) = d'_{(i)}(x'_1, \dots, x'_i)$$

for some $(x'_1, \dots, x'_i) \in L\Omega\rho^i_\bullet$. Thus

$$\begin{aligned} 0 &= dx_{i+1} - a_i d'_{(i)}(x'_1, \dots, x'_i) \\ &= d\big(x_{i+1} - a'_i(x'_1, \dots, x'_i)\big), \end{aligned}$$

where we used that $a_i d'_{(i)} = da'_i$ by $(2)_i$. Exactness of d^L for cochains on $L\rho^i_\bullet$ yields

$$x_{i+1} - a'_i(x'_1, \dots, x'_i) = dx'_{i+1}$$

for some $x'_{i+1} \in L\Omega^2\pi^i_\bullet$, hence $E^{i+1}_\bullet = im(d'_{(i+1)})$ and we have proved $(1)_{i+1}$.

Now we consider the diagram

$$\begin{array}{ccc} & & mor(L\rho_\bullet^{i+1}, L\pi_\bullet^{i+1}) \\ & & j_{i+1}^* \downarrow \\ mor(E_\bullet^{i+1}, K\pi_\bullet^{i+1}) & \longrightarrow & mor(E_\bullet^{i+1}, L\pi_\bullet^{i+1}) \end{array}$$

where the horizontal map is the inclusion of cocycles f_{i+1} in cochains on E^{i+1} and the vertical map is restriction of cochains a_{i+1} to the subspace $j_{i+1} : E_\bullet^{i+1} \subset L\rho_\bullet^{i+1}$. Restriction of a_{i+1} gives some cocycle f_{i+1} if and only if $da_{i+1}j_{i+1} = 0$. By $E_\bullet^{i+1} = im(d'_{(i+1)})$, this is equivalent to

$$da_{i+1}d'_{(i+1)} = 0.$$

Exactness of d^L for cochains on $L\Omega\rho_\bullet^{i+1}$ yields the equivalent statement $a_{i+1}d'_{(i+1)} = da'_{i+1}$ which proves $(2)_{i+1}$.

This shows that ϕ is represented by $a_1, \ldots, a_s$ and $a'_1, \ldots, a'_s$ satisfying the system of equations $(*)$. Evaluation of ϕ on a cohomology class ξ is defined by considering all lifts $(x_1, \ldots, x_i)$ of x_1 to $E_\bullet^i$ which successively are given by finding solutions of $(**)$ and then applying a_s to the top lifts $(x_1, \ldots, x_s)$.

It remains to show that every system of cochain operations $a_1, \ldots, a_s$, $a'_1, \ldots, a'_s$ satisfying $(*)$ defines an UHCO ϕ. We form $d_{(i)}$ and $d'_{(i)}$ as above and remark that $(*)$ for $1 \leq i \leq s$ is equivalent to

$$d_{(i)}d'_{(i)} = 0,$$

i.e. $im(d'_{(i)}) \subset ker(d_{(i)})$. As in the inductive proof of (1), it follows $im(d'_{(i)}) = ker(d_{(i)}) =: E_\bullet^i$ and a_i restricts to some map f_i by (2). □

We remark that the system of equations $(*)$ holds true for arbitrary cochains x_i on any space $X_\bullet$ if and only if it holds true for the universal cochains $x_1 = \iota_{\Omega\pi^0} \in C(L\Omega\pi_\bullet^0; \Omega\pi^0)$ and $x_i = \iota_{\Omega^2\pi^i} \in C(L\Omega^2\pi_\bullet^i; \Omega^2\pi^i)$ for $i > 1$.

There is a suggestive way of denoting $(*)$ for $1 \leq i \leq s$ in matrix notation:

$$0 = d_{(s)}d'_{(s)} = \begin{pmatrix} d & 0 & 0 & \dots & 0 \\ -a_1 & d & 0 & \dots & 0 \\ -a_2 & & d & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ -a_s & & & & d \end{pmatrix} \begin{pmatrix} d & 0 & 0 & \dots & 0 \\ a'_1 & d & 0 & \dots & 0 \\ a'_2 & & d & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ a'_s & & & & d \end{pmatrix}$$

Of course, one has to be careful with a matrix calculus over an 'algebra' where distributivity from the right fails.

