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LIMITS IN DOUBLE CATEGORIES

by Marco GRANDIS and Robert PARE (*)

Résumé. Dans le cadre des catégories doubles, on définit la *limite double* (*horizontale*) d'un foncteur double $F: \mathbb{I} \to \mathbb{A}$ et on donne un théorème de construction pour ces limites, à partir des produits doubles, égalisateurs doubles et *tabulateurs* (la limite double d'un morphisme vertical). Les limites doubles décrivent des outils importants: par exemple, la construction de Grothendieck pour un profoncteur est son tabulateur, dans la "catégorie double" $\mathbb{C}\mathrm{at}$ des catégories, foncteurs et profoncteurs. Si $\mathbb{A}$ est une 2-catégorie, notre résultat se réduit à la construction de Street des *limites pondérées* [22]; si, d'autre part, $\mathbb{I}$ n'a que des flèches verticales, on retrouve la construction de Bastiani-Ehresmann des *limites relatives aux catégories doubles* [2].

0. Introduction

Double categories where introduced by C. Ehresmann [7-8]. The notion is symmetric; nevertheless, we think of it in a non-symmetric way, giving priority to the *horizontal* direction. Thus, the usual morphisms between double functors will be the horizontal natural transformations, and the usual double comma categories will be the horizontal ones. The main examples will be organised consistently with this priority, choosing as horizontal arrows the ones "which preserve the structure" and have therefore good limits.

It was shown in [20] that weighted limits in a 2-category $\mathbf{A}$ can be described as limits of double functors $F: \mathbb{I} \to \mathbf{A}$, where $\mathbb{I}$ is a small double category; of course, $\mathbf{A}$ is viewed, as usual, as a double category whose vertical arrows are identities. *Persistent* limits, those invariant up to equivalence, were also characterised. Such results are proved in detail in the thesis of D. Verity [24].

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We define here the notion of *(horizontal) double limit* for a double functor $F: \mathbb{I} \to \mathbb{A}$ *with values in a double category* (4.2); it is based on *(horizontal) double cones* (4.1), consisting of horizontal maps $A \to Fi$ (for i in $\mathbb{I}$) and double cells depending on the vertical arrows of $\mathbb{I}$. And we give a construction theorem for such limits (5.5):

Theorem. *The double category* $\mathbb{A}$ *has all small double limits iff it has small double products, double equalisers and* tabulators.

The crucial new limit, the tabulator of a vertical map $u: A \twoheadrightarrow B$, is the double limit of the double diagram formed by u. It consists of an object Tu, universally equipped with two horizontal maps $p: Tu \to A$, $q: Tu \to B$ and a cell π

$$(1) \qquad \begin{tikzcd} & A \arrow[d, "u"] \\ Tu \arrow[ur, "p"] \arrow[dr, "q"'] \arrow[r, phantom, "\pi"] & B \end{tikzcd}$$

In particular, if $\mathbb{A}$ is a 2-category, tabulators (of vertical identities) reduce to cotensors $\mathbf{2}*A$; one recovers thus Street's construction ([22], Thm. 10) of *weighted limits* by such cotensors and ordinary limits (satisfying the corresponding two-dimensional universal property). As a more complex but close comparison (whose details can be found in 4.2 and 5.3), let us also recall that *limits relative to double categories*, or $\mathbb{A}$-*wise limits*, have been considered by Bastiani-Ehresmann ([2], p. 258), in a different sense. Their interest lies in what would be called here a "vertical" double functor in $\mathbb{A}$, defined over an ordinary category, and a "one-dimensional" notion of double limit for it. Under such restrictions (one-dimensional limits of vertical double functors), our result reduces to the construction of $\mathbb{A}$-wise limits in [2] (p. 265, Prop. 3).

The tabulator of a vertical map and its cotabulator (the corresponding colimit) describe some important, well-known constructions. For instance, in the prototype of most of our examples, the "double category" $\mathbb{C}$at of categories, functors and profunctors, the tabulator of a profunctor $u: A \twoheadrightarrow B$ is its category of elements, or Grothendieck construction, while its cotabulator $\perp u$ is the *gluing*, or *collage*, of A and B along u, i.e. the category consisting of the disjoint union $A+B$, together with new hom-sets $u(a, b)$, for a in A and b in B. In the double subcategory $\mathbb{R}$el of sets, mappings and relations, we get the "graph", or "tabulation", of a relation $u: A \twoheadrightarrow B$ (motivating the name "tabulator"), and its "cograph" (a quotient of $A+B$) for cotabulator.

As a preparatory lemma for the construction theorem, we show that the double category $\mathbb{A}$ has double limits of "horizontal functors" $\mathbb{H}\mathbf{I} \to \mathbb{A}$ iff it has double

products and double equalisers. The construction of such limits is the standard one. Here $\mathbb{H}\mathbf{I}$ is the obvious "horizontal" double category of an arbitrary category $\mathbf{I}$, obtained by adding identity vertical maps and cells. Then, the theorem is proved by constructing a new 1-dimensional graph $\mathbf{I}$ (a sort of "horizontal subdivision" of $\mathbb{I}$) where every vertical arrow u of $\mathbb{I}$ is replaced with a new object u^, simulating its tabulator, and every vertical composition v•u with a new object (u, v), simulating the "double tabulator" T(u, v) (5.4). The double functor $F: \mathbb{I} \to \mathbb{A}$ produces a "horizontal functor" $G: \mathbb{H}\mathbf{I} \to \mathbb{A}$, and its double limit (obtained from double products and equalisers, by the previous lemma) is the double limit of F.

The mere notion of double limit is unsatisfactory, because of two main problems: a) it is not sufficient to obtain the limit of vertical transformations (*vertical functoriality*); b) it is not vertically determined (*vertical uniqueness*). The first anomaly is shown, for instance, by double equalisers in the double category $\mathbb{A}$**dCat** of categories, functors and adjunctions (6.5). For b), consider for instance the double category $\mathbb{T}$g of topological groups, with algebraic homomorphisms as horizontal maps, continuous mappings as vertical ones and commutative squares (of mappings) as cells; then a horizontal double product G×H has the "right" group structure, but an arbitrary topology consistent with the former (a *vertical* double product would behave symmetrically). And we can provide various functorial choices of the horizontal double product, taking for instance the product topology, or the discrete one, or also the chaotic one, which are not even *vertically equivalent* (2.2).

The first problem leads us to the notion of a *functorial choice of* $\mathbb{I}$*-limits* (possibly a *lax*, or *pseudo* double functor, 4.3-4), to which the construction theorem is extended (5.5 ii). The second is solved by an elementary assumption on the ground double category $\mathbb{A}$, called *horizontal invariance*, so that "vertical maps are transportable along horizontal isos" (2.4). In this case, a functorial choice of $\mathbb{I}$-limits is also vertically determined, and we just speak of (invariant) *functorial* $\mathbb{I}$-*limits* (possibly lax, or pseudo; 4.5-6). This assumption is satisfied in all our examples of real interest (Section 3), but not in $\mathbb{T}$g. Our condition is also "necessary", being equivalent to having invariant unary limits (4.5). Marginally, we also consider a more complex solution to these problems, *regular* double limits, defined by a further universal property "of vertical terminality" (4.7).

Various relaxed notions impose themselves, as is usual in higher dimensional category theory. Some of our main examples, starting from $\mathbb{C}$at, are actually *pseudo* double categories, with a weakly associative vertical composition (as in bicategories; 1.9, 7.1). *Lax* and *pseudo* double functors, weakly preserving the

vertical structure, often appear as relevant constructions (Section 3).

A Strictification theorem (7.5) proves that pseudo double categories and functors can be replaced by strict versions, up to equivalence. However, *lax* double functors cannot be similarly replaced and the interest of constructing their double limits subsists. To treat everything in the widest generality would be obscure (e.g., see the definition of strong vertical transformation of lax double functors, which already "simplifies" an unmanageable general notion, 7.4). In theoretical parts, we shall therefore start from the strict case and give marginal indications for its extension; the main one, the extension of the construction theorem to double limits of lax double functors, is easy and based on the same elementary limits (5.5 iii). On the other hand, in concrete descriptions and computations (Sections 3, 6) it is simpler to use the natural pseudo double structures. Interestingly, all these relaxed constructs are intrinsically asymmetric (1.9): the breaking of symmetry mentioned above is "written in nature".

We conclude with some remarks about the motivation of this work. It's leitmotif can be summarised as follows: arrows which are *too relaxed* (like profunctors, spans, relations) or *too strict* (like adjunctions) to have limits, can be studied in a (pseudo) double category, correlating them with more ordinary (horizontal) arrows. (The structure of "2-equipment", introduced by Carboni, Kelly, Verity and Wood [5], is a different approach to a similar goal.) Note now that a notion of "symmetric double limit", based on "symmetric double cones" in some sense, apart from being confined to the strict case and formally suspect as highly overdetermined, would be of no use here: a double category whose category of vertical arrows lacks products cannot have "symmetric double products", in any reasonable sense. Finally, some points of the present work might be deduced from the theory of internal categories (cf. Street [23]) or indexed categories (Paré-Schumacher [21]), but we think that double categories are worth an independent treatment, founded on basic category theory.

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Outline. Basic notions of double categories are reviewed in Section 1. Sesqui-isomorphisms, linking horizontal isos and vertical equivalences, are introduced in Section 2. Examples of (pseudo) double categories are given in Section 3. The next two sections deal with double limits, their functoriality, regularity and construction theorem. Explicit calculations are considered in Section 6. Finally, Section 7 is an appendix on pseudo double categories, lax and pseudo double functors, and their transformations.

1. Double categories

Double functors will be mainly linked by *horizontal* transformations, providing (horizontal) comma double categories (1.7). The notion of (horizontal) *double terminal* (1.8) will give our universal notions.

1.1. Basic terminology. The notion of double category is well-known, we just specify here some basic terminology and notation. A double category $\mathbb{A}$ has horizontal morphisms $f: A \rightarrow B$ (with composition $g \circ f = gf$), vertical morphisms $u: A \twoheadrightarrow A'$ (with composition $v \bullet u$), and cells α

$$(1) \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle v} \\ A' & \xrightarrow[g]{} & B' \end{array}$$

where the horizontal map f is the *vertical* domain (the domain for vertical composition), and so on. The *boundary* of the cell (1) is written as $\alpha: (u\,{}^{f}_{g}\,v)$. Horizontal and vertical identities, of objects and maps, are denoted as follows

(2) $1_A: A \rightarrow A, \qquad 1_u: u \rightarrow u$

(3) $1^\bullet_A: A \twoheadrightarrow A, \qquad 1^\bullet_f: f \twoheadrightarrow f, \qquad \Box_A = 1_{1^\bullet_A} = 1^\bullet_{1_A}$.

Horizontal and vertical composition of cells will be written either in the "pasting" order, either in the "algebraic" one, respectively as

(4) $(\alpha \mid \beta) = \beta \circ \alpha = \beta\alpha, \qquad \left(\frac{\alpha}{\gamma}\right) = \gamma \bullet \alpha.$

The axioms essentially say that both laws are "categorical", and satisfy the interchange law. The expressions $(\alpha \mid f)$ and $(f \mid \beta)$ will stand for $(\alpha \mid 1^\bullet_f)$ and $(1^\bullet_f \mid \beta)$. The pasting satisfies a general associativity property, established in [6].

A double category is said to be *flat* if its cells are determined by their domains and codomains. The following notions are fairly obvious: *double functor* $F: \mathbb{A} \rightarrow \mathbb{B}$; *double subcategory*; *full* double subcategory (determined by a subset of objects); *cellwise-full* double subcategory (determined by a subcategory of horizontal maps and a subcategory of vertical maps having the same objects); *double graphs* and their morphisms; the latter are also called (double) diagrams. Unless otherwise stated, the letters $\mathbb{A}, \mathbb{B}, \mathbb{I}, \mathbb{J}, \mathbb{X}$ always denote double categories, while $\mathbb{D}$ denotes a double graph.

1.2. Dualities. The 8-element symmetry group of the square acts on a double category. We have thus the *horizontal opposite* $\mathbb{A}^h$, the *vertical opposite* $\mathbb{A}^v$, and the *transpose* $\mathbb{A}^t$, under the relations

(1) $hh = vv = tt = 1, \qquad hv = vh, \qquad ht = tv, \qquad vt = th.$

The prefix "co", as in *colimit* or *coequaliser* or *colax double functor* (7.2), will always refer to horizontal duality, except when the vertical direction is viewed as the main one; this only happens in 6.5 e, where we consider *vertical colimits*.

1.3. Categories and double categories. A double category $\mathbb{A}$ can be viewed as a 3×3 array of sets, connected by functions

$$
(1) \qquad
\begin{array}{ccccccc}
A_{00} & \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} & A_{01} & \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} & A_{02} & \qquad & \mathrm{hor}_0\mathbb{A} \\
\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow \\
A_{10} & \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} & A_{11} & \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} & A_{12} & & \mathrm{hor}_1\mathbb{A} \\
\uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow \\
A_{20} & \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} & A_{21} & \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} & A_{22} & & \mathrm{hor}_2\mathbb{A} \\
 & & & & & & \\
\mathrm{ver}_0\mathbb{A} & \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} & \mathrm{ver}_1\mathbb{A} & \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} & \mathrm{ver}_2\mathbb{A} & &
\end{array}
$$

Each row forms a category $\mathrm{hor}_i\mathbb{A}$, presenting $\mathbb{A}$ as a category object in **CAT**, as indicated at the right. Explicitly

- $\mathrm{hor}_0\mathbb{A}$ is the category of objects and horizontal maps of $\mathbb{A}$,
- $\mathrm{hor}_1\mathbb{A}$ is the category of vertical maps and cells $\alpha: u \to v$, with horizontal composition,
- $\mathrm{hor}_2\mathbb{A}$ is the analogous category, whose objects are the composable pairs of vertical maps of $\mathbb{A}$.

Similarly, each column is a category $\mathrm{ver}_i\mathbb{A} = \mathrm{hor}_i(\mathbb{A}^t)$, forming a second such presentation of $\mathbb{A}$. Giving priority to the horizontal composition, we consider the *first* presentation as the main one. We say that $\mathbb{A}$ has *small* (horizontal) *hom-sets* if the categories $\mathrm{hor}_0\mathbb{A}$, $\mathrm{hor}_1\mathbb{A}$ do. Our examples will always satisfy this property, but not necessarily the transposed one. A double functor $F: \mathbb{I} \to \mathbb{A}$ determines functors $\mathrm{hor}_iF: \mathrm{hor}_i\mathbb{I} \to \mathrm{hor}_i\mathbb{A}$.

The category $\mathrm{hor}_0\mathbb{A}$ becomes the *horizontal 2-category* of $\mathbb{A}$, written $\mathbf{H}\mathbb{A}$, when equipped with 2-cells $\alpha: f \to g$ provided by $\mathbb{A}$-cells $\alpha: (1^\bullet \,{}^{f}_{g}\, 1^\bullet)$, whose

vertical arrows are identities. The *vertical 2-category* $\mathbf{V}\mathbb{A} = \mathbf{H}(\mathbb{A}^t)$ of $\mathbb{A}$ will be of special interest (cf. Section 2); it has cells $\alpha\colon u \to v$ given by cells $\alpha\colon (u \,{}^{1}_{1}\, v)$ of $\mathbb{A}$. On the other hand, a 2-category $\mathbf{A}$ determines the double categories:

- $\mathbb{Q}\mathbf{A}$ (of *quintets* of $\mathbf{A}$, according to Ehresmann), whose horizontal and vertical maps are the maps of $\mathbf{A}$, the cells being defined by cells of $\mathbf{A}$

$$\text{(2)} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dl, phantom, "\swarrow\alpha"] \\ A' \arrow[r, "g"'] & B' \end{tikzcd} \qquad\qquad \alpha\colon vf \to gu\colon A \to B'$$

- $\mathbb{H}\mathbf{A}$, the double subcategory of $\mathbb{Q}\mathbf{A}$ with vertical maps the identities of $\mathbf{A}$;
- $\mathbb{V}\mathbf{A} = (\mathbb{H}\mathbf{A})^t$, the double subcategory of $(\mathbb{Q}\mathbf{A})^t$ having for horizontal maps the identities of $\mathbf{A}$ (for vertical maps the morphisms of $\mathbf{A}$ and cells $\alpha\colon (u \,{}^{1}_{1}\, v)$ produced by cells $\alpha\colon u \to v$ of $\mathbf{A}$).

There is a bijective correspondence $\mathbf{A} \mapsto \mathbb{H}\mathbf{A}$, $\mathbb{A} \mapsto \mathbf{H}\mathbb{A}$ between 2-categories, on the one hand, and *double categories where all vertical maps are identities*, on the other. Such a double category $\mathbb{A}$ will be said to be *horizontal*. And *1-horizontal* if moreover all its cells are vertical identities, i.e. if $\mathbb{A}$ is of the form $\mathbb{H}\mathbf{A}$ *for a category* $\mathbf{A}$ (viewed as a trivial 2-category).

1.4. Horizontal transformations. A *horizontal (natural) transformation* $H\colon F \to G\colon \mathbb{I} \to \mathbb{A}$ between double functors assigns

a) a horizontal map $Hi\colon Fi \to Gi$, for each object i in $\mathbb{I}$

b) a cell $Hu\colon (Fu \,{}^{Hi}_{Hj}\, Gu)$, for each vertical map $u\colon i \to\!\!\to j$ in $\mathbb{I}$

so that the following preservation and naturality conditions hold:

(ht.1) $H(1^\bullet_i) = 1^\bullet_{Hi}\colon (F1^\bullet_i \,{}^{Hi}_{Hi}\, G1^\bullet_i)$, for every i in $\mathbb{I}$

(ht.2) $H(v \bullet u) = Hv \bullet Hu$, for all u, v vertical in $\mathbb{I}$

$$\text{(1)} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "Fw"'] \arrow[dr, phantom, "H(v\bullet u)"] & \bullet \arrow[d, "Gw"] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "Fu"'] \arrow[dr, phantom, "Hu"] & \bullet \arrow[d, "Gu"] \\ \bullet \arrow[r] \arrow[d, "Fv"'] \arrow[dr, phantom, "Hv"] & \bullet \arrow[d, "Gv"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

(ht.3) $(F\alpha \mid Hv) = (Hu \mid G\alpha)$, for every $\alpha: (u \, {}^{f}_{g} \, v)$ in $\mathbb{I}$

$$\text{(2)} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "Fu"'] \arrow[dr, phantom, "F\alpha"] & \bullet \arrow[r] \arrow[d, "Fv"] \arrow[dr, phantom, "Hv"] & \bullet \arrow[d, "Gv"] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r] \arrow[d, "Fu"'] \arrow[dr, phantom, "Hu"] & \bullet \arrow[r] \arrow[d, "Gu"] \arrow[dr, phantom, "G\alpha"] & \bullet \arrow[d, "Gv"] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

Let $\mathbf{2}$ be the usual ordinal category, spanned by a map $z: 0 \to 1$ (with *faces* $\delta^-, \delta^+: \mathbf{1} \to \mathbf{2}$). A horizontal transformation can be viewed as a double functor $H: \mathbb{H}\mathbf{2}\times\mathbb{I} \to \mathbb{A}$; one recovers $F = H.(\delta^-\times\mathbb{I})$, $G = H.(\delta^+\times\mathbb{I})$, $Hi = H(z, 1^\bullet_i)$, $Hu = H(1^\bullet_z, 1_u)$. Similarly, a *vertical transformation* $U: F \twoheadrightarrow G: \mathbb{I} \to \mathbb{A}$ amounts to a double functor $\mathbb{V}\mathbf{2}\times\mathbb{I} \to \mathbb{A}$; it consists of vertical maps $Ui: Fi \twoheadrightarrow Gi$ and cells $Uf: (Ui \, {}^{Ff}_{Gf} \, Uj)$, under three axioms: $U(1_i) = 1_{Ui}$; $U(gf) = Ug \circ Uf$; $Ug \bullet F\alpha = G\alpha \bullet Uf$.