As in the primary case, we are also able to express the cohomology of the s-stage space $E^s_\bullet$ in terms of unstable cochain operations:

Theorem 6. *For an UHCO ϕ as above, the cohomology of the top space containing the element $[f_s]$ is given by*

$$H(E^s_\bullet;\pi^s) = \frac{\{a \in \mathcal{O}_{(s)} \mid dad_{(s)} = 0\}}{\{a \in \mathcal{O}_{(s)} \mid da = 0\} + \{a \in \mathcal{O}_{(s)} \mid ad_{(s)} = 0\}} = $$
$$= \frac{ker(d^L d^R_{(s)})}{ker(d^L) + ker(d^R_{(s)})}$$

where we denoted $\mathcal{O}(\rho^s, \pi^s)$ by $\mathcal{O}_{(s)}$.

Proof. We already have proved

$$(j^*_s)^{-1} Z(E^s_\bullet;\pi^s) = \{a \in \mathcal{O}_{(s)} \mid dad_{(s)} = 0\}$$

and it remains to determine the inverse image of the coboundaries

$$(j^*_s)^{-1} B(E^s_\bullet;\pi^s) \subset \mathcal{O}_{(s)}.$$

This is given by $(j^*_s)^{-1} d^L C(E^s_\bullet;\Omega\pi^s) = d^L C(L\rho^s_\bullet;\Omega\pi^s) + ker(j^*_s)$, but $im(d^L) = ker(d^L)$ on $L\rho^s_\bullet$ and $ker(j^*_s) = ker(d^R_{(s)})$ because $im(d_{(s)}) = E^s_\bullet$. □

It is also possible to stabilize these results as for primary operations which then leads to similar results as in [14], where the authors show that a generalized cohomology theory $h^*(-)$ with coefficient groups h^* non-trivial in only finitely many dimensions (that is, $h^*(-)$ is represented by a stable finite stage Postnikov system) can be represented by iteration of a non-linear cone construction for loop-valued cochain complexes. In fact, our description using triangular matrices for $d_{(i)}$ and $d'_{(i)}$ can be viewed as an unstable generalization of the results in [14].

We remark that another, independent approach to the main result in [14] is given in [4].

5. Pyramids of cocycle operations and higher cohomology operations

In this section, we introduce a second approach to higher cohomology operations which is based on cocycle operations instead of cochain operations. Because of the representability of the cocycle functor by Eilenberg-MacLane spaces, a cocycle operation is nothing else but a map between Eilenberg-MacLane spaces. Thus it is clear that any primary cohomology operation is given by a cocycle operation. Cocycle operations which induce vanishing cohomology operations are just given by zero-homotopies of cocycle operations.

In [17], secondary operations in homotopy theory are constructed by relations between primary operations, i.e. there are maps a, b of pointed spaces

$$A \xrightarrow{a} B \xrightarrow{b} C$$

and a zero-homotopy $c : A \to PC$ of the composition ba. This can be used to construct a secondary homotopy operation

$$\phi : ker\big(a_* : [-,A] \to [-,B]\big) \to coker\big(\Omega b_* : [-,\Omega B] \to [-,\Omega C]\big)$$

using a glueing construction for zero-homotopies (see also [2], chapter 5). In particular, for a relation between primary cohomology operations which is given by representing cocycle operations a, b and a zero-homotopy c, we get a secondary cohomology operation ϕ analogously to Kristensen's definition in the second section. We call the data a, b and c a pyramid of cocycle operations of height 1.

In [5], we generalized this glueing method to arbitrary order. The corresponding notion of a pyramid of larger height also includes higher zero-homotopies of mappings. Given a pyramid of height s, we defined a generalized glueing construction giving a tower of height $s+1$. If all spaces of the pyramid are Eilenberg-MacLane spaces, the associated tower represents an UHCO of order s.

Now we recall the basic definitions from [5] with some modifications as the spaces of the pyramids we are interested in are simplicial sets respectively simplicial Eilenberg-MacLane spaces.

Definition 6. Let $X_\bullet$ and $Y_\bullet$ be pointed simplicial sets where $Y_\bullet$ has to satisfy the Kan extension condition. A **zero-homotopy** a **of degree** s from $X_\bullet$ to $Y_\bullet$ is a map

$$a : X_\bullet \longrightarrow P^s Y_\bullet$$

where P^s denotes s-fold application of the simplicial path space functor P. (We need to assume the extension condition in order to guarantee that the simplicial path space behaves as in the topological category [15].) The set of zero-homotopies of degree s is denoted by

$$\mathcal{C}^s(X_\bullet, Y_\bullet).$$

We denote the map given by projection $p : P(-) \to (-)$ of the i-th factor P in P^s (counted from the right) by

$$d_i : \mathcal{C}^s(X_\bullet, Y_\bullet) \longrightarrow \mathcal{C}^{s-1}(X_\bullet, Y_\bullet).$$