1.5. Remarks. a) It follows that the maps $Hi: Fi \to Gi$ produce a natural transformation of the underlying functors hor_0F, hor_0G (and a 2-natural transformation $\mathbb{H}F \to \mathbb{H}G$).

b) *If* $\mathbb{A}$ *is flat*, (ht.1-2) are trivially satisfied while (ht.3) reduces to usual naturality. A horizontal transformation $H: F \to G: \mathbb{I} \to \mathbb{A}$ reduces thus to a natural transformation $H = (Hi)_i: \mathrm{hor}_0F \to \mathrm{hor}_0G: \mathrm{hor}_0\mathbb{I} \to \mathrm{hor}_0\mathbb{A}$ such that, for every vertical map $u: i \twoheadrightarrow j$ in $\mathbb{I}$, the boundary $(Fu \, {}^{Hi}_{Hj} \, Gu)$ *admits* a (unique) cell.

c) *If* $\mathbb{I}$ *is horizontal* (resp. *1-horizontal*), the conditions (ht.1-2) are again trivially satisfied and a horizontal transformation $H: F \to G: \mathbb{I} \to \mathbb{A}$ reduces to a 2-natural transformation $\mathbb{H}F \to \mathbb{H}G: \mathbb{H}\mathbb{I} \to \mathbb{H}\mathbb{A}$ (resp. to a natural transformation $\mathrm{hor}_0F \to \mathrm{hor}_0G: \mathrm{hor}_0\mathbb{I} \to \mathrm{hor}_0\mathbb{A}$).

d) If $\mathbb{D}$ is just a double graph, a *horizontal transformation* $H: F \to G: \mathbb{D} \to \mathbb{A}$ between double-graph morphisms, or diagrams, is formed of the same data a), b) above *under the unique condition* (ht.3); such a transformation is clearly the same as a horizontal transformation of double functors $\bar{H}: \bar{F} \to \bar{G}: \bar{\mathbb{D}} \to \mathbb{A}$, where $\bar{\mathbb{D}}$ is the free double category generated by $\mathbb{D}$.

e) In contrast with the 1-dimensional case, a horizontal transformation of double functors is a stronger notion than a horizontal transformation between the underlying diagrams, because of (ht.1-2). This fact will produce a distinction between limits of double functors and limits of diagrams (cf. 5.6-7, 6.6). But there is no difference whenever $\mathbb{A}$ is flat (by b) or $\mathbb{I}$ is horizontal (by c). Moreover, the stronger notion (on $\mathbb{I}$) can be reduced to the weaker one (on the double graph $\mathbb{D}$) when $\mathbb{I}$ is the free double category $\bar{\mathbb{D}}$ generated by $\mathbb{D}$ (by c).

1.6. Exponential. For a small $\mathbb{I}$, consider the double category $\mathbb{A}^{\mathbb{I}}$ of double functors $\mathbb{I} \to \mathbb{A}$, with horizontal and vertical morphisms respectively given by the horizontal and vertical transformations (written H, K and U, V respectively).

A cell $\mu: (U \frac{H}{K} V)$ is a *modification*, assigning to any object i of $\mathbb{I}$ a cell μi: $(Ui \frac{Hi}{Ki} Vi)$ so that:

(md.1) for every horizontal arrow $f: i \to j$

$$
\begin{tikzcd}
Fi \arrow[r, "Hi"] \arrow[d, "Ui"'] \arrow[dr, phantom, "\mu i"] & Gi \arrow[r, "Gf"] \arrow[d, "Vi"] \arrow[dr, phantom, "Vf"] & Gj \arrow[d, "Vj"] \\
F'i \arrow[r, "Ki"'] & G'i \arrow[r, "G'f"'] & G'j
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Fi \arrow[r, "Ff"] \arrow[d, "Ui"'] \arrow[dr, phantom, "Uf"] & Fj \arrow[r, "Hj"] \arrow[d, "Uj"] \arrow[dr, phantom, "\mu j"] & Gj \arrow[d, "Vj"] \\
F'i \arrow[r, "F'f"'] & F'j \arrow[r, "Kj"'] & G'j
\end{tikzcd}
\qquad (1)
$$

(md.2) for every vertical arrow $u: i \twoheadrightarrow j$, a symmetric pasting condition holds.

For $\mathbb{J}$ small, the double functors $F: \mathbb{J}\times\mathbb{I} \to \mathbb{A}$ correspond to double functors $G: \mathbb{I} \to \mathbb{A}^{\mathbb{J}}$, by the usual adjunction $G(i)(j) = F(j, i)$. It will be useful to note that a vertical transformation, viewed at the end of 1.4 as a double functor $\mathbb{V}2\times\mathbb{I} \to \mathbb{A}$, can also be viewed as a double functor $\mathbb{I} \to \mathbb{A}^{\mathbb{V}2}$, and a horizontal one as a double functor $\mathbb{I} \to \mathbb{A}^{\mathbb{H}2}$. Here, $\mathbb{V}2$ is the double category generated by one vertical arrow; $\mathbb{A}^{\mathbb{V}2}$ has for objects the vertical arrows of $\mathbb{A}$, for vertical morphisms their commutative squares, for horizontal morphisms the cells of $\mathbb{A}$, and finally for cells the cubes of $\mathbb{A}$ formed of two commutative squares of vertical arrows, linked by four cells of $\mathbb{A}$ which commute under vertical composition.

1.7. Comma double categories. Let two double functors $D: \mathbb{A} \to \mathbb{X}$, $F: \mathbb{B} \to \mathbb{X}$ with the same codomain be given. The (horizontal) *comma* $(D \Downarrow F)$ is a double category, equipped with two double functors and a *horizontal* transformation as below, universal in the usual sense

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "D"] & \mathbb{X} \\
(D \Downarrow F) \arrow[u, "P"] \arrow[r, "Q"'] \arrow[ur, phantom, "\searrow H"] & \mathbb{B} \arrow[u, "F"']
\end{tikzcd}
\qquad (1) \qquad H: DP \to FQ: (D \Downarrow F) \to \mathbb{X}.
$$

We describe the solution in the particular case we are interested in, for $\mathbb{B} = \mathbb{1}$, the free double category on one object, so that the double functor F reduces to an

object of $\mathbb{X}$. Then $(D \Downarrow F)$ has:

- objects $(A, x: DA \to F)$, where A is in $\mathbb{A}$ and x is horizontal in $\mathbb{X}$,
- horizontal arrows $f: (A, x: DA \to F) \to (B, y: DB \to F)$, where $f: A \to B$ is horizontal in $\mathbb{A}$ and $y \circ Df = x$ in $\mathbb{X}$ (so that $hor_0(D \Downarrow F) = (hor_0 D \downarrow hor_0 F)$),
- vertical arrows $(u, \xi): (A, x: DA \to F) \twoheadrightarrow (A', x': DA' \to F)$, where $u: A \twoheadrightarrow A'$ is vertical in $\mathbb{A}$ and $\xi: (Du \,{}^{x}_{y}\, 1^{\bullet}_{F})$ is in $\mathbb{X}$,
- cells α as below (left hand), where $\alpha: (u \,{}^{f}_{g}\, v)$ is in $\mathbb{A}$ and $(D\alpha \mid \eta) = \xi$,

$$(2) \qquad \begin{tikzcd} (A, x: DA \to F) \arrow[r, "f"] \arrow[d, "{(u, \xi)}"'] \arrow[dr, phantom, "\alpha"] & (B, y: DB \to F) \arrow[d, "{(v, \eta)}"] \\ (A', x': D'A \to F) \arrow[r, "g"'] & (B', y': DB' \to F) \end{tikzcd} \qquad \begin{tikzcd} DA \arrow[r, "Df"] \arrow[d, "Du"'] \arrow[dr, phantom, "D\alpha"] & DB \arrow[r, "y"] \arrow[d, "Dv"] \arrow[dr, phantom, "\eta"] & F \arrow[d, equal] \\ DA' \arrow[r, "Dg"'] & DB' \arrow[r, "y'"'] & F \end{tikzcd}$$

- compositions determined by the ones of $\mathbb{A}$ and $\mathbb{X}$, in the obvious way.

The "projections" P, Q are also obvious, and $H(A, x) = x$, $H(u, \xi) = \xi$.

1.8. Terminal object. A (horizontal) *double terminal* [20] of $\mathbb{A}$ is an object T such that:

(t.1) for every object A there is precisely one map $t: A \to T$ (also written t_A),

(t.2) for every vertical map $u: A \twoheadrightarrow A'$ there is precisely one cell τ (also written τ_u) with

$$(1) \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\tau"] & T \arrow[d, equal] \\ A' \arrow[r] & T \end{tikzcd}$$

(Actually, the 2-dimensional property (t.2) implies (t.1): apply it to 1_A.) A (horizontal) *universal arrow* from the double functor $D: \mathbb{A} \to \mathbb{X}$ to the object F of the codomain can now be defined as the double terminal $(A, x: DA \to F)$ of the comma $(D \Downarrow F)$, if such an object exists. If $D: \mathbb{A} \to \mathbb{A}^{\mathbf{I}}$ is the diagonal double functor into a double category of diagrams, this gives –as usual– the notion of double limit of the double functor $F \in \mathbb{A}^{\mathbf{I}}$, studied below.

1.9. Lax notions. We also need some relaxed notions, briefly reviewed here; a precise definition can be found in the appendix, Section 7. In the sequel, for the sake of simplicity, we shall generally start from the strict notions and give

marginal indications for the extension of the main results. A reader not familiar with bicategories might prefer to omit these indications, at first; then read the appendix and come back to them.

A *pseudo double category* (7.1) has an associative horizontal composition and a *weakly* associative vertical one, up to assigned, invertible comparison cells. Some of the main examples we are interested in are actually of this type, such as $\mathbb{S}\mathrm{et}$, the pseudo double category of sets, mappings and spans (3.2). For simplicity, a pseudo double category is always assumed to be *unitary*, i.e. with strict vertical identities (except in Section 7). The general case can be easily reduced to the latter, by adding new vertical identities; more simply, in concrete examples where the vertical composition is provided by some choice (of pullbacks, for $\mathbb{S}\mathrm{et}$), it may suffice to put some mild constraint on this choice.

A *lax* double functor (7.2) respects the horizontal structure in the usual strict sense, and the vertical one up to an assigned comparison; it is called a *pseudo* double functor if the latter is invertible. Various examples are given in Section 3. Again, a lax double functor is understood to be *unitary*, i.e. to respect strictly the vertical identities (except in Section 7); a motivation can be found in 4.3.

Note now that pseudo double categories have *no transposition* (1.2), for intrinsic reasons: *one must start from an ordinary category*, to be able to consider a second composition law, associative up to a natural isomorphism of the first.

Thus, reconsidering 1.3, a pseudo double category only admits the first presentation 1.3.1, as a *pseudo* category object in **CAT**. The associated vertical structure $\mathbf{V}\mathbb{A}$ is a *bi*category [3, 15, 19], whereas the horizontal one $\mathbf{H}\mathbb{A}$ is a 2-category (also because $1^\bullet_f \bullet 1^\bullet_f = 1^\bullet_f$, by unitarity). A vertical pseudo double category amounts to a *bicategory*, under the bijection $\mathbb{A} \mapsto \mathbf{V}\mathbb{A}$, $\mathbf{A} \mapsto \mathbb{V}\mathbf{A}$. On the other hand, a horizontal pseudo double category is necessarily strict, and amounts to a *2-category*, under the bijection $\mathbb{A} \mapsto \mathbf{H}\mathbb{A}$, $\mathbf{A} \mapsto \mathbb{H}\mathbf{A}$ (the constructs $\mathbb{H}\mathbf{A}$ and $\mathbb{Q}\mathbf{A}$ make no sense for a *bi*category). In particular, a monoidal category $\mathbf{A}$ can be viewed as a vertical pseudo double category, having one formal object $*$ and its horizontal identity, vertical arrows $A\colon * \twoheadrightarrow *$ coming from $\mathbf{A}$-objects (composed by tensoring) and cells $f\colon (A \,{}^{1}_{1}\, B)$ from $\mathbf{A}$-morphisms. This interpretation is consistent with viewing each monoidal category of R-modules within the pseudo double category $\mathbb{R}\mathrm{ng}$ of rings, homomorphisms and bimodules (5.3).

A *horizontal transformation* of lax double functors between pseudo double categories (7.3) can again be defined as a lax double functor $H\colon \mathbb{H}\mathbf{2}\times\mathbb{A} \to \mathbb{B}$; it is composed of the same data as its strict version, under coherence axioms with the comparison cells of the functors. Vertical transformations defined in a similar

way are an unmanageable tool, which we replace with a reduced version, a *strong vertical transformation* (7.4), having *one* system of *invertible* cells as a naturality comparison. *Strong modifications* are also introduced. For pseudo double functors, we use the transformations and modifications inherited from the lax case; but we drop the term *strong*, since here the previous restriction is (nearly) automatic.

We show in 7.5 that a pseudo double category always has an equivalent strict one. This strictification extends to pseudo double functors, but not to the lax ones.

2. Double isomorphisms and sesqui-isomorphisms

This section studies the connections between horizontal and vertical isomorphisms, or more generally between a horizontal iso and a vertical equivalence, whose linking forms a *sesqui-isomorphism*. Such phenomena, peculiar to double categories, will become important in Section 4, where we show that *invariant* double limits are determined up to sesqui-isomorphism. Everything is still valid for a *pseudo* double category $\mathbb{A}$ (and its *bi*category $\mathbf{V}\mathbb{A}$, 1.9).

2.1. Double isomorphisms. In the double category $\mathbb{A}$, the relation of being horizontally isomorphic objects (there exists a horizontal iso $f: A \to B$) may be very weakly correlated with the vertical analogue. Consider for instance the flat double category $\mathbb{T}\mathrm{g}$ of topological groups, with arbitrary homomorphisms as horizontal maps, continuous mappings as vertical ones and commutative squares (of the underlying mappings) as cells. Then the object A is horizontally (resp. vertically) isomorphic to B iff the underlying groups (resp. spaces) are isomorphic; the two facts only have in common the set-theoretical aspect.

However, there is an important symmetric notion, for a general $\mathbb{A}$. Consider a pair (f, u) consisting of a horizontal map $f: A \to B$ and a vertical map $u: A \twoheadrightarrow B$ (between the same objects), provided with a cell λ

$$
(1)\qquad
\begin{array}{ccc}
A & = & A \\
{\scriptstyle 1}\downarrow & \lambda & \downarrow{\scriptstyle u} \\
A & \xrightarrow[f]{} & B
\end{array}
\qquad\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle u}\downarrow & \lambda^* & \downarrow{\scriptstyle 1} \\
B & = & B
\end{array}
$$

(a *converging pairing*) both horizontally and vertically invertible, as in the left-hand diagram above (whence f is a horizontal iso and u a vertical one)

Equivalently, one can assign the right-hand diagram above, where λ^* (a *diverging pairing*) is again horizontally and vertically invertible: take $\lambda^* = (\lambda' \mid 1_f^\bullet)$, with λ' the horizontal inverse of λ. (It is easy to see that both compositions of λ and λ^* are identities: $(\lambda \mid \lambda^*) = 1_f^\bullet$, $\lambda^* \bullet \lambda = 1_u$.)

The pair (f, u), equipped with λ (or equivalently with λ^*), will be called a *double isomorphism* from A to B. We get an equivalence relation between objects, since double isomorphisms can be inverted (inverting the data horizontally and vertically) and composed (by a pasting).

In $\mathbb{T}$g, a double isomorphism consists of a pair (i, i), where i: A $\to$ B is an ordinary isomorphism of topological groups; moreover, λ and λ^* are uniquely determined, by flatness.

2.2. Vertical equivalences. In fact, we shall need a "vertically relaxed" notion of double isomorphism. To begin with, let us relax vertical isomorphisms. Recall that the *vertical 2-category* $\mathbf{A} = \mathbf{V}\mathbb{A}$ (1.3), consists of the vertical arrows of $\mathbb{A}$, with all cells whose horizontal arrows are identities

$$\text{(1)} \qquad \begin{array}{ccc} A & = & A \\ u\big\downarrow & \lambda & \big\downarrow v \\ B & = & B \end{array}$$

(The *vertical* composition of cells in $\mathbb{A}$ yields what is usually called the *horizontal* composition in the 2-category $\mathbf{A}$, and symmetrically. We shall always use horizontal and vertical *with respect to the original situation*, i.e. in $\mathbb{A}$; one should view $\mathbf{A}$ as a 2-category "disposed in vertical".)

Such a cell λ will be said to be *special*, and will also be written as λ: u $\to$ v: A $\twoheadrightarrow$ B, as appropriate within the 2-category $\mathbf{A}$. It is a *special isocell* if it is horizontally invertible in $\mathbb{A}$ (i.e., an isocell of $\mathbf{A}$); then, u and v are said to be *2-isomorphic* (u $\cong$ v). A *vertical equivalence* u: A $\twoheadrightarrow$ B is an equivalence of the vertical 2-category, i.e. a vertical morphism of $\mathbb{A}$ having a "quasi-inverse" v: B $\twoheadrightarrow$ A, with $v \bullet u \cong 1_A^\bullet$, $u \bullet v \cong 1_B^\bullet$.

2.3. Sesqui-isomorphisms. Finally, a *sesqui-isomorphism* from A to B consists of a horizontal iso linked to a vertical equivalence. Precisely, it may be assigned as a pair (λ, μ) of horizontally invertible cells, with boundary as in the

left-hand diagram below

$$(1)\qquad \begin{array}{ccc} A & = & A \\ {\scriptstyle 1}\downarrow & \lambda & \downarrow{\scriptstyle u} \\ A & \xrightarrow{f} & B \\ {\scriptstyle 1}\downarrow & \mu & \downarrow{\scriptstyle v} \\ A & = & A \end{array} \qquad\qquad\qquad \begin{array}{ccc} B & = & B \\ {\scriptstyle v}\downarrow & \mu^* & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & \lambda^* & \downarrow{\scriptstyle 1} \\ B & = & B \end{array}$$

Then, f is a horizontal iso, while the pair (u, v) is a vertical equivalence; in fact, the vertical composite of the given cells shows that $v{\bullet}u \cong 1^\bullet_A$, whereas the right-hand diagram (obtained as in 2.1.1, using the horizontal inverses of λ and μ) proves that $u{\bullet}v \cong 1^\bullet_B$. Equivalently, one can assign a pair (λ^*, μ^*) of horizontally invertible cells, as above. We get an equivalence relation between objects of $\mathbb{A}$, since sesqui-isomorphisms can be inverted (inverting the data horizontally), and composed (much as double isos, in 2.1).

The map f determines the *associated vertical equivalence* u up to special isocell. Moreover, A and B are sesqui-isomorphic iff there is a horizontally invertible cell λ as in (1), *where u is a vertical equivalence.* (Given a quasi-inverse v, we reconstruct $\mu^* = (\frac{1_v}{\lambda} \mid \alpha)$, with $\alpha\colon u{\bullet}v \cong 1^\bullet_B$.)

In $\mathbb{T}$g (2.1), a special cell is necessarily a horizontal identity 1_u. A vertical equivalence is the same as a vertical iso, i.e. a homeomorphism. Thus, a sesqui-isomorphism is the same as a double iso and amounts to an isomorphism of topological groups. In the double category $\mathbb{Q}\mathbf{A}$ of quintets of the 2-category $\mathbf{A}$ (1.3), the vertical equivalences are the equivalences of $\mathbf{A}$ (having an inverse up to isocell); a horizontal iso is an isomorphism of $\mathbf{A}$, and can always be completed to a double iso; the latter property is investigated below, in a relaxed form.

2.4. Horizontal invariance. Say that the double category $\mathbb{A}$ is *horizontally invariant* if vertical arrows are transportable along horizontal isomorphisms. Precisely, given two horizontal isos f, g and a vertical morphism u disposed as below, there always is a horizontally invertible cell λ (a "filler")

$$(1)\qquad \begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ \vdots & \lambda & \downarrow{\scriptstyle u} \\ \bullet & \xrightarrow[g]{} & \bullet \end{array}$$

as in the well-known Kan extension property. This condition is horizontally and vertically selfdual. If it holds, *two objects* A, B *horizontally isomorphic are always sesqui-isomorphic*, hence vertically equivalent; in fact, any horizontal iso $f: A \to B$ produces two horizontally invertible cells λ^*, μ^* as in 2.3.1.

This shows that $\mathbb{T}g$ is not horizontally invariant; on the other hand, every double category of quintets $\mathbb{Q}A$ and all the examples of the Section 3 are horizontally invariant. Finally, we give a functorial version of the previous property, which will be of use for limit functors.

2.5. Lemma. *If* $\mathbb{A}$ *is a horizontally invariant, two lax double functors* $F, G: \mathbb{I} \to \mathbb{A}$ (7.2) *which are horizontally isomorphic are also "vertically equivalent". Precisely, a horizontal iso* $H: F \to G$ *produces a strong vertical transformation* $U: F \twoheadrightarrow G$ (7.4), *whose components* $Ui: Fi \twoheadrightarrow Gi$ *are* vertical equivalences associated to $Hi: Fi \cong Gi$ (2.3), *and determined as such up to special isocell.*

(In fact, we prove more: F and G are sesqui-isomorphic objects, in a suitable pseudo double category of lax double functors $\mathbb{I} \to \mathbb{A}$; but we prefer not to rest on this complicated structure, described in 7.4.)

Proof. Choose, for every i in $\mathbb{I}$, a sesqui-isomorphism extending Hi

$$(1)\qquad \begin{array}{ccc} Fi & = & Fi \\ 1\downarrow & \lambda i & \downarrow Ui \\ Fi & \xrightarrow{Hi} & Gi \\ 1\downarrow & \mu i & \downarrow Vi \\ Gi & = & Gi \end{array}$$

with horizontally invertible cells λi, μi. Now, the family of vertical equivalences Ui can be canonically extended to a strong vertical transformation of lax double functors $U: F \twoheadrightarrow G$ (7.4), as we verify below. Similarly we form $V: G \twoheadrightarrow F$; and λ, μ are horizontally invertible modifications.

To complete U, let $Uf: (Ui \,{}^{Ff}_{Gf}\, Uj)$ (for $f: i \to j$ in $\mathbb{I}$) be the composed cell

$$(2)\qquad \begin{array}{ccccccc} Fi & = & Fi & \xrightarrow{Ff} & Fj & = & Fj \\ Ui\downarrow & (\lambda i)^{-1} & 1\downarrow & 1^\bullet & \downarrow 1 & \lambda j & \downarrow Uj \\ Gi & \xrightarrow[(Hi)^{-1}]{} & Fi & \xrightarrow[Ff]{} & Fj & \xrightarrow[Hj]{} & Gj \end{array}$$

and, for a vertical $u\colon i \twoheadrightarrow j$, let the comparison special isocell $Uu\colon Uj{\bullet}Fu \to Gu{\bullet}Ui\colon Fi \twoheadrightarrow Gj$ be the following vertical composite (λ^*j is obtained as in 2.1.1, using the horizontal inverse of λj)

$$
(3) \qquad
\begin{tikzcd}
Fi \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "\lambda i"] & Fi \arrow[d, "Ui"] \\
Fi \arrow[r, "Hi"] \arrow[d, "Fu"'] \arrow[dr, phantom, "Hu"] & Gi \arrow[d, "Gu"] \\
Fj \arrow[r, "Hj"] \arrow[d, "Uj"'] \arrow[dr, phantom, "\lambda^*j"] & Gj \arrow[d, "1"] \\
Gj \arrow[r, equal] & Gj
\end{tikzcd}
$$

In the extension to a pseudo double category $\mathbb{A}$ (1.9), one should note that the *two* possible vertical pastings of (3) yield the same result: the associativity isocells which link them, $\alpha(1^\bullet, Fu, Uj)$ and $\alpha(Ui, Gu, 1^\bullet)$, are identities (7.1), by the unitarity assumption.

3. Examples of double and pseudo double categories

Some examples are considered, mostly related to two prototypes (also considered in Street [23] and Kelly-Street [15]), the pseudo double category $\mathbb{C}\mathrm{at}$ of categories, functors and profunctors and the double category $\mathbb{A}\mathbf{dCat}$ of categories, functors and adjunctions. All the examples are horizontally invariant (so that sesqui-isomorphisms reduce to horizontal isos, by 2.4).

3.1. Functors and profunctors. Our first prototype is the *pseudo* double category $\mathbb{C}\mathrm{at}$ of (small) *categories, functors and profunctors* (or distributors, or bimodules; cf. Bénabou [4], Lawvere [16]). In a general cell

$$
(1) \qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\sigma"] & A' \arrow[d, "v"] \\
B \arrow[r, "g"'] & B'
\end{tikzcd}
$$

an object is a category, a horizontal arrow is a functor, a vertical arrow $u\colon A \twoheadrightarrow B$

is a profunctor $u: A^{op}{\times}B \to \mathbf{Set}$, and $\sigma: u \to v(f, g): A^{op}{\times}B \to \mathbf{Set}$ is a natural transformation. The composition of u with $v: B \twoheadrightarrow C$ is given by a coend

(2) $(v{\bullet}u)(a, c) = \int^b v(b, c) \times u(a, b)$.

It is useful to view the elements $\lambda \in u(a, b)$ as new formal arrows $\lambda: a \twoheadrightarrow b$ from the objects of A to the ones of B. Together with the objects and old arrows of A and B, we form thus a new category $A{+_u}B$ known as the *gluing*, or *collage*, of A and B along u (which will be shown below to be a double colimit, the *cotabulator of* u in $\mathbb{C}\mathrm{at}$, 6.3); the composition between old and new arrows is determined by the action of u on the old ones

(3) $\beta\lambda\alpha = u(\alpha, \beta)(\lambda)$ $\qquad\qquad$ $(\alpha: a' \to a,\ \beta: b \to b')$.

Thus, the profunctor u amounts to a category C containing $A{+}B$ and, possibly, additional arrows from objects of A to objects of B; or, more formally, to a *category over* $\mathbf{2}$, $C \to \mathbf{2}$ (with A and B over 0 and 1, respectively). An element of $(v{\bullet}u)(a, c)$ is an equivalence class $\mu{\otimes}\lambda: a \twoheadrightarrow c$, where $\lambda: a \twoheadrightarrow b$, $\mu: b \twoheadrightarrow c$, and the equivalence relation is generated by $\mu'\beta{\otimes}\lambda \sim \mu'{\otimes}\beta\lambda$ (β in B). The cell σ, in (1), corresponds to a functor over $\mathbf{2}$ (with f and g over 0 and 1)

(4) $\sigma: A{+_u}B \to A'{+_v}B'$, $\qquad\qquad$ $(\lambda: a \twoheadrightarrow b) \mapsto (\sigma(\lambda): fa \twoheadrightarrow gb)$.

(The cell σ amounts also to a morphism of profunctors $g_*{\bullet}u \to v{\bullet}f_*$, where $f_*: A^{op}{\times}A' \to \mathbf{Set}$ is the associated profunctor $f_*(a, a') = B(fa, a')$.)

To simplify things, we assume that the choice of the coend in (2) is bound "to make vertical identities strict". It is also possible to turn the pseudo double category $\mathbb{C}\mathrm{at}$ into a *strict* one, by letting a profunctor $u: A \twoheadrightarrow B$ be defined by a cocontinuous functor

(5) $u^\wedge: \mathbf{Set}^A \to \mathbf{Set}^B$, $\qquad\qquad$ $u^\wedge(\Phi)(b) = \int^a u(a, b) \times \Phi(a)$

in the same way as a relation $r: X \twoheadrightarrow Y$ can be defined as a sup-preserving map $r^\wedge: 2^X \to 2^Y$. However, we shall keep the usual setting, where it is simpler to do computations.

3.2. Spans. The full substructure of $\mathbb{C}\mathrm{at}$ determined by the discrete categories gives the pseudo double category $\mathbb{S}\mathrm{et} = \mathbb{S}\mathrm{p}\mathbf{Set}$ of *sets, mappings and spans*. In fact, a profunctor $u: A^{op}{\times}B \to \mathbf{Set}$ between discrete categories is a family of sets indexed on $A{\times}B$, and amounts to the usual presentation of a span as a pair of mappings

(1) $A \leftarrow \Sigma\, u(a, b) \rightarrow B$

while a cell $\sigma: u \rightarrow v(f, g)$ is a family of mappings $\sigma(a, b): u(a, b) \rightarrow v(fa, gb)$ and defines a mapping $\sigma: \Sigma\, u(a, b) \rightarrow \Sigma\, v(fa, gb) \rightarrow \Sigma\, v(a', b')$.

We obtain thus a direct description of $\mathbb{S}$et. The horizontal morphisms belong to **Set**; a vertical arrow $u = (u_1, u_2): A \twoheadrightarrow B$ is a span; a cell σ is a commutative diagram of mappings

$$(2) \qquad \begin{tikzcd} A \arrow[r, "f"] & A' \\ U \arrow[u, "u_1"] \arrow[d, "u_2"'] \arrow[r, "\sigma"] & V \arrow[u, "v_1"'] \arrow[d, "v_2"] \\ B \arrow[r, "g"] & B' \end{tikzcd}$$

spans are composed by a choice of pullbacks (under the constraint that "pullbacks preserve identities"); the horizontal and vertical compositions of cells are obvious. The first law gives an ordinary category, while the second behaves in a bicategorical way, with the usual equations satisfied up to *special isocells* (2.2), i.e. horizontally invertible cells whose horizontal arrows are identities.

Here, a special isocell is given by a bijective mapping $\sigma: U \rightarrow V$ (with $v_i\sigma = u_i$). An endospan $u = (u_1, u_2): A \twoheadrightarrow A$ is equivalent to the vertical identity iff u_1, u_2 are the same bijection. A vertical equivalence is a span whose components are bijections.

The embedding $D: \mathbb{S}\mathrm{et} \rightarrow \mathbb{C}\mathrm{at}$ is a pseudo double functor. Its horizontal 1-category level D: **Set** $\rightarrow$ **Cat** has a left adjoint π_0: **Cat** $\rightarrow$ **Set**, associating to a category its set of connected components and to a functor the induced mapping. It is natural to define π_0 on profunctors: the span $(\pi_0 u)(\bar{a}, \bar{b})$ is a quotient of $\Sigma u(a, b)$, where a formal arrow $\lambda: a \twoheadrightarrow b$ is identified to all composites $\beta\lambda\alpha$: $a' \rightarrow a \twoheadrightarrow b \rightarrow b'$. (The theory of adjoints for double functors, to be developed in a sequel, gives a *colax* left adjoint $\pi_0: \mathbb{C}\mathrm{at} \rightarrow \mathbb{S}\mathrm{et}$, with comparison special cells $\pi_0(v \bullet u) \rightarrow \pi_0 v \bullet \pi_0 u$.)

Similarly, we have the pseudo double category $\mathbb{C}$**ospSet**, where the vertical arrows are *cospans* $u = (u_1, u_2) = (A \rightarrow U \leftarrow B)$ and the cells are mappings σ: $U \rightarrow V$ producing two commutative squares. The pseudo double category $\mathbb{C}$**ospTop** of *topological spaces, continuous mappings and topological cospans* contains a pseudo double subcategory $\mathbb{M}$nf related with cobordism and topological quantum field theory [1, 25]: an object is a topological manifolds, a horizontal

arrow is a continuous mapping between such manifolds, a vertical arrow is a cospan u which is either an identity, or subject to the following conditions: U is a manifold with boundary, the u_i are homeomorphisms onto their images, and the boundary of U is the disjoint union of these two images; a cell is continuos mapping $\sigma: U \to V$ as above.

3.3. Additive categories and metric spaces. Let a symmetric monoidal category $\mathcal{V}$ be given, having all coends, preserved by the functors $v\otimes-$, $-\otimes v$, for all v in $\mathcal{V}$. The prototype $\mathbb{C}$at considered above can be generalised, forming the pseudo double category $\mathcal{V}$-$\mathbb{C}$at of *$\mathcal{V}$-categories, $\mathcal{V}$-functors* and *$\mathcal{V}$-profunctors* (cf. Kelly [14]).

In particular, $\mathcal{V}$ = **Ab** gives the pseudo double category of ***preadditive categories, with additive functors and profunctors.*** $\mathcal{V}$ = **2** gives the (ordinary) double category $\mathbb{P}$os of *posets* (preordered sets) with *monotonic functions* and ***poset-profunctors*** $u: X^{op}\times Y \to \mathbf{2}$, consisting of relations $u \subset X\times Y$ down-closed in X and up-closed in Y (also called *order ideals*); they compose as relations. $\mathbb{P}$os is flat, and has one cell with boundary $(u\,{}^{f}_{g}\,v)$ iff $gu \leq vf$.

It is also interesting to work out the case introduced by Lawvere [16] to formalise (extended) metric spaces as categories enriched in the strict monoidal category $\mathbf{R}_+ = [0, +\infty]$, with arrows given by the order relation $\geq$ and tensor product given by the sum. The double category $\mathbb{M}$tr = $\mathbf{R}_+$-$\mathbb{C}$at has for objects the metric spaces X, *in the generalised* (non symmetric) *sense of* $\mathbf{R}_+$-categories

(1) $d(x, y) \in [0, +\infty]$, $\qquad d(x, x) = 0$, $\qquad d(x, y) + d(y, z) \geq d(x, z)$

for horizontal arrows the (weak) contractions $f: X \to X'$ $(d(x, y) \geq d(fx, fy))$, for vertical arrows the profunctors $u: X \twoheadrightarrow Y$, represented by $\mathbf{R}_+$-functors

(2) $u: X^{op}\times Y \to \mathbf{R}_+$,

$$u(x, y) + d(y, y') \geq u(x, y'), \qquad d(x, x') + u(x', y) \geq u(x, y)$$

which compose by a coend, calculated as a greatest lower bound in $[0, +\infty]$

(3) $(v\bullet u)(x, z) = \bigwedge_y (u(x, y) + v(y, z))$.

As in 3.1, u(x, y) should be viewed as the distance from a point of X to a point of Y, defining a new metric space, the collage $X+_uY$ (with $d(y, x) = +\infty$). $\mathbb{M}$tr is flat, as the cell $\alpha: g_*u \to vf_*$ just corresponds to the following inequalities

(4)
$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle u}\downarrow & \geq & \downarrow{\scriptstyle v} \\ Y & \xrightarrow[g]{} & Y' \end{array} \qquad u(x, y) \geq v(fx, gy) \qquad (x \in X,\ y \in Y)$$

or, equivalently, $u(x, y) + d(gy, y') \geq v(fx, y')$.

The embedding of strict monoidal categories

(5) $\mathbf{2} \rightarrow \mathbf{R}_+$ $\qquad 0 \mapsto +\infty,\ 1 \mapsto 0$

gives an embedding of double categories (reflective and lax coreflective, according to a notion of adjoint double functors to be studied in a sequel)

(6) $M: \mathbb{P}\mathrm{os} \rightarrow \mathbb{M}\mathrm{tr}$, $\qquad d_{MX}(x, x') = 0\ (+\infty) \quad \Leftrightarrow \quad x \prec x'$ (otherwise)

identifying $\mathbb{P}\mathrm{os}$ to the double subcategory of $\mathbb{M}\mathrm{tr}$ consisting of those metric spaces whose distance takes values in $\{0, +\infty\}$, their weak contractions and their profunctors with values in $\{0, +\infty\}$

(7) $Mu: X{\times}Y \rightarrow \{0, +\infty\}$, $\qquad Mu(x, y) = 0 \quad \Leftrightarrow \quad (x, y) \in u$.

(In this respect, a generalised metric space X can be viewed as a preordered set equipped with further information $d(x, x')$.)

3.4. Relations. In $\mathbb{P}\mathrm{os}$, the full double subcategory of *discrete* posets ($x \prec x'$ iff $x = x'$) is the (flat) double category $\mathbb{R}\mathrm{el} = \mathbb{R}\mathrm{el}\mathbf{Set}$ of *sets, mappings and relations*.

By the previous embedding, $\mathbb{R}\mathrm{el}$ can also be viewed as the double subcategory of $\mathbb{M}\mathrm{tr}$ consisting of *discrete* metric spaces, with $d(x, x') = +\infty$ if $x \neq x'$ (and 0 otherwise), their weak contractions and their profunctors with values in $\{0, +\infty\}$. There is also a split *lax embedding* (identifying relations with jointly monic spans)

(1) $S: \mathbb{R}\mathrm{el} \rightarrow \mathbb{S}\mathrm{et}$, $\qquad R: \mathbb{S}\mathrm{et} \rightarrow \mathbb{R}\mathrm{el}$, $\qquad RS = 1$

where R is the obvious double functor taking a span to the associated relation, while S is the *lax* double functor taking a relation $u \subset A{\times}B$ to the jointly monic span $Su = (A \leftarrow u \rightarrow B)$, having a comparison cell $(Sv)\bullet(Su) \rightarrow S(v\bullet u)$ from a (composed) span to the jointly monic span defining the same relation.

Similarly, we have the double category $\mathbb{R}\mathrm{el}\mathbf{Ab}$ of *abelian groups, homomorphisms and relations*. More generally, any abelian category $\mathbf{A}$ has an associated

double category $\mathbb{R}\mathrm{el}A$ of morphisms and relations (and this can be extended in various directions, e.g. regular categories or Puppe-exact ones).

This construction arises naturally. Indeed, if $F: A \to B$ is an exact functor between abelian categories, F extends to a unique functor $\mathrm{Rel}F: \mathrm{Rel}A \to \mathrm{Rel}B$ preserving order and involution, hence to a double functor $\mathbb{R}\mathrm{el}F: \mathbb{R}\mathrm{el}A \to \mathbb{R}\mathrm{el}B$. Further, if $\alpha: F \to G: A \to B$ is any natural transformation between such functors, the *same* components form a *lax*-natural transformation $\mathrm{Rel}\alpha: \mathrm{Rel}F \to \mathrm{Rel}G$ and a horizontal transformation $\mathbb{R}\mathrm{el}\alpha: \mathbb{R}\mathrm{el}F \to \mathbb{R}\mathrm{el}G$. *Regular inductive squares* (providing regularly induced morphisms between subquotients) are cells of the previous type, where $u: A \twoheadrightarrow A'$ and $v: B \twoheadrightarrow B'$ are monorelations (subquotients), $g: A' \to B'$ is the inducing morphism and $f: A \to B$ the induced one, coinciding with $v^{\#}gu$ [17, 10].

3.5. Functors and adjunctions. We consider now a second prototype, the double category $\mathbb{A}\mathbf{dCat}$ of (small) *categories, functors and adjunctions*. In a general cell

$$(1) \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\downarrow\alpha"] & A' \arrow[d, "v"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

each object is a category, a horizontal arrow is a functor, a vertical arrow is an adjunction directed as its *left*-hand component

$$(2) \quad u = (u_\bullet, u^\bullet, \eta, \varepsilon), \qquad (u_\bullet: A \to B) \dashv (u^\bullet: B \to A)$$
$$\eta: 1 \to u^\bullet u_\bullet, \qquad \varepsilon: u_\bullet u^\bullet \to 1$$

and finally $\alpha = (\alpha_\bullet, \alpha^\bullet)$ is a pair of natural transformations (*from* f *to* g, as made precise below) each of them determining the other one, via the units and counits of the two adjunctions

$$(3) \quad \alpha_\bullet: v_\bullet f \to g u_\bullet, \qquad \alpha^\bullet: f u^\bullet \to v^\bullet g$$
$$\alpha^\bullet = (fu^\bullet \to v^\bullet v_\bullet f u^\bullet \to v^\bullet g u_\bullet u^\bullet \to v^\bullet g),$$
$$\alpha_\bullet = (v_\bullet f \to v_\bullet f u^\bullet u_\bullet \to v_\bullet v^\bullet g u_\bullet \to g u_\bullet).$$

Horizontal isos are isomorphisms of categories, vertical equivalences are adjoint pairs which are equivalences. There is an obvious forgetful double functor $\mathbb{A}\mathbf{dCat}^t \to \mathbb{Q}\mathbf{Cat}$, which takes u to $u_\bullet$ and α to $\alpha_\bullet$. Abelian categories, with their exact functors and adjunctions, form a cellwise-full double subcategory

$\mathbb{A}$**dAbc** of the previous one. Similarly, we write $\mathbb{A}$**dTp** the double subcategory of $\mathbb{A}$**dCat**$^{\mathrm{v}}$ (vertically reversed) consisting of toposes, logical morphisms (functors which preserve the topos structure) and geometrical morphisms (adjunctions whose left-hand part preserves finite limits, directed according to the right adjoint, as usual).