If $Y_\bullet = K\pi_\bullet$ is a (generalized) Eilenberg-MacLane spaces, a is called a **cocycle of degree** s and the set of these is denoted by

$$\mathcal{C}^s(X_\bullet; \pi) := \mathcal{C}^s(X_\bullet, K\pi_\bullet).$$

If both $X_\bullet = K\rho_\bullet$ and $Y_\bullet = K\pi_\bullet$ are (generalized) Eilenberg-MacLane spaces, a is called a **cocycle operation of degree** s and the set of these is denoted by

$$\mathcal{C}^s(\rho, \pi) := \mathcal{C}^s(K\rho_\bullet, K\pi_\bullet).$$

Clearly, the d_i satisfy the simplicial relations $d_i d_j = d_j d_{i+1}$ for $i \geq j$. For $s = 0, 1$, we have

$$\mathcal{C}^0(X_\bullet; \pi) = Z(X_\bullet; \pi),$$

$$\mathcal{C}^1(X_\bullet; \pi) = C(X_\bullet; \Omega\pi),$$

and d_1 is just the coboundary map. The structure of higher degree cocycles $\mathcal{C}^s(X_\bullet, K\pi_\bullet)$ for $s > 1$ (and thus also of higher degree cocycle operations!) can be reduced to the case $s = 1$ of usual cocycles:

Theorem 7. *For $s \geq 1$, there is a natural isomorphism*

$$\mathcal{C}^s(X_\bullet, K\pi_\bullet) = \bigoplus_{i=1}^{s} C(X_\bullet; \Omega^i \pi)^{\binom{s-1}{i-1}}.$$

Proof. For any simplicial abelian group $A_\bullet$, we have a short exact sequence of simplicial abelian groups

$$0 \to \Omega A_\bullet \xrightarrow{i} PA_\bullet \xrightarrow{p} A_\bullet \to 0$$

which in general is not split. But this is the case if $A_\bullet$ is linearly contractible, i.e. there is a split homomorphism $c : A_\bullet \to PA_\bullet$. In this case, we have an isomorphism

$$(c, i) : A_\bullet \oplus \Omega A_\bullet \longrightarrow PA_\bullet$$

and $PA_\bullet$ is linearly contractible by $(c, \Omega c)$, again. Now, we show by Dold-Kan equivalence that $L\rho_\bullet$ is linearly contractible, what we need to check for each factor $L(\rho_n, n+1)$ only. Hence, we have to show that the short exact sequence of chain complexes

$$0 \to l(\rho_n, n)_* \xrightarrow{i} N_* \xrightarrow{p} l(\rho_n, n+1)_* \to 0$$

is split, where $N_* := N_*(PL(\rho_n, n+1)_\bullet)$ denotes the associated Moore complex. This is true by straightforward diagram chasing. The statement follows inductively as

$$P^s K\pi_\bullet = P^{s-1} L\Omega\pi_\bullet.$$

□

The proof also shows how to describe the i-th projection d_i from $\mathcal{C}^s(X_\bullet; \pi)$ to $\mathcal{C}^{s-1}(X_\bullet; \pi)$ with respect to this decomposition. For example, the projections d_1, d_2 from $\mathcal{C}^2(X_\bullet; \pi) = C(X_\bullet; \Omega\pi) \oplus C(X_\bullet; \Omega^2\pi)$ to $\mathcal{C}^1(X_\bullet; \pi) = C(X_\bullet; \Omega\pi)$ are given by the differential on the second factor and the projection to the first factor, respectively.

Definition 7. If $a \in \mathcal{C}^s(X_\bullet, Y_\bullet)$ and $b \in \mathcal{C}^t(Y_\bullet, Z_\bullet)$, where $Y_\bullet$ and $Z_\bullet$ satisfy the Kan extension condition, we define the **composition of higher degree zero-homotopies** $b \,\widehat{\circ}\, a \in \mathcal{C}^{s+t}(X_\bullet, Z_\bullet)$ by

$$X_\bullet \xrightarrow{a} P^s Y_\bullet \xrightarrow{P^s b} P^{s+t} Z_\bullet.$$

In particular, we get a **higher degree evaluation of cocycle operations on cocycles**

$$\widehat{\circ}\ : \mathcal{C}^t(\rho, \pi) \times \mathcal{C}^s(X_\bullet; \rho) \longrightarrow \mathcal{C}^{s+t}(X_\bullet; \pi)$$

and a **composition operation for higher degree cocycle operations**

$$\widehat{\circ}\ : \mathcal{C}^t(\pi^2, \pi^3) \times \mathcal{C}^s(\pi^1, \pi^2) \longrightarrow \mathcal{C}^{s+t}(\pi^1, \pi^3).$$

As this composition is associative, we get a **category $\mathcal{C}^*$ of higher degree cocycle operations** with objects being graded abelian groups and morphisms from π to π' being the graded sets $\coprod_{s\geq 0} \mathcal{C}^s(\pi, \pi')$.

Now we are ready for the definition of a pyramid:

Definition 8. A **simplicial pyramid of height** s (in [5], we called this 'length $s+1$') is a system of pointed simplicial sets $X^0_\bullet, X^1_\bullet, \dots, X^{s+1}_\bullet$ satisfying the Kan extension condition (with the possible exception of $X^0_\bullet$) and higher degree zero-homotopies

$$a^{i,j} \in \mathcal{C}^{j-i-1}(X^i_\bullet, X^j_\bullet)$$

such that

$$d_k(a^{i,j}) = a^{i+k,j}\ \widehat{\circ}\ a^{i,i+k}$$

for all $0 \leq i < j \leq s$ and $k = 1, \dots, j-i-1$. If all $X^i_\bullet$ are (generalized) Eilenberg-MacLane spaces, we call this a **pyramid of cocycle operations of height** s.

In [5], we constructed from a topological pyramid $Y^0, Y^1, \dots, Y^{s+1}$ of height s (= length $s+1$) a tower of height $s+1$

$$\begin{array}{ccc}
E^{s+1} & \longrightarrow & \bar{\Omega}^s Y^{s+1} \\
\downarrow & & \\
\vdots & & \\
E^2 & \longrightarrow & \bar{\Omega} Y^2 \\
\downarrow & & \\
Y^0 = E^1 & \longrightarrow & Y^1
\end{array}$$

Here, $\bar{\Omega}^s$ denotes a homeomorphic, but combinatorially different version of the s-fold loop space functor. Our construction uses certain coordinate transformations of parameter spaces of higher degree zero-homotopies and a higher dimensional glueing process. Hence it seems to be difficult to carry out this construction in the simplicial category without subdivision of the parameter spaces, which would change our models of path and loop spaces of Eilenberg-MacLane spaces into non-standard ones. This would destroy representability of cochains by these spaces and thus also the advantage of working in the simplicial category.

Of course, geometric realization of a simplicial pyramid $Pyr_\bullet := (K\pi^i_\bullet, a^{i,j})$ of cochain operations gives a topological pyramid $Pyr := (K\pi^i, \|a^{i,j}\|)$ of cochain operations, which by our topological construction has an associated topological UHCO Φ.

In the following theorem, we give a simplicial criterion for the case that Φ applied to the geometric realization of a cocycle $x \in Z(X_\bullet, \pi^0)$ vanishes (i.e., $\Phi(\|x\|)$ is defined and contains 0).

Theorem 8. *Let $K\pi^0_\bullet, K\pi^1_\bullet, \ldots, K\pi^{s+1}_\bullet$, $a^{i,j}$ be a pyramid of cochain operations of height s and $x^0 \in Z(X_\bullet, \pi^0)$ be a cocycle on some simplicial set $X_\bullet$. Then the evaluation of the associated topological UHCO Φ vanishes on the geometric realization of x^0 if there exist higher degree cocycles $x^i \in C^i(X_\bullet, K\pi^i_\bullet)$, $i = 0, \ldots, s+1$, such that*

$$d_i x^k = a^{i-1,k} \,\hat{\circ}\, x^{i-1} \quad \textit{for all } 1 \leq i \leq k \textit{ and } 0 \leq k \leq s+1,$$

i.e., the pyramid $(K\pi^i_\bullet, a^{i,j})$ can be extended to a pyramid consisting of $(X_\bullet, K\pi^i_\bullet, x^i, a^{i,j})$.

Proof. We denote the geometric realization of the simplicial pyramid $(K\pi^i_\bullet, a^{i,j})$ as $(Y^i, b^{i,j})$. By [5], the associated tower is constructed as follows. The elements $(y^0, y^1, \ldots, y^{k-1})$ of

$$E^k \subset Y^0 \times PY^1 \times P\bar{\Omega}^1 Y^2 \times \ldots \times P\bar{\Omega}^{k-2} Y^{k-1}$$

are characterized by the condition

$$g^{i-1}\big(b^{0,i} \,\hat{\circ}\, \phi^0(y^0), b^{1,i} \,\hat{\circ}\, \phi^1(y^1), \ldots, b^{i-1,i} \,\hat{\circ}\, \phi^{i-1}(y^{i-1})\big) = dy^i$$

for $i = 1, \ldots, k$, where g^j denotes the j-dimensional glueing map and $\phi^j : P\bar{\Omega}^{j-1}(-) \to P^j(-)$ is a homeomorphism induced by a certain coordinate transformation. In fact, the left hand side of this condition gives the map

$$f^i : E^i \to \bar{\Omega}^{i-1} Y^i$$

in the i-th stage of the tower.