3.6. Ordered sets and adjunctions. The double category $\mathbb{A}$**dOrd** of (small) *ordered sets, functors* (monotonic functions) *and adjunctions* (or "covariant connections") is a full double subcategory of $\mathbb{A}$**dCat**. It is flat, since a cell α: $(u \, {}^{f}_{g} \, v)$ as above (3.5.1) exists iff $v_\bullet f \leq g u_\bullet$ (or equivalently $f u^\bullet \leq v^\bullet g$) and is then determined by its boundary.

Important double subcategories of $\mathbb{A}$**dOrd**, full with respect to *vertical arrows* and *cells*, are:

- $\mathbb{A}$**dLt**: lattices (with 0 and 1), homomorphisms and adjunctions;
- $\mathbb{A}$**dMl**: modular lattices, homomorphisms and *modular* connections.

A *modular connection* $u: X \twoheadrightarrow Y$ is a pair of monotonic functions $u_\bullet: X \to Y$, $u^\bullet: Y \to X$ between modular lattices, which satisfies the following conditions, stronger than the adjunction ones: $u^\bullet u_\bullet(x) = x \vee u^\bullet(0)$, $u_\bullet u^\bullet(y) = y \wedge u_\bullet(1)$ (for $x \in X$, $y \in Y$) [9].

We also consider further restrictions, of more direct interest in homological algebra, the double subcategories $\mathbb{A}$**d$_0$Lt** and $\mathbb{A}$**d$_0$Ml** having the same objects and arrows but only *bicommutative cells*, with $v_\bullet f = g u_\bullet$ and $f u^\bullet = v^\bullet g$. $\mathbb{A}$**d$_0$Ml** was used to treat formally direct and inverse images of subobjects for categories of modules, or more generally for abelian or Puppe-exact categories [9], while $\mathbb{A}$**d$_0$Lt** plays a similar role in the more general context of "semiexact" and "homological" categories, in the sense of [11]. As proved in 6.5 e, $\mathbb{A}$**d$_0$Ml** is "vertically Puppe-exact"; its category of "vertical relations" was studied in [9], Section 3 (*modular relations*).

4. Double limits

Double limits for double functors $F: \mathbb{I} \to \mathbb{A}$ are considered. $\mathbb{I}$ is assumed to be small, and F is viewed as an object in the double category $\mathbb{A}^{\mathbf{I}} = \mathbb{X}$.

4.1. Cones. Consider the *diagonal* double functor

(1) $D\colon \mathbb{A} \to \mathbb{A}^{\mathbb{I}} = \mathbb{X}$

taking each object A to the constant double functor $DA\colon \mathbb{I} \to \mathbb{A}$.

A (horizontal) *double cone* for the double functor $F \in \mathbb{A}^{\mathbb{I}}$ is a horizontal transformation $x\colon DA \to F\colon \mathbb{I} \to \mathbb{A}$, where A (the *vertex* of the cone) is in $\mathbb{A}$. By definition (1.4), this amounts to assigning the following data a), b), subject to the axioms (dc.1-3):

a) horizontal maps $xi\colon A \to Fi$, for i in $\mathbb{I}$,

b) cells $xu\colon (1^\bullet_A \, {}^{xi}_{xj} \, Fu)$, for $u\colon i \twoheadrightarrow j$ in $\mathbb{I}$,

(dc.1) $x(1^\bullet_i) = 1^\bullet_{xi}\colon (1^\bullet_A \, {}^{xi}_{xi} \, 1^\bullet_{Fi})$, for i in $\mathbb{I}$,

(dc.2) $x(v\bullet u) = xv\bullet xu$, for u, v vertical in $\mathbb{I}$,

(dc.3) $(xu \mid F\alpha) = xv$, for $\alpha\colon (u \, {}^f_g \, v)$ in $\mathbb{I}$.

More precisely (as $\mathbb{I}$ might be empty, in which case DA does not determine A), a double cone of F is a *pair* $(A, x\colon DA \to F)$ as above, i.e. an object of the comma $(D \Downarrow F)$, described in 1.7.

If we allow $\mathbb{A}$ to be a (unitary) *pseudo* double category (7.1) and $F = (F, \varphi)\colon \mathbb{I} \to \mathbb{A}$ a (unitary) *lax* double functor (7.2), with comparison special cells $\varphi(u, v)\colon Fv\bullet Fu \to F(v\bullet u)$, then the axiom (dc.2) is to be replaced with

(dc.2') $x(v\bullet u) = (\frac{xu}{xv} \mid \varphi(u, v))$, for u, v vertical in $\mathbb{I}$.

This more general situation will only be considered marginally (at the end of 4.2, 4.3, 5.5 and 5.7). No modification is needed when $\mathbb{I}$ is just a double graph.

4.2. Limits. A (horizontal) *double limit* $\lim(F) = (A, x)$ of the double functor $F \in \mathbb{A}^{\mathbb{I}}$ is a universal such cone $(A, x\colon DA \to F)$, i.e. a double terminal of $(D \Downarrow F)$, if such exists. Specifying the conditions (t.1-2) in 1.8, this means that

(dl.0) $x\colon DA \to F$ is a horizontal transformation (4.1),

(dl.1) for every double cone $(A', x'\colon DA' \to F)$ there is precisely one horizontal map $t\colon A' \to A$ in $\mathbb{A}$ such that $x\circ Dt = x'$,

(dl.2) for every vertical arrow in $(D \Downarrow F)$

(1) $(u, \xi)\colon (A', x'\colon DA' \to F) \twoheadrightarrow (A'', x''\colon DA'' \to F)$

where $u\colon A' \twoheadrightarrow A''$ is vertical in $\mathbb{A}$ and $\xi\colon (Du \, {}^{x'}_{x''} \, 1^\bullet_F)$ is an $\mathbb{X}$-cell, there is precisely one cell τ in $(D \Downarrow F)$ with boundary

$$(2) \qquad \begin{tikzcd} (A', x'\colon DA' \to F) \arrow[r, "t'"] \arrow[d, "{(u,\xi)}"'] \arrow[dr, phantom, "\tau"] & (A, x\colon DA \to F) \arrow[d, equal] \\ (A'', x''\colon DA'' \to F) \arrow[r, "t''"'] & (A, x\colon DA \to F) \end{tikzcd}$$

or, in other words, precisely one $\mathbb{A}$-cell τ such that

(3) $\tau\colon (u \,{}^{t'}_{t''}\, 1^\bullet_A)$, $\qquad\qquad$ $(D\tau \,|\, x) = \xi$ (in $\mathbb{A}^{\mathbb{I}}$)

i.e. $(\tau \,|\, xi) = \xi i$ in $\mathbb{A}$, for all i in $\mathbb{I}$

$$(4) \qquad \begin{tikzcd} A' \arrow[r, "t'"] \arrow[d, "u"'] \arrow[dr, phantom, "\tau"] & A \arrow[r, "xi"] \arrow[d, equal] \arrow[dr, phantom, "1^\bullet"] & Fi \arrow[d, equal] \\ A'' \arrow[r, "t''"'] & A \arrow[r, "xi"'] & Fi \end{tikzcd} \quad = \quad \begin{tikzcd} A' \arrow[r, "x'i"] \arrow[d, "u"'] \arrow[dr, phantom, "\xi i"] & Fi \arrow[d, equal] \\ A'' \arrow[r, "x''i"'] & Fi \end{tikzcd}$$

a) Note that (dl.2) implies (dl.1), by 1.8. The conditions (dl.0-2) only depend, as far as $\mathbb{I}$ is concerned, on its *double graph* structure plus the *composition of vertical maps*; and the latter is only relevant for (dc.1-2).

b) The uniqueness part in (dl.2) means that the projections $xi\colon A \to A_i$ are jointly monic with respect to $\mathbb{A}$-cells $\tau\colon (u \,{}^{t'}_{t''}\, 1^\bullet_A)$.

c) If $\mathbb{A} = \mathbb{H}\mathbf{A}$ is the 1-*horizontal* double category associated to a category $\mathbf{A}$, the double limits in $\mathbb{A}$ reduce to ordinary limits in $\mathbf{A}$. If $\mathbf{A}$ is a 2-category (and $\mathbb{A} = \mathbb{H}\mathbf{A}$ is *horizontal*), a double limit in $\mathbb{A}$ can be viewed as a *weighted limits* in $\mathbf{A}$ and conversely ([20, 24]; cf. 6.6). On the other hand, the restriction to 1-horizontal $\mathbb{I} = \mathbb{H}\mathbf{I}$ yields double limits of *horizontal* functors, considered in 5.1.

d) If F and G have a double limit, a horizontal transformation $H = ((Hi), (Hu))\colon F \twoheadrightarrow G\colon \mathbb{I} \to \mathbb{A}$ determines a horizontal arrow $\lim H\colon \lim F \to \lim G$. Vertical transformations are considered below.

e) No modification is needed for the lax case (4.1), except what we already said about cones.

In a double category, a double limit will also be called a *limit*, provided no ambiguity may arise. When, occasionally, we want to refer uniquely to the first universal property (dl.0-1), we speak of a *one-dimensional limit.* Let us recall now that Bastiani-Ehresmann ([2], p. 258) introduced the notion of an $\mathbb{A}$-*wise limit* of a functor $f\colon \mathbf{I} \to \mathrm{ver}_0\mathbb{A}$, defined over an ordinary category, with values in the category of objects and vertical arrows of a double category. This notion

corresponds, in the present terminology, to the *one-dimensional limit* of the associated *vertical double functor* $F: \mathbf{VI} \to \mathbb{A}$ (see also 5.3.)

4.3. Vertical functoriality. The notion of double limit presents two problems with respect to the vertical structure, namely "functoriality" and "uniqueness", which we turn now to investigate. First, the existence of $\mathbb{I}$-limits does not automatically produce a (lax) double functor, because the limit of vertical transformations may fail (as happens for equalisers in $\mathbb{A}\mathbf{dCat}$ or $\mathbb{A}\mathbf{dOrd}$; 6.5).

To be precise, the *limit of a vertical transformation* $U = ((Ui), (Uf)): F \twoheadrightarrow G: \mathbb{I} \to \mathbb{A}$ is a pair $(u, \pi: Du \to U)$, universal with respect to the horizontal composition of modifications (formally, a one-dimensional limit for $U: \mathbb{I} \to \mathbb{A}^{V2}$, see 1.6). In other words, we have a modification $\pi: (Du\ {}^{p}_{q}\ U)$ such that every $\xi: (Dv\ {}^{x}_{y}\ U)$ factors uniquely as $(D\varphi \mid \pi)$

$$(1)\qquad \begin{array}{ccccc} X & \xrightarrow{f} & A & \xrightarrow{pi} & Fi \\ {\scriptstyle v}\downarrow & \varphi & {\scriptstyle u}\downarrow & \pi i & \downarrow{\scriptstyle Ui} \\ Y & \xrightarrow[g]{} & B & \xrightarrow[qi]{} & Gi \end{array} \quad = \quad \begin{array}{ccc} X & \xrightarrow{xi} & Fi \\ {\scriptstyle v}\downarrow & \xi i & \downarrow{\scriptstyle Ui} \\ Y & \xrightarrow[yi]{} & Gi \end{array}$$

The solution π is unique up to horizontal composition with a horizontally invertible cell φ; and the latter is necessarily special, if π is confined to have as horizontal arrows $p: DA \to F$, $q: DB \to G$ two *fixed* (even one-dimensional) double limits of F and G. The universal property (dl.2) amounts to saying that the vertical identity of the one-dimensional limit of F is the limit of $1^{\bullet}_{F}$.

We say that $\mathbb{A}$ has *a lax* (resp. *pseudo, strict*) *functorial choice* of $\mathbb{I}$-limits if there is a lax (resp. pseudo, ordinary) double functor $L = (L, \varphi): \mathbb{A}^{\mathbb{I}} \to \mathbb{A}$ (7.2) with a horizontal transformation $p = ((p^F), (\pi^U)): DL \to 1: \mathbb{A}^{\mathbb{I}} \to \mathbb{A}^{\mathbb{I}}$ satisfying the following conditions

i) for any double functor $F \in \mathbb{A}^{\mathbb{I}}$, $(LF, p^F: DLF \to F)$ is a double limit in $\mathbb{A}$, with components $p^F i = pi: LF \to Fi$, $p^F u = pu: (1^{\bullet}\ {}^{pi}_{pj}\ u)$),

ii) for any vertical transformation $U = ((Ui), (Uf)): F \twoheadrightarrow G$, $(LU, \pi^U: DLU \to U)$ is a limit.

Thus, L takes all families (of objects, arrows and cells) to their limit, coherently with domains and codomains, identities and compositions, in the appropriate way –lax, or pseudo, or strict– with respect to the vertical composition. If (L', q) is a second lax functorial choice, there is a unique horizontal transformation of

lax double functors (7.3) $H: L \to L'$ such that $(DH \mid q) = p: DL \to 1$, and H is horizontally invertible.

Dually, for double $\mathbb{I}$-*colimits*, a *colax functorial choice* consists of a colax double functor $C = (C, \gamma): \mathbb{A}^{\mathbb{I}} \to \mathbb{A}$ with a horizontal transformation $k = ((k^F), (\kappa^U)): 1 \to DC$.

For a pseudo double category $\mathbb{A}$, the same terminology will be adopted (with some abuse), replacing the exponential $\mathbb{A}^{\mathbb{I}}$ with the pseudo double category $Ps(\mathbb{I}, \mathbb{A})$ of pseudo double functors (7.4); but the existence of a *strict* functorial choice has no longer any interest (nor practically any sense, when the vertical composition is defined by invoking the axiom of choice). Finally, as a partial motivation of the unitarity assumption, note that a lax double functor L satisfying i), ii) is necessarily *pseudo unitary*, and can always be made unitary (as in 7.5).

4.4. Lemma. *A* lax *functorial choice of* $\mathbb{I}$*-limits in* $\mathbb{A}$ *can be equivalently reduced to the following choices*:

a) *for every double functor* $F: \mathbb{I} \to \mathbb{A}$, *a "one-dimensional limit"* (LF, p^F), *satisfying* (dl.1),

b) *for every vertical transformation* $U = ((Ui), (Uf)): F \twoheadrightarrow G: \mathbb{I} \to \mathbb{A}$, *a limit*

(1) $LU: LF \twoheadrightarrow LF$, $\qquad\qquad \pi^U: (DLU \; {}^{p^F}_{p^F} \; U)$

with respect to the horizontal composition of modifications (as specified above, 4.3.1), *so that vertical identities are preserved* $(L1^\bullet_F = 1^\bullet_{LF})$.

Our choice L *is* pseudo, *or* strict, *iff its comparison special cells* $\varphi(U, V)$: $LV \bullet LU \to LW$ *(determined below, in* (2)) *are horizontally invertible, or identities, respectively.*

Proof. In fact, we can (uniquely) complete the data a) b), to form a lax double functor $L = (L, \varphi): \mathbb{A}^{\mathbb{I}} \to \mathbb{A}$ and a horizontal transformation $p = ((p^F), (\pi^U))$: $DL \to 1: \mathbb{A}^{\mathbb{I}} \to \mathbb{A}^{\mathbb{I}}$, which satisfy the definition 4.3:

- for any horizontal transformation $H: F \to F'$, let $LH: LF \to LF'$ be its limit (4.2 d), using a); the horizontal composition and identities of such transformations are automatically preserved;

- for any cell α in $\mathbb{A}^{\mathbb{I}}$, let $L\alpha$ be its limit, using b), consistently with the previous choices (by hypothesis); again, horizontal composition and identities are preserved;

- for any vertical composition $W = V \bullet U: F \twoheadrightarrow G \twoheadrightarrow H$ in $\mathbb{A}^{\mathbb{I}}$, let the

comparison special cell $\varphi(U, V)$ be determined by the universal property of $(LW, (\pi^W i))$

(2) $\varphi(U, V)\colon LV \bullet LU \to LW\colon LF \twoheadrightarrow LH, \qquad (D\varphi(U, V) \mid \pi^W i) = \pi^V i \bullet \pi^U i.$

The coherence equations follow now from the cancellation property of universal solutions.

4.5. Invariant functorial limits. The second problem, "vertical uniqueness" of double $\mathbb{I}$-limits, is still unsolved. In fact, it is true that, given a pseudo (resp. strict) functorial choice L, a limit of vertical isos is a vertical equivalence (resp. a vertical iso), so that *–inside this choice–* a limit is also vertically determined. However, there may exist *various choices* for L, vertically non-equivalent; this is the case of binary products in $\mathbb{T}$g (2.1), where we may choose to equip the algebraic product $G{\times}H$ with the product topology, or the discrete one, or the chaotic one...

A solution is provided by assuming that our double category $\mathbb{A}$ be *horizontally invariant* (2.4), as are all our examples of real interest (Section 3). Then, for any two limits A, B of the same double functor $F\colon \mathbb{I} \to \mathbb{A}$, the canonical horizontal iso $A \to B$ has an associated vertical equivalence $A \twoheadrightarrow B$ determined up to special isocell (2.3). By Lemma 2.5, this fact can be extended to two lax functorial choices (L, p) and (L', q) of $\mathbb{I}$-limits in $\mathbb{A}$: the canonical (and invertible) horizontal transformation $H\colon L \to L'$ (7.3) produces a strong vertical transformation $U\colon L \twoheadrightarrow L'$, whose general component $UF\colon LF \twoheadrightarrow L'F$ is a vertical equivalence associated to HF.

Thus, in a horizontally invariant double category $\mathbb{A}$, a functorial lax choice of $\mathbb{I}$-limits *is also vertically determined*, and we just speak of (invariant) *lax functorial $\mathbb{I}$-limits*, or *pseudo functorial $\mathbb{I}$-limits*, or *functorial $\mathbb{I}$-limits*. (In the last case, we mean that we have *pseudo* functorial $\mathbb{I}$-limits, and we are actually exhibiting a realisation which is an ordinary double functor.) The characterisation of Lemma 4.4 can now be expressed more simply, as in theorem 4.6, below. All this extends to pseudo double categories and limits of lax double functors; except that, now, a strict solution may make no sense (see 6.3).

Finally, let us remark that there is scarce interest in giving a direct notion of "invariant lax functorial $\mathbb{I}$-limits", independent of the horizontal invariance of $\mathbb{A}$ and based on the existence of "horizontally invariant" limits of vertical transformations for $\mathbb{I}$ (as in 4.6a). In fact, and plainly, $\mathbb{A}$ has *unary* limits ($\mathbb{I} = \mathbf{1}$) of this kind *precisely iff it is horizontally invariant.* (Unary *co*limits give the same.)

4.6. Invariance Theorem. *Assume that the double category* $\mathbb{A}$ *is horizontally invariant* (2.4).

a) *The existing* $\mathbb{I}$*-limits of vertical transformations are also* horizontally invariant, *in the sense that we can modify their domain and codomain up to horizontal isomorphism. Precisely, let a vertical transformation* $U\colon F \twoheadrightarrow G\colon \mathbb{I} \to \mathbb{A}$ *be given, with double limits* (A, p) *of* F *and* (B, q) *of* G, *and a consistent limit* (u, π) *of* U *(i.e.,* $u\colon A \twoheadrightarrow B$ *and* $\pi\colon (Du\,{}^{p}_{q}\,U)$). *If also* (A', h) *and* (B', k) *are double limits of* F *and* G, *there is a limit* (v, ρ) *of* U *consistent with them.*

b) $\mathbb{A}$ *has lax functorial* $\mathbb{I}$*-limits iff every double functor* $F\colon \mathbb{I} \to \mathbb{A}$ *has a double limit and every vertical transformation* $U = ((Ui), (Uf))\colon F \twoheadrightarrow G\colon \mathbb{I} \to \mathbb{A}$ *has a limit* (4.3.1).

Proof. a) It is sufficient to prove that, in the limit of $U = ((Ui), (Uf))$, the given limit (B, q) of G can be replaced by any other limit (B', k); similarly, one can modify the domain. First, there is a unique horizontal iso $b\colon B \to B'$ coherent with the cones q and k $(q = (Db \mid k))$. By horizontal invariance, this b can be embedded in a cell λ, horizontally invertible (write λ' for its inverse)

$$
(1) \qquad
\begin{tikzcd}
A \arrow[rr, "pi"] \arrow[d, "u"'] \arrow[drr, phantom, "\pi i"] & & Fi \arrow[d, "Ui"] \\
B \arrow[r, "b"] \arrow[d, "y"'] \arrow[dr, phantom, "\lambda"] & B' \arrow[r, "ki"] \arrow[d, "1"] \arrow[dr, phantom, "1"] & Gi \arrow[d, "1"] \\
B' \arrow[r, "1"'] & B' \arrow[r, "ki"'] & Gi
\end{tikzcd}
$$

producing cells $\lambda i = (\lambda \mid 1^{\bullet}_{ki})$, again invertible, and $\rho i = \left(\frac{\pi i}{\lambda i}\right)$; as a pasting of modifications, this family is a modification $\rho\colon (D(y{\bullet}u)\,{}^{p}_{k}\,U)$.

We have to prove that $y{\bullet}u\colon A \twoheadrightarrow B'$ is a limit of U, with projections (ρi). Take a cone $\alpha i\colon x \to Ui$, with factorisation $\alpha i = (\alpha \mid \pi i)$

$$
(2) \qquad
\begin{tikzcd}
X' \arrow[r, "f"] \arrow[d, "x"'] \arrow[dr, phantom, "\alpha"] & A \arrow[r, "p_i"] \arrow[d, "u"] \arrow[dr, phantom, "\pi i"] & Fi \arrow[d, "Ui"] \\
X'' \arrow[r, "g"'] & B \arrow[r, "qi"'] & Gi
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "fi"] \arrow[d, "x"'] \arrow[dr, phantom, "\alpha i"] & Fi \arrow[d, "Ui"] \\
A \arrow[r, "gi"'] & Gi
\end{tikzcd}
$$

There is a unique cell μ such that $(\mu \mid \lambda i) = 1^{\bullet}_{gi}$, precisely $\mu = (1^{\bullet}_{bg} \mid \lambda')$

$$(3) \qquad \begin{array}{ccccc} X'' & \xrightarrow{g} & B & \xrightarrow{qi} & Gi \\ 1\downarrow & \mu & y\downarrow & \lambda i & \downarrow 1 \\ X'' & \xrightarrow[bg]{} & B' & \xrightarrow[ki]{} & Gi \end{array} \quad = \quad \begin{array}{ccc} X'' & \xrightarrow{gi} & Gi \\ 1\downarrow & 1 & \downarrow 1 \\ X'' & \xrightarrow[gi]{} & Gi \end{array}$$

and the vertical pasting of the diagrams (2) and (3) provides the solution of our problem, $\mu \bullet \alpha$. Its uniqueness can be proved similarly. Finally, b) is a rewriting of Lemma 4.4, once we know that, in the limit of a horizontal transformation, "domains and codomains can be modified".

4.7. Regular limits. We end this Section introducing a stronger notion of double limit, defined by a further universal property of "vertical terminality" and providing a second solution to the two problems considered above.

Let $F: \mathbb{I} \to \mathbb{A}$ be a double functor. We say that the cone $(X, p: DX \to F)$ is a *regular (double) limit* if it is a double limit (satisfies (dl.1-2) and moreover

(dl.3) for any double functor $G \in \mathbb{A}^{\mathbf{I}}$, any vertical transformation $U = ((Ui), (Uf)): G \twoheadrightarrow F$ and any horizontal cone $q: DY \to G$, there exists a vertical arrow $u: Y \twoheadrightarrow X$ and a modification $\pi: (Du\,{}^{q}_{p}\,U)$ (consisting of cells πi as in the middle square below, coherent with the cells Uf of U), which are terminal in the following sense

$$(1) \qquad \begin{array}{ccccc} Y' & \xrightarrow{g} & Y & \xrightarrow{qi} & Gi \\ v\downarrow & \varphi & \downarrow u & \pi i & \downarrow Ui \\ X' & \xrightarrow[f]{} & X & \xrightarrow[pi]{} & Fi \end{array} \quad = \quad \begin{array}{ccc} Y' & \xrightarrow{yi} & Gi \\ v\downarrow & \xi i & \downarrow Ui \\ X' & \xrightarrow[xi]{} & Fi \end{array}$$

- for any $g: Y' \to Y$, $v: Y' \twoheadrightarrow X'$ and any modification $\xi: (Dv\,{}^{y}_{x}\,U)$ (consisting of cells ξi as in the right-hand square above, coherent with the cells of U) *having vertical domain* $y = q \circ Dg$, there exists a unique cell $\varphi: (v\,{}^{g}_{f}\,u)$ such that $(D\varphi \mid \pi) = \xi$.

Plainly, the existence of regular double $\mathbb{I}$-limits in $\mathbb{A}$ produces a lax functorial choice of them; it is also evident that $\mathbb{T}\mathrm{g}$ has (vertically unique) regular products, the chaotic-algebraic ones, by the terminality of the chaotic topology. Further study of regular double limits, deferred to a sequel, will show that, again, they are determined up to sesqui-isomorphism and can be reduced to the "unary case". This shows immediately that in $\mathbb{C}\mathrm{at}$ all the existing (co)lax functorial (co)limits

(see 6.3) are regular, whereas this is not true in $\mathbb{A}$**dCat**.

But note that this further universal property (dl.3) has a different status from the previous ones, (dl.1-2), which keeps us from inserting it in the definition of double limit.

a) As category objects in **Cat**, small double categories give rise to **Cat**-indexed categories, and the indexed-category-theory notion of limit merely amounts to double limits. In fact, we get a horizontal cone which is universal with respect to all cones "whose vertex is an arbitrary vertical diagram". But this is equivalent to universality for cones "with vertex an object" (dl.1) and "cones with vertex one vertical arrow" (dl.2).

b) The property (dl.3) cannot be expressed within the comma $(D \Downarrow F)$, since it appeals to other double functors $G \in \mathbb{A}^{\mathbf{I}}$; thus, a regular limit is not the same as a regular terminal in $(D \Downarrow F)$.

c) While, in any double category $\mathbb{A}$, the unary limit of the object X is any horizontal iso $X' \to X$, the existence of *regular unary limits* in $\mathbb{A}$ is a non-trivial property (which can be used to produce regular $\mathbb{I}$-limits from any lax functorial choice of them).

5. The construction of double limits

All double limits can be constructed from products, equalisers and tabulators (Thm. 5.5).

5.1. Limits of horizontal functors. Let us treat first the easier case of a *horizontal functor* $F: \mathbb{I} \to \mathbb{A}$, which means that $\mathbb{I} = \mathbb{H}\mathbf{I}$ is 1-horizontal (for $\mathbf{I}$ a category), and F can be thought of as an ordinary functor $\bar{F} = \mathrm{hor}_0 F: \mathbf{I} \to \mathbf{A} = \mathrm{hor}_0\mathbb{A}$.

Then a double cone of F is the same as an ordinary cone of $\bar{F}$, and the first universal property (dl.1) for $\lim F = (A, x: DA \to F)$ amounts to saying that $(A, x) = \lim \bar{F}$. A double limit of a horizontal functor is thus an ordinary limit which also satisfies (dl.2), i.e. is preserved by the functor $1^\bullet_-: \mathrm{hor}_0\mathbb{A} \to \mathrm{hor}_1\mathbb{A}$. Moreover, *if* $\mathbb{A}$ *is horizontally invariant*, then (by 4.6b) it has lax functorial $\mathbb{H}\mathbf{I}$-limits iff:

(a) the categories $\mathrm{hor}_0\mathbb{A}$ and $\mathrm{hor}_1\mathbb{A}$ have (ordinary) $\mathbf{I}$-limits, preserved by the three structural functors $\mathrm{hor}_0\mathbb{A} \leftrightarrows \mathrm{hor}_1\mathbb{A}$ (identity, domain and codomain).

In particular, consider the ordinary product $(A, (pi\colon A \to A_i))$ in $hor_0\mathbb{A}$, of a (small) family $(A_i)_{i\in I}$ of objects of $\mathbb{A}$ (our indexing double category $\mathbb{I}$ is discrete). Then this cone $(A, (pi)_i)$ is the double product of the family in $\mathbb{A}$ iff

(dp.2) given two cones $(X, (xi)_i)$, $(Y, (yi)_i)$, a vertical arrow $u\colon X \twoheadrightarrow Y$ and cells $\xi i\colon (u\ {}^{xi}_{yi}\ 1^\bullet_{A_i})$, there is precisely one cell τ such that, for all i, $(\tau \mid pi) = \xi i$

$$(1) \qquad \begin{array}{ccccc} X & \xrightarrow{t'} & A & \xrightarrow{pi} & A_i \\ u\downarrow & \tau & \| & 1^\bullet & \| \\ Y & \xrightarrow[t'']{} & A & \xrightarrow[pi]{} & A_i \end{array} \quad = \quad \begin{array}{ccc} X & \xrightarrow{xi} & A_i \\ u\downarrow & \xi i & \| \\ Y & \xrightarrow[yi]{} & A_i \end{array}$$

Similarly, consider now the ordinary equaliser $e\colon E \to A$ (in $hor_0\mathbb{A}$) of a pair of horizontal maps $f, g\colon A \rightrightarrows B$. Then (A, e) is the *double equaliser* of the two maps in $\mathbb{A}$ iff

(de.2) for every cell $\xi\colon (u\ {}^{x}_{y}\ 1^\bullet_A)$ of $\mathbb{A}$ which equalises f and g $((\xi \mid f) = (\xi \mid g))$ there is precisely one cell $\tau\colon (u\ {}^{t'}_{t''}\ 1^\bullet_E)$ such that $(\tau \mid e) = \xi$.

The existence and functoriality of (co)products and (co)equalisers in our standard examples will be studied in Section 6.

5.2. Lemma. *The double category* $\mathbb{A}$ *has all small (resp. finite) double limits of* horizontal *functors iff it has all small (resp. finite) double products and double equalisers; the construction is the standard one. If* $\mathbb{A}$ *has a lax (pseudo, strict) functorial choice of double products and double equalisers, this construction provides a similar choice of* $\mathbb{I}$*-limits, for any small* $\mathbb{I} = \mathbb{H}\mathbf{I}$ *(for* $\mathbf{I}$ *a category).*

Proof. We already remarked (5.1) that the double limit of a horizontal functor amounts to an ordinary limit in $hor_0\mathbb{A}$ which is preserved by the functor $1^\bullet_-\colon hor_0\mathbb{A} \to hor_1\mathbb{A}$. Thus, the first statement is a straightforward consequence of the construction theorem for ordinary limits. The second as well, by Lemma 4.4.

5.3. Tabulators and cotabulators. We call (horizontal) *tabulator* of a vertical arrow $u\colon A \twoheadrightarrow B$ the double limit of the double diagram consisting of u. The object Tu is thus equipped with two horizontal maps p, q and a cell π

$$
(1) \qquad \begin{tikzcd} Tu \arrow[r, "p"] \arrow[d, "1"'] \arrow[dr, phantom, "\pi"] & A \arrow[d, "u"] \\ Tu \arrow[r, "q"'] & B \end{tikzcd}
$$

which determines the whole cone; π is therefore monic on horizontal maps into Tu. The corresponding double colimit is called *cotabulator* and written $\perp u$; it comes equipped with

(2) $i: A \to \perp u, \qquad j: B \to \perp u, \qquad \iota: (u \,{}^{i}_{j}\, 1^{\bullet})$.

In particular, the *cotensor* $2{*}A$ (resp. the *tensor* $2 \otimes A$) of the object A by **2** is the tabulator (resp. cotabulator) of its vertical identity. There is a canonical *diagonal* (horizontal) map, provided by the identity cell of A

(3) $dA: A \to 2{*}A, \qquad (dA \mid \pi_A) = \Box_A$.

The one-dimensional property of tabulators amounts to a right adjoint to the functor $1^{\bullet}_{-}: \mathrm{hor}_0\mathbb{A} \to \mathrm{hor}_1\mathbb{A}$. Thus, if $\mathbb{A}$ has one-dimensional *co*tabulators, $1^{\bullet}_{-}$ is a right adjoint and the double limits of *horizontal* functors in $\mathbb{A}$ just amount to their one-dimensional property (5.1).

All the (pseudo) double categories considered in Section 3 have tabulators and cotabulators, most of which are calculated in 6.1-5. In particular, in $\mathbb{R}$el, $Tu \subset A \times B$ is the "graph", or "tabulation", of the relation u, motivating the general name; in $\mathbb{C}$at, Tu is the Grothendieck construction on the profunctor u and $\perp u$ its collage $A +_u B$; in $\mathbb{A}$dCat we have the comma category $Tu = (u_\bullet \downarrow B) = (A \downarrow u^\bullet)$. Restricting **Ab-$\mathbb{C}$at** to the full substructure of preadditive categories having one object, we obtain the pseudo double category $\mathbb{R}$ng of rings, with homomorphisms and bimodules $u: R \twoheadrightarrow S$ (u is a left-R, right-S bimodule). Then, Tu is a ring of matrices, with "matrix" product

(4) $\begin{pmatrix} r & x \\ 0 & s \end{pmatrix} \begin{pmatrix} r' & x' \\ 0 & s' \end{pmatrix} = \begin{pmatrix} rr' & rx' + xs' \\ 0 & ss' \end{pmatrix} \qquad (r, r' \in R;\ s, s' \in S;\ x, x' \in u)$.

On the other hand, $\mathbb{T}$g (2.1) lacks tabulators: if the continuous mapping $u: A \twoheadrightarrow B$ does not preserve the unit, there are no cells having the boundary required for (1).

Finally, completing (after 4.2) a comparison with "$\mathbb{A}$-wise limits" in the sense of Bastiani-Ehresmann, let us recall that the tabulator, more precisely its one-dimensional version, was introduced in [2] (p. 260) as the $\mathbb{A}$-wise 2-limit of a vertical arrow u, and called "representation" of u; the double category $\mathbb{A}$ is

called "representable" if each vertical arrow has a representation. The construction of double limits which we give below (5.5i), restricted to one-dimensional limits for "vertical" double functors $F: \mathbf{VI} \to \mathbb{A}$, coincides with the construction of $\mathbb{A}$-wise limits given in [2] (p. 265, Prop. 3), by means of representations and limits of ordinary functors in the category $\mathrm{hor}_0\mathbb{A}$ of horizontal arrows of $\mathbb{A}$.

5.4. Tabulators and composition. We already know how the tabulator $2{*}A$ of a vertical identity is related to the object A, through the diagonal map dA (5.3.3). Given now a vertical composite $w = v{\bullet}u: A \twoheadrightarrow B \twoheadrightarrow C$, we need to know how the three tabulators of u, v, w are related.

This will use the pullback $T(u, v)$ of Tu and Tv, over the middle object B (which we assume to exist)

$$
(1) \qquad
\begin{tikzcd}
 & & & A \arrow[r, equal] \arrow[dd, "u", "\shortmid" marking] & A \arrow[dddd, "w", "\shortmid" marking] & \\
 & Tu \arrow[urr, "p_u"] \arrow[drr, "q_u"'] \arrow[rr, phantom, "\pi_u"] & & & & \\
T(u, v) \arrow[ur, "p_{uv}"] \arrow[dr, "q_{uv}"'] & & & B \arrow[dd, "v", "\shortmid" marking] & & Tw \arrow[uul, "p_w"'] \arrow[ddl, "q_w"] \arrow[l, phantom, "\pi_w"] \\
 & Tv \arrow[urr, "p_v"] \arrow[drr, "q_v"'] \arrow[rr, phantom, "\pi_v"] & & & & \\
 & & & C \arrow[r, equal] & C &
\end{tikzcd}
$$

and the *diagonal* map $d_{uv}: T(u, v) \to Tw$ given by the universal property of Tw

$$(2) \quad (d_{uv} \mid \pi_w) = \left(\frac{p_{uv} \mid \pi_u}{q_{uv} \mid \pi_v}\right).$$

It is easy to show (and it also follows from the construction theorem below) that $T(u, v)$ is the double limit of the diagram consisting of the *consecutive vertical arrows* u, v (with projections in A, B, C and structural cells in u, v given by the left-hand part of diagram (1)).

5.5. Theorem: the construction of double limits. i) *The double category* $\mathbb{A}$ *has all small (resp. finite) double limits iff it has small (resp. finite) double products, double equalisers and tabulators. The construction is explicitly described in the proof below* (5.6 *for the free case,* 5.7 *for the general one).*

ii) *If* $\mathbb{A}$ *has a lax (or pseudo, strict) functorial choice of the basic limits (double products, double equalisers and tabulators), this construction provides a similar choice of* $\mathbb{I}$*-limits, for any small* $\mathbb{I}$.

iii) *Finally, if* $\mathbb{A}$ *is a pseudo double category, double limits for* lax *double functors* $\mathbb{I} \to \mathbb{A}$ *(end of* 4.1*) are still obtained from the same basic limits, under*

a slightly generalised construction (5.7 F).

5.6. Proof, Part I. (*The double limit of a double diagram*) Of course one needs only to prove the "sufficiency" part of the statement. We consider first the "free case", which is considerably simpler, contains various interesting non-horizontal cases and allows for a more economical algorithm.

We assume thus that $\mathbb{I}$ is just a double graph, and $F: \mathbb{I} \to \mathbb{A}$ a (double) diagram; or equivalently, we consider the double functor $\bar{F}: \bar{\mathbb{I}} \to \mathbb{A}$ spanned by F, on the free double category generated by $\mathbb{I}$ (1.5 d). We already know that, in this case, the cones $x: DA \to F: \mathbb{I} \to \mathbb{A}$ are just subject to *one* naturality condition (dc.3) (1.5), which simplifies the problem.

The solution is based on turning F into a 1-horizontal functor $G: \mathbb{H}\mathbf{I} \to \mathbb{A}$, and taking its double limit; the graph $\mathbf{I}$ is a sort of "horizontal subdivision" of $\mathbb{I}$. The procedure is similar to computing the end of a functor $S: \mathbf{C}^{op}{\times}\mathbf{C} \to \mathbf{D}$ as the limit of the associated functor $S^{\S}: \mathbf{C}^{\S} \to \mathbf{D}$ based on Kan's *subdivision category* of $\mathbf{C}$ ([13], 1.10; [18], IX.5).

A) Form a new 1-dimensional graph $\mathbf{I}$, the *horizontal subdivision* of $\mathbb{I}$, by replacing every vertical arrow of $\mathbb{I}$ with a new object, *simulating its tabulator.* Precisely, $\mathbf{I}$ is formed by the following objects and arrows (and is finite whenever $\mathbb{I}$ is so):

a) all the objects and horizontal arrows of $\mathbb{I}$,

b) for every vertical map $u: i_u \twoheadrightarrow j_u$ of $\mathbb{I}$, a new formal object $u^\wedge$, also written u, together with two new arrows $p_u: u^\wedge \to i_u$, $q_u: u^\wedge \to j_u$

c) for every cell $\alpha: (u \,{}^{f}_{g}\, v)$ of $\mathbb{I}$, a new arrow $\alpha^\wedge: u^\wedge \to v^\wedge$, also written α.

B) Let $G: \mathbb{H}\mathbf{I} \to \mathbb{A}$ be the 1-horizontal functor naturally deriving from F and the tabulator-construction for vertical maps:

a) G coincides with F on the objects and horizontal arrows of $\mathbb{I}$,

b) for every vertical map $u: i_u \twoheadrightarrow j_u$ of $\mathbb{I}$,

(1) $Gu = \top(Fu)$,

while $Gp_u: Gu \to Gi_u$ and $Gq_u: Gu \to Gj_u$ are the projections of $\top(Fu)$ in $\mathbb{A}$; these projections will again be written p_u and q_u, while we write $\pi_u: (1^\bullet \,{}^{p_u}_{q_u}\, Fu)$ the structural cell of the tabulator,

c) for every cell $\alpha: (u \,{}^{f}_{g}\, v)$ of $\mathbb{I}$, $G\alpha$ is the horizontal map of $\mathbb{A}$ such that

(2) $G\alpha\colon T(Fu) \to T(Fv)$, $\qquad (G\alpha \mid \pi_v) = (\pi_u \mid F\alpha)$

according to the universal property of π_v with respect to the cell $(\pi_u \mid F\alpha)$.