We denote the geometric realization of $X_\bullet$ by X and of the maps x^i by the same symbols. In particular, we have the map

$$x^0 : X \longrightarrow Y^0 = E^1$$

which we also denote by e^1. Now, we define lifts $e^k : X \to E^k$ of e^1 by

$$e^k := \big((\phi^0)^{-1}(x^0), (\phi^1)^{-1}(x^1), \dots, (\phi^{k-1})^{-1}(x^{k-1})\big).$$

Actually, these maps are lifts as the glueing condition

$$f^i e^i = g^{i-1}\big(b^{0,i} \,\widehat{\circ}\, x^0, b^{1,i} \,\widehat{\circ}\, x^1, \dots, b^{i-1,i} \,\widehat{\circ}\, x^{i-1}\big) = d(\phi^i)^{-1}(x^i)$$

(with $i = 1, \dots, s$) is by [5], lemma 4.12, equivalent to the equations

$$d_i x^k = a^{i-1,k} \,\widehat{\circ}\, x^{i-1}.$$

Furthermore, the glueing condition in the case of $i = s+1$ gives a zero-homotopy for the top composition $f^{s+1}e^{s+1}$, hence the vanishing of the evaluation $\Phi(||x^0||)$. □

It would be interesting to have a simplicial formula for evaluation of Φ on x^0 also in the non-vanishing case.

References

[1] F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. 72, 20-104 (1960)

[2] S. Klaus, *Brown-Kervaire invariants*, Dissertation Univ. Mainz (1995), Shaker Verlag (1995), Aachen

[3] S. Klaus, *The Ochanine k-invariant is a Brown-Kervaire invariant*, Topology 36, 257-270 (1997)

[4] S. Klaus, *On simplicial loops and H-spaces* Preprint Mathematisches Forschungsinstitut Oberwolfach (1999), to appear in Topology and its Applications

[5] S. Klaus, *Towers and pyramids I*, Preprint Mathematisches Forschungsinstitut Oberwolfach (1999), to appear in Forum Mathematicum

[6] S. Klaus, *A category of higher cohomology operations and finite spectra*, Preprint Mathematisches Forschungsinstitut Oberwolfach (2000)

[7] S. Klaus, *Cochain operations and subspace arrangements*, Preprint Mathematisches Forschungsinstitut Oberwolfach (2000)

[8] L. Kristensen, *On secondary cohomology operations*, Math. Scand. 12, 57-82 (1963)

[9] L. Kristensen, *On a Cartan formula for secondary cohomology operations*, Math. Scand. 16, 97-115 (1965)

[10] L. Kristensen, *A secondary product structure in cohomology theory*, Math. Scand. 17, 113-149 (1965)

[11] L. Kristensen, *Massey products in Steenrod's algebra*, Proc. Advanced Study Inst. on Algebraic Topology, Aarhus (1970), Vol. II, 240-255

[12] L. Kristensen, I. Madsen, *On evaluation of higher order cohomology operations*, Math. Scand. 20, 114-130 (1967)

[13] L. Kristensen, I. Madsen, *On the structure of the operation algebra for certain cohomology theories*, Conf. on Algebraic Topology, University of Illinois at Chicgo Circle, Proceedings (1968), 134-160

[14] A. Kock, L. Kristensen, I. Madsen, *Cochain functors for general cohomology theories I, II*, Math. Scand. 20, 131-176 (1967)

[15] J.P. May, *Simplicial objects in algebraic topology*, Chicago Lectures in Mathematics, Univ. of Chicago Press (1992), Chicago

[16] L. Smith, *Secondary cohomology theories*, Indiana Univ. Math. J. 23, 899-923 (1974)

[17] E. Spanier, *Secondary operations on mappings and cohomology*, Ann. of Math. 75, 260-282 (1962)

[18] N.E. Steenrod, D.B.A. Epstein, *Cohomology operations*, Annals of Math. Studies 50, Princeton Univ. Press (1962), Princeton

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