The double limit of this horizontal diagram $G\colon \mathbb{HI} \to \mathbb{A}$ exists, by hypotheses and the previous lemma 5.2. We want now to prove that it gives the double limit of F; in fact, we construct an isomorphism of double categories between $(D \downdownarrows F)$ and $(D' \downdownarrows G)$, whose double terminals yield our two limits. Here, $D'\colon \mathbb{A} \to \mathbb{A}^{\mathbb{HI}}$ is the new diagonal double functor.

C) The canonical double functor $(D \downdownarrows F) \to (D' \downdownarrows G)$.

a) Let $(A, x\colon DA \to F)$ be a double cone of F. Its "ordinary part" $(A, (xi)_i)$ can be extended to an ordinary cone $(A, x'\colon D'A \to G)$ of G, using the non-ordinary part $(xu)_u$: define $x'(u)\colon A \to T(Fu)$ as the horizontal map of A determined by the cell xu, via the tabulator-property

(3) $(1^\bullet_{x'(u)} \mid \pi_u) = xu.$

And x' is indeed a cone, as it is coherent with the new arrows p_u, q_u, $G\alpha$:

(4) $p_u\, x'(u) = x(i_u)$, $\qquad q_u\, x'(u') = x(j_u)$, $\qquad G\alpha \circ x'(u) = x'(v)$

where the first two properties follow from (3), the third from the cancellation property of π_v

(5) $(x'(u) \mid G\alpha \mid \pi_v) = (x'(u) \mid \pi_u \mid F\alpha) = (xu \mid F\alpha) = xv = (x'(v) \mid \pi_v).$

b) A horizontal map of $(D \downdownarrows F)$, $a\colon (A, x\colon DA \to F) \to (B, y\colon DB \to F)$, determines a horizontal map $a\colon (A, x') \to (B, y')$ of $(D' \downdownarrows G)$, since (using again the cancellation property of π_u)

(6) $(a \mid y'(u) \mid \pi_u) = (a \mid yu) = xu = (x'(u) \mid \pi_u).$

c) A vertical map of $(D \downdownarrows F)$, $(s, \xi)\colon (A, x\colon DA \to F) \twoheadrightarrow (C, z\colon DC \to F)$, where $s\colon A \twoheadrightarrow C$ is vertical in $\mathbb{A}$ and $\xi\colon (Ds\ {}^{x}_{y}\ 1^\bullet_F)$ is an $\mathbb{X}$-cell, determines a vertical map $(s, \xi')\colon (A, x') \to (B, y')$ of $(D' \downdownarrows G)$, extending the cell ξ to ξ': $(D's\ {}^{x'}_{y'}\ 1^\bullet_G)$ so that $\xi'(u)$ satisfies (setting $\eta u = (zu) \bullet (\xi i_u) = (\xi j_u) \bullet (xu)$)

$$
(7) \qquad
\begin{tikzcd}
A \arrow[r, "x'u"] \arrow[d, "s"'] \arrow[dr, phantom, "\xi'u"] & T(Fu) \arrow[r, "p_u"] \arrow[d, "1"'] \arrow[dr, phantom, "\pi_u"] & Fi_u \arrow[d, "Fu"] & & A \arrow[r, "xi_u"] \arrow[d, "s"'] \arrow[dr, phantom, "\eta u"] & Fi_u \arrow[d, "Fu"] \\
C \arrow[r, "z'u"'] & T(Fu) \arrow[r, "q_u"'] & Fj_u & = & C \arrow[r, "zj_u"'] & Fj_u
\end{tikzcd}
$$

d) Finally, it follows that a cell of $(D \downdownarrows F)$ determines one of $(D' \downdownarrows G)$.

D) In the reverse direction, one constructs a canonical double functor $(D'\Downarrow G) \to (D \Downarrow F)$ inverse to the former, by similar arguments. We just specify its action on objects. Given a cone $(A, (xi: A \to Fi)_i, (x'u: A \to Gu)_u)$ of G, one forms a double cone $(A, x: DA \to F)$ by letting

(8) $xu = (x'u \mid \pi_u)$

which satisfies (dc.3) since, for $\alpha: (u \,{}^{f}_{g}\, v)$ in $\mathbb{I}$

(9) $(xu \mid F\alpha) = (x'u \mid \pi_u \mid F\alpha) = (x'u \mid G\alpha \mid \pi_v) = (x'v \mid \pi_v) = xv.$

E) Assume now, for ii), that we have, in $\mathbb{A}$, a lax (pseudo, strict) functorial choice of the basic limits and let us construct a similar choice for $\mathbb{I}$-limits; by Lemma 4.4, this can be reduced (both on hypotheses and conclusion) to two more elementary choices a), b).

The first choice is given by the "one-dimensional part" of what we have already proven. As to the second, a vertical transformation $U = ((Ui), (Uf)): F \twoheadrightarrow F': \mathbb{I} \to \mathbb{A}$ has a natural extension to a vertical transformation $U: G \twoheadrightarrow G': \mathbb{H}\mathbf{I} \to \mathbb{A}$, which is defined on the new objects $u^\wedge$ through the fact that tabulators have been assigned a choice of limits of vertical transformations

(10) $U(u^\wedge): \mathsf{T}(Fu) \twoheadrightarrow \mathsf{T}(F'u)$ (u vertical in $\mathbb{I}$)

so that also the value on the new arrows $\alpha^\wedge: u^\wedge \to v^\wedge$ is uniquely determined. But this vertical transformation of horizontal functors $U: G \twoheadrightarrow G'$ has an assigned limit, because of 5.2. Finally, since both steps respect the structural functors $(1^\bullet_-, \mathrm{dom}, \mathrm{cod})$, so does their result.

5.7. Proof, Part II. (*The general case*) Now let $\mathbb{I}$ be an arbitrary double category. Then the previous construction of the graph $\mathbf{I}$ (step A), of G (step B) and of the isomorphism $(D \Downarrow F) \cong (D'\Downarrow G)$ (steps C-D) has to be supplemented as follows.

A) $\mathbf{I}$ has some supplementary objects and arrows:

- for every i in $\mathbb{I}$, a new arrow $d_i: i \to 1^\bullet_i{}^\wedge$ (simulating the diagonal map 5.3.3),
- for every vertical composition $w = v \bullet u$ in $\mathbb{I}$, a new object (u, v) and three arrows $p_{uv}: (u, v) \to u^\wedge$, $q_{uv}: (u, v) \to v^\wedge$, $d_{uv}: (u, v) \to w^\wedge$ (simulating the object $\mathsf{T}(u, v)$ of 5.4 and its arrows).

B) G is extended to these objects and arrows, by the objects and maps of $\mathbb{A}$ they simulate:

- $G(d_i)$ is the diagonal map $dFi: Fi \to 2*Fi = G(1_i^{\bullet}\wedge)$ (5.3.3); again, for the sake of simplicity, we write $G(d_i)$ as d_i, and π_i the structural cell of the tabulator $2*Fi$ (i.e., π_u for $u = 1_i^{\bullet}$)

- $G_{uv} = T(Fu, Fv)$ is the double limit of the composable pair Fu, Fv (5.4); the arrows p_{uv}, q_{uv}, d_{uv} of $\mathbb{I}$ are taken by G to the projections and the diagonal of G_{uv} (5.4.1-2), which we simply write as $p_{uv}: G_{uv} \to Gu$, $q_{uv}: G_{uv} \to Gv$, $d_{uv}: G_{uv} \to Gw$; note that (G_{uv}, p_{uv}, q_{uv}) is the pullback of (q_u, p_v) in $\mathrm{hor}_0\mathbb{A}$.

C) Given a double cone $(A, x: DA \to F)$, extend our previous $x': D'A \to G$ (5.6 C)) to the new objects (u, v) by letting $x'_{uv}: A \to G_{uv}$ be defined by the pullback-property of G_{uv}

(1) $p_{uv} . x'_{uv} = x'u: A \to Gu, \qquad q_{uv} . x'_{uv} = x'v: A \to Gv.$

In order to prove that the new x' is a cone, it suffices to prove its coherence with the new arrows $d_i, p_{uv}, q_{uv}, d_{uv}$; two conditions hold by definition (1) above, the remaining two follow from the definition of the diagonal maps d_i, d_{uv} and the structural cells π_i, π_u (together with their usual cancellation property)

(2) $(xi \mid d_i \mid \pi_i) = (xi \mid \Box_{Fi}) = 1_{xi}^{\bullet} = x(1_i^{\bullet}) = (x'(1_i^{\bullet}) \mid \pi_i)$

(3) $(x'_{uv} \mid d_{uv} \mid \pi_w) = (x'_{uv} \mid \left(\dfrac{p_{uv} \mid \pi_u}{q_{uv} \mid \pi_v}\right)) = \left(\dfrac{x'u \mid \pi_u}{x'v \mid \pi_v}\right) = \left(\dfrac{xu}{xv}\right) =$

$= xw = (x'w \mid \pi_w).$

D) Given an ordinary cone $(A, x': D'A \to G)$, we have to prove that the old associated double cone $(A, x: DA \to F)$, defined by letting $xu = (x'u \mid \pi_u)$, is indeed a double cone for the new situation, i.e. satisfies also the conditions (dc.1-2) concerning the vertical composition in $\mathbb{I}$; this proceeds much as above (let $w = v\bullet u$, in the second case)

(4) $1_{x_i}^{\bullet} = (x_i \mid d_i \mid \pi_i) = (x'(1_i^{\bullet}) \mid \pi_i) = x(1_i^{\bullet})$

(5) $\left(\dfrac{xu}{xv}\right) = \left(\dfrac{x'u \mid \pi_u}{x'v \mid \pi_v}\right) = (x'_{uv} \mid \left(\dfrac{p_{uv} \mid \pi_u}{q_{uv} \mid \pi_v}\right)) = (x'_{uv} \mid d_{uv} \mid \pi_w) =$

$= (x'w \mid \pi_w) = xw.$

E) For the lax functoriality part ii), take a vertical transformation $U = ((Ui), (Uf)): F \twoheadrightarrow F': \mathbb{I} \to \mathbb{A}$. The corresponding $U: G \twoheadrightarrow G': \mathbb{HI} \to \mathbb{A}$ is defined on the new objects (u, v)

(6) $U(u, v): T(Fu, Fv) \twoheadrightarrow T(F'u, F'v)$

through the fact that tabulators and pullbacks have been assigned a choice of limits for vertical transformations. Again, the extension to the new arrows $d_i: i \to 1_i^{\bullet\wedge}$, $p_{uv}: (u, v) \to u^\wedge$, $q_{uv}: (u, v) \to v^\wedge$, $d_{uv}: (u, v) \to w^\wedge$ is uniquely determined.

F) Finally, it is easy to modify the previous steps for the more general relaxed case iii). Now, x preserves the vertical composition up to the special comparison cells of F, $\varphi = \varphi(u, v): Fv \bullet Fu \to Fw$ (4.1, (dc.2')) and d_{uv} are defined coherently

$$(7)\quad (d_{uv} \mid \pi_w) = \Big(\frac{p_{uv} \mid \pi_u}{q_{uv} \mid \pi_v} \mid \varphi(u, v)\Big).$$

Therefore, we just replace (3), (5) above with (3'), (5'):

$$(3')\quad (x'_{uv} \mid d_{uv} \mid \pi_w) = (x'_{uv} \mid \Big(\frac{p_{uv} \mid \pi_u}{q_{uv} \mid \pi_v}\Big) \mid \varphi) = \Big(\frac{xu}{xv} \mid \varphi\Big) = (xw) =$$

$$= (x'w \mid \pi_w)$$

$$(5')\quad \Big(\frac{xu}{xv} \mid \varphi\Big) = (x'_{uv} \mid \Big(\frac{p_{uv} \mid \pi_u}{q_{uv} \mid \pi_v}\Big) \mid \varphi\Big) = (x'_{uv} \mid d_{uv} \mid \pi_w) = (x'w \mid \pi_w) =$$

$$= xw.$$

6. Explicit constructions

After computing limits and colimits in double categories of quintets (6.1) and in the examples of Section 3 (6.2-5), we end by an example showing the difference between double limits of *double diagrams* and *double functor* (6.6). Recall that the term *double limit* is often replaced with limit, and note that all (pseudo) double categories studied below are horizontally invariant, so that the results of 4.5-6 apply.

6.1. Limits for quintets. The 2-category **A** is 2-complete iff the associated double category $\mathbb{Q}$**A** has all double limits; and dually.

In fact, **A** is 2-complete iff it has 2-products, 2-equalisers and cotensors by **2** [22]. First, it is easy to see that 2-products (resp. 2-equalisers) in **A** produce double products (resp. double equalisers) in $\mathbb{Q}$**A**; and conversely. Second, if the **A**-morphism $u: A \to B$ is viewed as vertical in $\mathbb{Q}$**A**, its tabulator $(Tu; p, q; \pi)$ can be constructed as the inserter

$$(1)\quad Tu \xrightarrow{\ i\ } A\times B \underset{p''}{\overset{up'}{\rightrightarrows}} B, \qquad\qquad \pi\colon up'i \to p''i\colon Tu \to B$$

with $p = p'i$, $q = p''i$. Conversely, the cotensor $\mathbf{2}*A$ is obtained as the tabulator of the identity of A.

6.2. Metric spaces, posets and relations. Examining the examples of Section 3, let us begin to study the profunctor-based, flat double categories $\mathbb{M}\mathrm{tr} \supset \mathbb{P}\mathrm{os} \supset \mathbb{R}\mathrm{el}$ (3.3-4). We prove here that all of them have lax functorial limits and colax functorial colimits. Moreover:

- the double category $\mathbb{M}\mathrm{tr}$ of metric spaces has functorial sums and cotabulators;

- the double subcategory $\mathbb{P}\mathrm{os}$ of preordered sets (identified with metric spaces with distance in $\{0, +\infty\}$) is closed in $\mathbb{M}\mathrm{tr}$ under limits and colimits; it has functorial products, sums and cotabulators;

- the double subcategory $\mathbb{R}\mathrm{el}$ (a set being viewed as a discrete metric space with values in $\{0, +\infty\}$) is closed in both under limits, sums and coequalisers; products and sums are functorial; on the other hand, cotabulators are quotients of the corresponding ones in $\mathbb{M}\mathrm{tr}$ and $\mathbb{P}\mathrm{os}$, and are not functorial.

a) *Products and sums*. For products, recall that ΠA_i has the l_∞-metric $d((x_i), (y_i)) = \vee_i\, d(x_i, y_i)$; the product of profunctors $u_i\colon A_i \twoheadrightarrow B_i$ is $\Pi u_i((a_i), (b_i)) = \vee_i\, u_i(a_i, b_i)$. For a sum ΣA_i, the distance within each component is completed by setting $d(x, y) = +\infty$ when x, y are in different components; similarly for a sum of profunctors. $\mathbb{P}\mathrm{os}$ and $\mathbb{R}\mathrm{el}$ are closed under such constructs.

Arbitrary sums are obviously functorial, whereas even finite products in $\mathbb{M}\mathrm{tr}$ are not; it is sufficient to consider the square $P\times P$ of the singleton, and the following profunctors $P \twoheadrightarrow P$ (each of them, being defined on *one* pair, amounts to a constant)

$$\begin{array}{lll}(1) & u = v' = 0\colon P \twoheadrightarrow P, & u' = v = 1\colon P \twoheadrightarrow P\\ & v\bullet u = v'\bullet u' = 1\colon P \twoheadrightarrow P, & u\times u' = v\times v' = 1\colon P\times P \twoheadrightarrow P\times P\\ & (v\bullet u)\times(v'\bullet u') = 1, & (v\times v')\bullet(u\times u') = 2.\end{array}$$

This counterexample would fail for $u' = v = +\infty$. In fact, a product of profunctors $(u_i\colon A_i \twoheadrightarrow B_i)$ in $\mathbb{R}\mathrm{el}$ (and $\mathbb{P}\mathrm{os}$) is represented by the obvious relation $\Pi u_i\colon \Pi A_i \twoheadrightarrow \Pi B_i$ (down-closed in domain and up-closed in codomain, 3.3) whose graph is the product of the graphs of all u_i; and this procedure is functorial.

b) *Equalisers and coequalisers.* In $\mathbb{M}$tr, the equaliser of a pair of weak contractions $f, g \colon A \to B$ is the set-theoretical equaliser, say $\bar{A}$, with the restricted metric; for a vertical transformation

(2) $(u, v) \colon (f, g) \twoheadrightarrow (f', g'), \qquad (u \geq v(f, f'), \quad u \geq v(g, g'))$

take $\bar{u} \colon \bar{A} \twoheadrightarrow \bar{A}'$ the restriction of u. Similarly, the coequaliser $\bar{B}$ is the set-theoretical one, a quotient of B with the induced metric $d(\beta, \beta') = \wedge_{b,b'} d(b, b')$; to prove colax functoriality, take $\bar{v} \colon \bar{B} \twoheadrightarrow \bar{B}'$ as induced by v $(\bar{v}(\beta, \beta') = \wedge_{b,b'} v(b, b'))$. Again, $\mathbb{P}$os and $\mathbb{R}$el are closed under such constructs.

It is easy to give examples showing that equalisers and coequalisers do not preserve the vertical composition in $\mathbb{R}$el, whence in $\mathbb{P}$os and $\mathbb{M}$tr, as below

(3)
$$\begin{array}{ccccc} \{0\} & \longrightarrow & \{0\} & \rightrightarrows & \{0\} \\ \downarrow & \leq & {\scriptstyle 1}\downarrow & \leq & \downarrow{\scriptstyle y} \\ \emptyset & \longrightarrow & \{0\} & \rightrightarrows & \{0, 1\} \\ \downarrow & \leq & {\scriptstyle 1}\downarrow & \leq & \downarrow{\scriptstyle y'} \\ \{0\} & \longrightarrow & \{0\} & \rightrightarrows & \{0\} \end{array} \qquad \begin{array}{l} y = \{0\}\times\{0, 1\} \\ f'(0) = 0, \quad g'(0) = 1 \\ y' = \{0, 1\}\times\{0\} \end{array}$$

(4)
$$\begin{array}{ccccc} \emptyset & \rightrightarrows & \{0\} & \longrightarrow & \{0\} \\ {\scriptstyle x}\downarrow & \leq & {\scriptstyle y}\downarrow & \leq & \downarrow{\scriptstyle 1} \\ \{0\} & \rightrightarrows & \{0, 1\} & \longrightarrow & \{0\} \\ {\scriptstyle x'}\downarrow & \leq & {\scriptstyle y'}\downarrow & \leq & \downarrow{\scriptstyle 1} \\ \emptyset & \rightrightarrows & \{0\} & \longrightarrow & \{0\} \end{array} \qquad \begin{array}{l} y = \{(0, 0)\} \\ f'(0) = 0, \quad g'(0) = 1 \\ y' = \{(1, 0)\} \end{array}$$

c) *Tabulators and cotabulators, I.* In $\mathbb{M}$tr, the tabulator of a profunctor $u \colon A \twoheadrightarrow B$ is a sort of "graph" of u (viewing $u(a, b)$ as a distance, as in 3.3)

(5) $Tu = \{(a, b) \in A{\times}B \mid u(a, b) = 0\},$
$\quad d((a, b), (a', b')) = d(a, a') \vee d(b, b')$

(6)
$$\begin{array}{ccc} Tu & \xrightarrow{p} & A \\ {\scriptstyle 1}\downarrow & \geq & \downarrow{\scriptstyle u} \\ Tu & \xrightarrow[q]{} & B \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle 1}\downarrow & \geq & \downarrow{\scriptstyle u} \\ X & \xrightarrow[g]{} & B \end{array}$$

(in fact, given the right-hand cell above, we have $u(fx, gx) \leq d(x, x)$). Given a

commutative square of vertical arrows $(x, y)\colon u \twoheadrightarrow v$, its limit $T(x, y)$ is the restriction of $x{\times}y\colon A{\times}B \twoheadrightarrow A'{\times}B'$.

The cotabulator of u is the collage $A{+}_uB$ (3.3), with $d(a, b) = u(a, b)$ and $d(b, a) = +\infty$ for $a \in A$, $b \in B$. Given a commutative square of vertical arrows $(x, y)\colon u \twoheadrightarrow v$, its colimit $s = \bot(x, y)$ is calculated on pairs (a, b') by means of the diagonal of the square, $z = v{\bullet}x = y{\bullet}u\colon A \twoheadrightarrow B'$ (the other values of s being obvious)

$$\begin{aligned}
(7)\quad & s(a, b') \;=\; z(a, b') \;=\; \wedge_{a'}\,(x(a, a') + v(a', b')) \;=\; \wedge_b\,(u(a, b) + y(b, b'))\\
& s(a, a') \;=\; x(a, a'), \qquad s(b, b') \;=\; y(b, b'), \qquad s(b, a') \;=\; +\infty.
\end{aligned}$$

Moreover, this procedure is functorial. Given a second square $(x', y')\colon v \twoheadrightarrow w$, with $s' = \bot(x', y')$ and $\bar{s} = \bot(x'{\bullet}x, y'{\bullet}y)$, the only non trivial verification concerns the pairs $(a, b'') \in A{\times}B''$; noting that $y'{\bullet}z = z'{\bullet}x$, we have

$$\begin{aligned}
(8)\quad & s'{\bullet}s(a, b'') \;=\; \wedge_\xi\,(s(a, \xi) + s'(\xi, b'')) \;=\; z'{\bullet}x(a, b'') \wedge y'{\bullet}z(a, b'') \;=\\
& =\; z'{\bullet}x(a, b'') \;=\; (w{\bullet}x'{\bullet}x)(a, b'') \;=\; \bar{z}(a, b'') \;=\; \bar{s}(a, b'').
\end{aligned}$$

d) *Tabulators and cotabulators, II.* $\mathbb{P}$os is closed under both constructs, whereas $\mathbb{R}$el is just closed under tabulators: the cotabulator of a profunctor $u\colon A \twoheadrightarrow B$ in $\mathbb{M}$tr may have $d(a, b) = u(a,b) = 0$, in which case a and b must be identified for the $\mathbb{R}$el-cotabulator.

Giving a direct description for $\mathbb{R}$el, the tabulator $Tu \subset A{\times}B$ is the graph of the relation u, whereas the cotabulator $\bot u$ is its cograph, a quotient of $A{+}B$. Given a commutative square of vertical arrows $(x, y)\colon u \twoheadrightarrow v$, its limit $T(x, y)$

$$(9)\qquad \begin{tikzcd}
Tu \arrow[rr, "\pi" '] \arrow[d, "1"] \arrow[dr] & & A \arrow[rr, "\iota" '] \arrow[d, dotted, "u"] \arrow[dr, "x"] & & \bot u \arrow[d, dotted, "1"] \arrow[dr, "s"] & \\
Tu \arrow[rr, dashed] \arrow[dr, "t" '] & Tv \arrow[rr] \arrow[d] & B \arrow[rr, dashed] \arrow[dr, dotted, "y" '] & A' \arrow[rr] \arrow[d, "v"] & \bot u \arrow[dr, dotted, "s" '] & \bot v \arrow[d, "1"] \\
 & Tv \arrow[rr] & & B' \arrow[rr] & & \bot v
\end{tikzcd}$$

is the induced relation $t = m'^{\#}(x{\times}y)m = (Tu{\times}Tv) \cap (x{\times}y)$ (where $m = \langle p, q\rangle\colon Tu \rightarrowtail A{\times}B$); its colimit is the induced relation $s = h'(x{+}y)h^{\#}$ (for $h\colon A{+}B \twoheadrightarrow \bot u$). Again, these constructs need not preserve vertical composition (which also proves that tabulators are not functorial in $\mathbb{P}$os and $\mathbb{M}$tr); below, $(0, 0; 1, 0)$ belongs to $T(x'x, y'y)$, but not to $T(x', y'){\bullet}T(x, y)$

$$(10) \qquad \begin{tikzcd} \{0\} \arrow[r, "x"] \arrow[d, "u"'] & \{0,1\} \arrow[r, "1"] \arrow[d, "1"] & \{0,1\} \arrow[d, "w"] \\ \{0,1\} \arrow[r, "1"'] & \{0,1\} \arrow[r, "y'"'] & \{0\} \end{tikzcd} \qquad \begin{array}{l} x = u = \{0\}\times\{0,1\} \\ y' = w = \{0,1\}\times\{0\} \end{array}$$

while in (11) all cotabulators of u, v, w are the singleton, $\perp(x, y)$ and $\perp(x', y')$ are identities, but $\perp(x'\bullet x, y'\bullet y)$ is empty

$$(11) \qquad \begin{tikzcd} \emptyset \arrow[r, "x"] \arrow[d, "u"'] & \{0\} \arrow[r, "1"] \arrow[d, "v"] & \{0\} \arrow[d, "w=1"] \\ \{0\} \arrow[r, "y"'] & \{0,1\} \arrow[r, "y'"'] & \{0\} \end{tikzcd} \qquad \begin{array}{l} v = \{0\}\times\{0,1\} \\ y = \{(0,0)\} \\ y' = \{(1,0)\}. \end{array}$$

6.3. Limits of profunctors. We study now the pseudo double category $\mathbb{C}$at of categories, functors and profunctors (3.1), proving that it has all lax functorial limits and colax functorial colimits, actually pseudo functorial for *finite products*, *arbitrary sums* and *cotabulators*. For horizontal double functors, our (co)limits are given by the ones of **Cat**. (Note also that, in the ordinary presentation of profunctors, the vertical composition is defined by some "unknown" choice, so that here a *strict* functorial construction of (co)limits would have no sensible meaning.)

a) *Products and sums*. In $\mathbb{C}$at, the product of a family of profunctors u_i: $A_i \to B_i$ is $\Pi u_i((a_i), (b_i)) = \Pi u_i(a_i, b_i))$. Their sum Σu_i takes a pair of objects x, y of ΣA_i to $u_i(x, y)$ (resp. $\emptyset$) when x and y belong to the i-th component (resp. to different components).

Vertical composition is plainly preserved by sums (up to natural iso); it is also preserved by finite products, as it follows from the cartesian closedness of **Set** (take $a = (a_1, a_2)$ in $A_1{\times}A_2$, etc.)

$$(1) \quad ((v_1{\times}v_2)\bullet(u_1{\times}u_2))\,(a, c) = \int^b v_1(b_1, c_1){\times}v_2(b_2, c_2){\times}u_1(a_1, b_1){\times}u_2(a_2, b_2)$$
$$\cong \Big(\int^{b_1} v_1(b_1, c_1){\times}u_1(a_1, b_1)\Big) \times \Big(\int^{b_2} v_2(b_2, c_2){\times}u_2(a_2, b_2)\Big) =$$
$$= (v_1\bullet u_1)(a_1, c_1) \times (v_2\bullet u_2)(a_2, c_2) = ((v_1\bullet u_1) \times (v_2\bullet u_2))\,(a, c).$$

In the infinite case, the components of the canonical cell are surjective mappings

(2) $(\Pi v_i \bullet \Pi u_i)((a_i), (c_i)) \to \Pi(v_i \bullet u_i)((a_i), (c_i))$

$$(\Pi v_i \bullet \Pi u_i)((a_i), (c_i)) = \int^b \Pi(v_i(b_i, c_i) \times u_i(a_i, b_i))$$

$$\Pi(v_i \bullet u_i)((a_i), (c_i)) = \Pi \left(\int^{b_i} v_i(b_i, c_i) \times u_i(a_i, b_i)\right)$$

which are not injective, in general. In fact, if $u: \mathbf{1} \twoheadrightarrow B$ and $v: B \twoheadrightarrow \mathbf{1}$ are terminal profunctors (with constant value the singleton), then $v \bullet u: \mathbf{1} \twoheadrightarrow \mathbf{1}$ amounts to the set $\pi_0 B$ of connected components of the middle category. It is well-known that π_0: **Cat** $\to$ **Set** does not preserve infinite products, as shown for instance by a sequence B_n $(n \geq 0)$ of "zig-zag" categories

(3)
$$\begin{tikzcd}[column sep=small] & 1' \arrow[dl] \arrow[dr] & & \cdot \arrow[dl] & \cdots & \cdot \arrow[dr] & & n' \arrow[dl] \arrow[dr] & \\ 0 & & 1 & & & & n-1 & & n \end{tikzcd} \qquad B_n$$

of unbounded length, so that there is no (finite!) path in ΠB_n connecting the objects (0) and (n).

b) *Equalisers and coequalisers.* Equalisers and coequalisers in **Cat** are well-known; they produce (co)lax functorial (co)limits in $\mathbb{C}$at: we already noted that a vertical transformation $(u, v; \alpha, \beta): (f, g) \twoheadrightarrow (f', g')$ amounts to a pair of functors $\alpha, \beta: A+_uB \to A'+_vB'$ over **2** (3.1.4); their (co)equaliser is a category over **2**, yielding the (co)limit of the transformation.

All this is better verified in a formal way. Recall our analysis of a double category as a category object within categories (1.3). Here, the category $\mathrm{hor}_0\mathbb{C}$at of objects and horizontal arrows is **Cat**, and we have seen that the category $\mathrm{hor}_1\mathbb{C}$at of vertical arrows and cells is **Cat/2** (3.1). The structural functors

(4) $\mathbf{1}^\bullet_- : \mathbf{Cat} \rightleftarrows \mathbf{Cat/2} : \partial^i$

are pulling back along the face and degeneracy functors $\delta^i : \mathbf{1} \rightleftarrows \mathbf{2} : p$. As **Cat** and **Cat/2** are complete and cocomplete, and all their structural functors have left and right adjoints, and finally $\mathbb{C}$at is horizontally invariant (2.4), it follows that $\mathbb{C}$at has lax functorial limits and colax functorial colimits for all horizontal pseudo double functors $\mathbb{H}\mathbf{I} \to \mathbb{C}$at, constructed as in **Cat** (5.1a).

Also here one proves that equalisers are not pseudo functorial using the fact that π_0: **Cat** $\to$ **Set** does not preserve them; for coequalisers, this is proved below (6.4a).

c) *Tabulators and cotabulators.* In $\mathbb{C}$at, $\top u$ is the *category of elements*, or Grothendieck construction, of the profunctor $u: A \twoheadrightarrow B$. It has objects (a, b, λ)

with $a \in \mathrm{Ob}A$, $b \in \mathrm{Ob}B$, $\lambda \in u(a, b)$ and maps given by pair of maps of $A{\times}B$ which form a commutative square in the collage $A{+_u}B$ (3.1)

(5) $(f, g): (a, b, \lambda) \to (a', b', \lambda')$

$\quad f: a \to a', \qquad g: b \to b', \qquad\qquad u(1,g)(\lambda) = u(f,1)(\lambda') \in u(a, b');$

the functors p, q are obvious, and the natural transformation π is

(6) $\pi: q_* \to up_*: \mathrm{T}u^{op} \times B \to \mathbf{Set}$

$\quad \pi(a, b, \lambda; b'): B(b, b') \to u(a, b'), \qquad (g: b \to b') \mapsto u(1,g)(\lambda) \in u(a, b').$

The cotabulator $\perp u = A{+_u}B$ is the *gluing*, or *collage*, of A and B along u (with new maps given by $(\perp u)(a, b) = u(a, b)$, 3.1), with the obvious inclusions i, j and structural cell ι

(7) $\iota: u \to 1^\bullet_{\perp u}(i, j): A^{op}{\times}B \to \mathbf{Set}, \qquad \iota(a, b): u(a, b) = \perp u(a, b).$

A vertical transformation of *pseudo* double functors (7.4) amounts here to a square of vertical arrows $(x, y; \varphi): u \twoheadrightarrow v$, commutative up to a special isocell $\varphi: y{\bullet}u \cong v{\bullet}x: A \twoheadrightarrow B'$. Its limit $\mathrm{T}(x, y): \mathrm{T}u \twoheadrightarrow \mathrm{T}v$ (omitting φ for simplicity) is induced by the product profunctor $x{\times}y: A{\times}B \twoheadrightarrow A'{\times}B'$

(8) $\mathrm{T}(x, y): \mathrm{T}u^{op} \times \mathrm{T}v \to \mathbf{Set}, \qquad (a, b, \lambda;\ c, d, \mu) \mapsto x(a, c){\times}y(b, d)$

$\quad (f, g; h, k) \mapsto x(f, h){\times}y(g, k): x(a, c){\times}y(b, d) \to x(a', c'){\times}y(b', d')$

with $(f, g): (a', b', \lambda') \to (a, b, \lambda)$ in $\mathrm{T}u$ and $(h, k): (c, d, \mu) \to (c', d', \mu')$ in $\mathrm{T}v$. This is not consistent with vertical composition, as proved below (6.4b).

Its colimit $s = \perp(x, y)$ (determined up to special isocell), is calculated on pairs (a, b') by means of the diagonal of the square, $z = v{\bullet}x \cong y{\bullet}u: A \twoheadrightarrow B'$

(9) $s(a, b') = z(a, b') = v{\bullet}x(a, b') \cong y{\bullet}u(a, b')$

$\quad s(a, a') = x(a, a'), \qquad s(b, b') = y(b, b'), \qquad s(b, a') = \emptyset.$

This procedure is pseudo functorial. Given a second square $(x', y'; \varphi'): v \twoheadrightarrow w$, with $s' = \perp(x', y')$ and $\bar{s} = \perp(x'{\bullet}x, y'{\bullet}y)$, the only non trivial verification concerns pairs $(a, b'') \in A{\times}B''$ (as in 6.2.8); but, since $y'{\bullet}z \cong z'{\bullet}x$, the value of $s'{\bullet}s$ on such pairs is given by $z'{\bullet}x = (w{\bullet}x'){\bullet}x \cong w{\bullet}(x'{\bullet}x) = \bar{z}$.

6.4. Limits for spans. The pseudo double category $\mathbb{S}\mathrm{et}$ of sets, mappings and spans is identified to the full double subcategory of $\mathbb{C}\mathrm{at}$ consisting of discrete categories (3.2). We prove now that also $\mathbb{S}\mathrm{et}$ has all lax functorial limits and colax functorial colimits; for horizontal functors, (co)limits are given by the ones of **Set**. The embedding $D: \mathbb{S}\mathrm{et} \to \mathbb{C}\mathrm{at}$ preserves all limits, as well as all

colimits of horizontal functors, while cotabulators in $\mathbb{S}$et are quotients of the corresponding ones in $\mathbb{C}$at. Moreover, $\mathbb{S}$et has pseudo functorial products, equalisers, sums and cotabulators.

a) *Horizontal double functors*. The "positive" results of 6.3 a,b) restrict to the *full* double subcategory $\mathbb{S}$et, since the embedding **Set** $\to$ **Cat** preserves limits and colimits.

But here, arbitrary products are pseudo functorial in $\mathbb{S}$et. In fact, the canonical mappings 6.3.2 are bijective as soon as all categories B_i are discrete, so to reduce all coends to sums of sets. Moreover, (co)equalisers in $\mathbb{S}$et have an obvious, simpler construction, given by the (co)equaliser of the mappings α, β: $U \to V$ representing the cells

$$\begin{array}{ccc} A & \overset{f,\, g}{\rightrightarrows} & A' \\ {\scriptstyle u_1}\uparrow & & \uparrow{\scriptstyle v_1} \\ U & \overset{\alpha,\, \beta}{\rightrightarrows} & V \\ {\scriptstyle u_2}\downarrow & & \downarrow{\scriptstyle v_2} \\ B & \overset{f',\, g'}{\rightrightarrows} & B' \end{array} \tag{1}$$

Equalisers in $\mathbb{S}$et are pseudo functorial, because in **Set** "pullbacks preserve equalisers". On the other hand, coequalisers do not preserve the vertical composition in $\mathbb{S}$et; it is sufficient to take the similar counterexample for $\mathbb{R}$el in 6.2.4, and transfer it by the lax double functor S: $\mathbb{R}$el $\to$ $\mathbb{S}$et taking a relation to the associated jointly monic span (3.5), which happens to preserve the vertical compositions and coequalisers of that diagram. This also shows that coequalisers are not pseudo functorial in $\mathbb{C}$at.

b) *Tabulators and cotabulators*. In $\mathbb{S}$et, the tabulator $\top u$ is plainly the "graph" of the span, i.e. the middle object in the usual representation of a span; this is consistent with the embedding in $\mathbb{C}$at, which gives the discrete category $\top u = \Sigma u(a, b)$, and with the lax double functor S: $\mathbb{R}$el $\to$ $\mathbb{S}$et. Thus, tabulators in $\mathbb{S}$et are lax functorial (as in $\mathbb{C}$at); but they are not pseudo functorial (nor in $\mathbb{C}$at), as proved by our counterexample in $\mathbb{R}$el (6.2.10), noting that S preserves the compositions of profunctors appearing there.

Finally, the cotabulator $\bot u$ in $\mathbb{S}$et is given by the pushout of the span u, which is the set of connected components $\pi_0(\bot Du)$ of its cotabulator in $\mathbb{C}$at. The lax double functor S: $\mathbb{R}$el $\to$ $\mathbb{S}$et preserves them. Again, cotabulators in $\mathbb{S}$et are colax functorial: take a vertical transformation $(x, y; \varphi)$: $u \twoheadrightarrow v$ in $\mathbb{S}$et,

with $\varphi: y \bullet u \cong v \bullet x: A \twoheadrightarrow B'$ (6.3c), its colimit $\perp(Dx, Dy): \perp Du \twoheadrightarrow \perp Dv$ in $\mathbb{C}$at, and realise its colimit in $\mathbb{S}$et as π_0 of the last profunctor (3.2.4). Again, they are not pseudo functorial, as proved by our counterexample in $\mathbb{R}$el (6.2.11).

6.5. Limits and colimits for adjoints. Let us examine now the double category $\mathbb{A}$**dCat** of categories, functors and adjunctions, together with its double subcategories considered in 3.5-6.

a) *Products and sums*. The prototype $\mathbb{A}$**dCat** has functorial products and sums. It is sufficient to note that the product $u = (u_\bullet, u^\bullet, \eta, \varepsilon)$ of a family of adjunctions $(u_i: A_i \twoheadrightarrow B_i)$ can be formed by (standard) products in **Cat**

(1) $u_\bullet = \Pi u_{i\bullet}: \Pi A_i \to \Pi B_i,\qquad u^\bullet = \Pi u_i^\bullet: \Pi B_i \to \Pi A_i$

$\quad\ \eta = \Pi \eta_i: 1 \to u^\bullet u_\bullet: \Pi A_i \to \Pi A_i,\qquad \varepsilon = \Pi \varepsilon_i: u_\bullet u^\bullet \to 1: \Pi B_i \to \Pi B_i$

equipped with the obvious cells π_i: $(u \, {}^{pi}_{pi} \, u_i)$, whose covariant and contravariant part are identities

(2) $\pi_{i\bullet}: u_{i\bullet}\, p_i = p_i\, u_\bullet: \Pi A_j \to B_i,\qquad \pi_i^\bullet: u_i^\bullet\, p_i = p_i\, u^\bullet: \Pi B_j \to A_i.$

Similarly for sums. The double subcategories $\mathbb{A}$**dAbc**, $\mathbb{A}$**dTp**, $\mathbb{A}$**dOrd**, $\mathbb{A}$**dLt** and $\mathbb{A}$**dMl** are closed under such functorial products. $\mathbb{A}$**dOrd** is also closed under functorial sums.

b) *Equalisers*. $\mathbb{A}$**dCat**, $\mathbb{A}$**dOrd**, $\mathbb{A}$**dLt** and $\mathbb{A}$**dMl** have equalisers, supplied by the equalisers of **Cat**; but they are not even lax functorial. Plainly, equalisers do not exist in $\mathbb{A}$**dAbc**, but isoinserters do (6.6).

c) *Tabulators*. In $\mathbb{A}$**dCat**, the tabulator $\mathrm{T}u$ is the comma category

(3) $\mathrm{T}u = (u_\bullet \downarrow B) = (A \downarrow u^\bullet)\qquad (a, b; f: u_\bullet a \to b) \longleftrightarrow (a, b; f': a \to u^\bullet b)$

$\quad\ \pi: u_\bullet p \to q: \mathrm{T}u \to B,\qquad \pi_\bullet(a, b; f: u_\bullet a \to b) = f: u_\bullet a \to b.$

This realisation is functorial: given a commutative square of vertical arrows $(x, y): u \twoheadrightarrow v$, as in diagram 6.2.9, let $\mathrm{T}(x, y) = t$ be the induced adjunction, which it is convenient to write using both descriptions of the tabulator given above, in (3):

(4) $t_\bullet(a, b; f: u_\bullet a \to b) = (x_\bullet a, y_\bullet b;\ y_\bullet(f): y_\bullet u_\bullet a \to y_\bullet b)$

$\quad\ t^\bullet(a', b'; f': a' \to v^\bullet b') = (x^\bullet a', y^\bullet b';\ x^\bullet(f'): x^\bullet a' \to x^\bullet v^\bullet b').$

The unit of $t_\bullet \dashv t^\bullet$ consists of morphisms of $(u_\bullet \downarrow B))$ induced by the units $\eta_\cdot$ of x and y

(5) $\eta_\bullet(a, b;\ f: u_\bullet a \to b) = (\eta_\bullet a, \eta_\bullet b): (a, b; f) \to t^\bullet t_\bullet(a, b; f)$

as shown by the outer square of the following commutative diagram

(6)
$$
\begin{tikzcd}
u_\bullet a \arrow[rr, "u_\bullet\eta"] \arrow[dd, "u_\bullet\eta"'] & & u_\bullet u^\bullet u_\bullet a \arrow[r, "\varepsilon u_\bullet"] \arrow[d] & u_\bullet a \arrow[r, "f"] & b \arrow[dd, "\eta"] \\
& u_\bullet x^\bullet v^\bullet v_\bullet x_\bullet a \arrow[r, equal] \arrow[d, equal] & u_\bullet u^\bullet y^\bullet y_\bullet u_\bullet a \arrow[r] & u_\bullet u^\bullet y^\bullet y_\bullet b \arrow[d, equal] \arrow[dr] & \\
u_\bullet x^\bullet x_\bullet a \arrow[r, dashed] \arrow[ur] & u_\bullet x^\bullet v^\bullet y_\bullet u_\bullet a \arrow[rr, "u_\bullet x^\bullet v^\bullet y_\bullet f"] & & u_\bullet x^\bullet v^\bullet y_\bullet b \arrow[r, dashed] & y^\bullet y_\bullet b
\end{tikzcd}
$$

(7) $t^\bullet t_\bullet(a, b;\ f\colon u_\bullet a \to b) = t^\bullet(x_\bullet a, y_\bullet b;\ x_\bullet a \to v^\bullet v_\bullet x_\bullet a \to v^\bullet y_\bullet b) =$
$= (x^\bullet x_\bullet a, y^\bullet y_\bullet b;\ u_\bullet x^\bullet x_\bullet a \to u_\bullet x^\bullet v^\bullet v_\bullet x_\bullet a \to u_\bullet x^\bullet v^\bullet y_\bullet b \to y^\bullet y_\bullet b).$

Tabulators are inherited by **$\mathbb{A}$dAbc**. In fact, if A and B have some type of limits, or colimits, the same holds for $\mathrm{T}u$ (since $u_\bullet$ preserves the existing colimits and $u^\bullet$ the existing limits), and the projections preserve them. It is thus easy to show that, if A and B are abelian, so is $\mathrm{T}u$ (use also the fact that the functor $(p, q)\colon \mathrm{T}u \to A{\times}B$ is faithful and reflects isos). They are also inherited by **$\mathbb{A}$dTp$^\vee$** (by Artin gluing, [12], 4.27), as well as **$\mathbb{A}$dOrd**, **$\mathbb{A}$dLt** and **$\mathbb{A}$dMl**, where

(8) $\mathrm{T}u = \{(a, b) \in A{\times}B \mid u_\bullet a \leq b\} = \{(a, b) \mid a \leq u^\bullet b\}.$

d) *Cotabulators*. **$\mathbb{A}$dCat** has functorial cotabulators, where $C = \perp u = A{+}_u B$ is the category consisting of the disjoint union $A{+}B$, together with new maps $\hat{h} \in C(a, b)$ (from objects of A to objects of B) represented by elements $h \in B(u_\bullet a, b)$, and the natural composition between new maps $\hat{h}$ and old maps f, g

(9) $(g\hat{h}f) = ((gh)u_\bullet f)^\wedge\colon u_\bullet a' \to u_\bullet a \to b \to b' \quad (f \in A(a', a),\ g \in B(b, b'));$

the cotabulator-cell $\iota\colon i \to ju_\bullet\colon A \to \perp u$ is given by $\iota a = (1_{u_\bullet a})^\wedge\colon a \to u_\bullet a$.

Restricting to **$\mathbb{A}$dOrd**, we get the disjoint union $A{+}_u B$, with the order relations of A and B, and $a \leq b$ iff $u_\bullet a \leq b$ (iff $a \leq u^\bullet b$).

e) Finally, the double subcategories **$\mathbb{A}$d$_0$Lt** and **$\mathbb{A}$d$_0$Ml** of bicommutative cells (3.6) have also some *vertical* (co)limits of interest. To begin with, **$\mathbb{A}$d$_0$Lt** and **$\mathbb{A}$d$_0$Ml** have a vertical zero-object (trivially functorial), the singleton

(10) $X \xrightarrow{t} \{*\} \xrightarrow{s} X, \qquad t^\bullet(*) = 1, \qquad s_\bullet(*) = 0.$

The kernel and cokernel of a covariant connection $u = (u_\bullet, u^\bullet)\colon X \to Y$, in the categories **Ltc** and **Mlc**

$$
\begin{array}{llll}
(11) & m\colon {\downarrow}(u^\bullet 0) \twoheadrightarrow X, & m_\bullet(x) = x, & m^\bullet(x) = x \wedge u^\bullet 0 \\
& p\colon Y \twoheadrightarrow {\uparrow}(u_\bullet 1), & p_\bullet(y) = y \vee u_\bullet 1, & p^\bullet(y) = y
\end{array}
$$

become functorial vertical kernels and cokernels in our double categories (and $\mathbb{A}\mathrm{d}_0\mathbf{Ml}$ is "vertically Puppe-exact", in an obvious sense: the induced cell from cok∘ker to ker∘cok is vertically invertible).

$\mathbb{A}\mathrm{d}_0\mathbf{Lt}$ has also functorial vertical products and sums (while these do not exist in $\mathbb{A}\mathrm{d}_0\mathbf{Ml}$), both constructed with the cartesian product of lattices

$$
(12) \quad \Pi X_i \xrightarrow{p_i} X_i \xrightarrow{m_i} \Pi X_i, \qquad m_{i\bullet} \dashv m_i^\bullet = pr_i = p_{i\bullet} \dashv p_i^\bullet .
$$

6.6. Diagrams versus functors. Finally, we want to clarify the difference between the double limit of a *double diagram* and a *double functor*, showing that the latter is indeed a richer concept.

The *isoinserter* (X, x, ξ) of a pair of horizontal arrows $f, g\colon A \to B$, in a double category $\mathbb{A}$

$$
(1) \quad x\colon X \to A, \qquad \xi\colon (1^\bullet_X \,{}^{fx}_{gx}\, 1^\bullet_B)
$$

universally inserts a vertically invertible cell between fx and gx. It is (a well-known weighted limit in the 2-category $\mathbf{H}\mathbb{A}$ and) the double limit of the following double functor $F\colon \mathbb{I} \to \mathbb{A}$ (in $\mathbb{I}$, $v \bullet u = 1^\bullet_j$, $u \bullet v = 1^\bullet_k$)

$$
(2) \quad
\begin{tikzcd}
 & j \arrow[d, shift right, "u"'] \\
i \arrow[ur, "f"] \arrow[dr, "g"'] & k \arrow[u, shift right, "v"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & B \arrow[d, shift right, "1"'] \\
A \arrow[ur, "f"] \arrow[dr, "g"'] & B \arrow[u, shift right, "1"']
\end{tikzcd}
$$

Now, take $\mathbb{A} = \mathbb{A}\mathrm{d}\mathbf{Cat}$. Let $\mathbb{I}_0$ be the double graph explicitly shown in (2) (omitting the identities of $\mathbb{I}$ and the conditions on $v \bullet u$, $u \bullet v$) and $F_0\colon \mathbb{I}_0 \to \mathbb{A}\mathrm{d}\mathbf{Cat}$ the restriction of F. Then, the double limit X of the double diagram F_0 is the full subcategory of $(f{\downarrow}g) \times (g{\downarrow}f)$ over the objects

$$
(3) \quad (a,\ \beta\colon fa \to ga,\ \beta'\colon ga \to fa) \in \mathrm{Ob}A \times B \times B,
$$

while the double limit of F is the isoinserter of f and g, i.e. the full subcategory of X of those objects where β and β' are reciprocal isos. In the *flat* double subcategory $\mathbb{A}\mathrm{d}\mathbf{Ord}$ of monotonic functions and adjunctions the distinction disappears: in both cases we get the equaliser of f and g.

7. Appendix: Lax notions

This section contains the definition of pseudo double categories, lax and pseudo double functors, their horizontal and vertical transformations, and their modifications. It ends with a Strictification Theorem, showing that *pseudo* double categories and functors can be replaced with ordinary ones.

7.1. Pseudo double categories. A *pseudo double category* $\mathbb{A}$ is a "pseudo category object" in **CAT**. Accordingly, it is equipped with a horizontal partial composition $(b\circ a)$ satisfying the category laws, and a vertical partial composition $(b\bullet a)$, functorial on the previous structure, which satisfies such laws *up to three natural isomorphisms* λ, ρ, α (natural with respect to horizontal composition), whose horizontal domain and codomain are identities.

To give an explicit formulation, we adopt here the "one-sort formulation" where everything is a double cell, and write their compositions as $a\otimes_1 b = b\circ a$, $a\otimes_2 b = b\bullet a$. $\mathbb{A}$ is thus a system $(A; \partial_i^\varepsilon, \otimes_i; \lambda, \rho, \alpha)$ (with $i = 1, 2$; $\varepsilon = -, +$) such that:

(pdc0) (*basic properties*) A is a set, whose elements are called (double) *cells* of $\mathbb{A}$, equipped with four *faces* $\partial_i^\varepsilon \colon A \to A$ satisfying

(1) $\partial_i^\eta\,\partial_i^\varepsilon = \partial_i^\varepsilon$, $\qquad\qquad \partial_j^\eta\,\partial_i^\varepsilon = \partial_i^\varepsilon\,\partial_j^\eta \qquad (i \neq j)$;

given two cells a, b, the i-composition $a\otimes_i b$ is defined iff $\partial_i^+(a) = \partial_i^-(b)$, and then

(2) $\partial_i^-(a\otimes_i b) = \partial_i^-(a)$, $\qquad\qquad \partial_i^+(a\otimes_i b) = \partial_i^+(b)$

(3) $\partial_j^\varepsilon(a\otimes_i b) = (\partial_j^\varepsilon a)\otimes_i(\partial_j^\varepsilon b) \qquad (i \neq j)$

(pdc1) (*the main structure*) $\mathbf{A}_1 = (A; \partial_1^\varepsilon, \otimes_1)$ is a category, called the *main structure*, or 1-*structure* (or *horizontal structure*) of $\mathbb{A}$. The main faces of a cell a are written $a^\varepsilon = \partial_1^\varepsilon a$; they form the objects of $\mathbf{A}_1$ (vertical arrows of $\mathbb{A}$), also denoted by letters u, v, w...

(pdc2) (*the two-dimensional structure*) $\mathbf{A}_{12} = (\mathbf{A}_1; \partial_2^\varepsilon, \otimes_2; \lambda, \rho, \alpha)$ is a pseudo-category object in **Cat**.

Explicitly, the last axiom (after what was already anticipated in (pdc0)) means that:

(pdc2.1) (*interchange*) $\qquad (a\otimes_2 b)\otimes_1(a'\otimes_2 b') = (a\otimes_1 a')\otimes_2(b\otimes_1 b')$

(pdc2.2) (*left unit comparison*, in $\mathbf{A}_1$) $\qquad \lambda u\colon (\partial_2^- u)\otimes_2 u \cong u$

$$\partial_2^\varepsilon \lambda u = \partial_2^\varepsilon u, \qquad ((\partial_2^- a)\otimes_2 a) \otimes_1 (\lambda a^+) = (\lambda a^-) \otimes_1 a$$

(pdc2.3) (*right unit comparison*, in $\mathbf{A}_1$) $\qquad \rho u\colon u\otimes_2(\partial_2^+ u) \cong u$

$$\partial_2^\varepsilon \rho u = \partial_2^\varepsilon u, \qquad (a\otimes_2(\partial_2^+ a)) \otimes_1 (\rho a^+) = (\rho a^-) \otimes_1 a$$

(pdc2.4) (*associativity comparison*, in $\mathbf{A}_1$)

$$\alpha(u,v,w)\colon (u\otimes_2 v)\otimes_2 w \cong u\otimes_2(v\otimes_2 w)$$

$$\partial_2^- \alpha(u,v,w) = \partial_2^- u, \qquad \partial_2^+ \alpha(u,v,w) = \partial_2^+ w$$

$$((a\otimes_2 b)\otimes_2 c)) \otimes_1 (\alpha(a^+,b^+,c^+)) = (\alpha(a^-,b^-,c^-)) \otimes_1 (a\otimes_2(b\otimes_2 c))$$

(pdc2.5) (*coherence triangle for* λ, ρ, α)

$$\begin{tikzcd}
(u\otimes_2\partial_2^+ u)\otimes_2 v \arrow[rr, "\alpha"'] \arrow[dr, "\rho\otimes v"'] & & u\otimes_2(\partial_2^- v\otimes_2 v) \arrow[dl, "u\otimes\lambda"] \\
& u\otimes_2 v &
\end{tikzcd}$$

(pdc2.6) (*coherence pentagon for* α)

$$\begin{tikzcd}
& (u\otimes_2 v) \otimes_2 (w\otimes_2 x) \arrow[dr, "\alpha"] & \\
((u\otimes_2 v)\otimes_2 w)\otimes_2 x \arrow[ur, "\alpha"] \arrow[d, "\alpha\otimes 1"'] & & u\otimes_2(v\otimes_2(w\otimes_2 x)) \\
(u\otimes_2(v\otimes_2 w))\otimes_2 x \arrow[rr, "\alpha"] & & u\otimes_2((v\otimes_2 w)\otimes_2 x) \arrow[u, "1\otimes\alpha"']
\end{tikzcd}$$

The above axioms can be represented by diagrams, either in $\mathbf{A}_1$ (as above), either in $\mathbb{A}$. Thus, in (pdc2.2), the naturality condition $((\partial_2^- a)\otimes_2 a) \otimes_1 (\lambda a^+) = (\lambda a^-) \otimes_1 a$ amounts to a commutative square of $\mathbf{A}_1$ (under $\otimes_1$) and also to an equality of pasting diagrams in $\mathbb{A}$ (write $u = a^-$, $v = a^+$)

(4)
$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\partial_2^- u"'] \arrow[dr, phantom, "\partial_2^- a"] & \bullet \arrow[r, equal] \arrow[d, "\partial_2^- v"] \arrow[ddr, phantom, "\lambda v"] & \bullet \arrow[dd, "v"] \\
\bullet \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "a"] & \bullet \arrow[d, "v"] & \\
\bullet \arrow[r] & \bullet \arrow[r, equal] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "\partial_2^- u"'] \arrow[ddr, phantom, "\lambda u"] & \bullet \arrow[r] \arrow[dd, "u"] \arrow[ddr, phantom, "a"] & \bullet \arrow[dd, "v"] \\
\bullet \arrow[d, "u"'] & & \\
\bullet \arrow[r, equal] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

Note also that the formula $\partial_2^\varepsilon \lambda u = \partial_2^\varepsilon u$ in (pdc2.2) says that the horizontally invertible cells λu are special (iso)cells (2.2), i.e. their vertical domain and codomain are identities. A bicategory [3, 15, 19] is the same as a *vertical* pseudo double category: all horizontal arrows are identities.

$\mathbb{A}$ is said to be *unitary*, or *normalised*, if the unit comparisons λ, ρ are identities. In the previous sections this assumption, which is useful and no real restriction (1.9; 3.1-2), is always understood.

7.2. Lax double functors. A *lax double functor* $F\colon \mathbb{A} \to \mathbb{B}$ between pseudo double categories takes "items" of $\mathbb{A}$ to "items" of $\mathbb{B}$ of the corresponding type, respecting the horizontal structure in the usual strict sense, and the vertical one up to an assigned *comparison* $\varphi = \varphi^F$.

Thus, every object A is equipped with a special cell $\varphi A\colon 1^\bullet_{FA} \to F(1^\bullet_A)\colon FA \twoheadrightarrow FA$ (the *identity comparison*), and every vertical composition $v{\bullet}u$ is equipped with a special cell $\varphi(u, v)\colon Fv{\bullet}Fu \to F(v{\bullet}u)\colon FA \twoheadrightarrow FC$ (the *composition comparison*), in a coherent way. This means the following:

i) (*naturality*) for a vertical composition $b{\bullet}a$, $(\frac{Fa}{Fb} \mid \varphi(u',v')) = (\varphi(u,v) \mid F(b{\bullet}a))$

$$(1) \qquad \begin{tikzcd} FA \arrow[r] \arrow[d, "Fu"'] \arrow[dr, phantom, "Fa"] & FA' \arrow[r, equal] \arrow[d, "Fu'"] & FA' \arrow[dd] \\ FB \arrow[r] \arrow[d, "Fv"'] \arrow[dr, phantom, "Fb"] & FB' \arrow[d, "Fv'"] \arrow[r, phantom, "\varphi"] & {} \\ FC \arrow[r] & FC' \arrow[r, equal] & FC' \end{tikzcd} \quad = \quad \begin{tikzcd} FA \arrow[r, equal] \arrow[d, "Fu"'] & FA \arrow[r] \arrow[dd, "F(v\bullet u)"] \arrow[ddr, phantom, "F(b\bullet a)"] & FA' \arrow[dd, "F(v'\bullet u')"] \\ FB \arrow[d, "Fv"'] \arrow[r, phantom, "\varphi"] & {} & {} \\ FC \arrow[r, equal] & FC \arrow[r] & FC' \end{tikzcd}$$

ii) (*coherence laws for* φ, λ *and* φ, ρ) for a vertical morphism u, the following diagrams of special cells are commutative (under horizontal composition)

$$(2) \qquad \begin{tikzcd} Fu{\bullet}1^\bullet \arrow[r, "\lambda Fu"] \arrow[d, "1\bullet\varphi"'] & Fu \\ Fu{\bullet}F1^\bullet \arrow[r, "\varphi"] & F(u{\bullet}1^\bullet) \arrow[u, "F\lambda"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} 1^\bullet{\bullet}Fu \arrow[r, "\rho Fu"] \arrow[d, "\varphi\bullet 1"'] & Fu \\ F1^\bullet{\bullet}Fu \arrow[r, "\varphi"] & F(1{\bullet}u) \arrow[u, "F\rho"'] \end{tikzcd}$$

iii) (*coherence hexagon for* φ, α) for consecutive vertical morphisms u, v, w, the following diagram of special cells is commutative

$$
(3) \qquad \begin{array}{ccccc}
Fw\bullet(Fv\bullet Fu) & \xrightarrow{1\bullet\varphi} & Fw\bullet F(v\bullet u) & \xrightarrow{\varphi} & F(w\bullet(v\bullet u)) \\
{\scriptstyle\alpha}\downarrow & & & & \downarrow{\scriptstyle F\alpha} \\
(Fw\bullet Fv)\bullet Fu & \xrightarrow{\varphi\bullet 1} & F(w\bullet v)\bullet Fu & \xrightarrow{\varphi} & F((w\bullet v)\bullet u)
\end{array}
$$

The composition of lax double functors, being based on the horizontal composition of arrows and cells, is strictly associative:

(4) $\varphi^{GF}A = (G\varphi^F(A))(\varphi^G(FA))\colon 1^\bullet_{GFA} \to FG(1^\bullet_A)$

$\varphi^{GF}(u, v) = (G\varphi^F(u, v))\circ(\varphi^G(Fu, Fv))\colon GFv\bullet GFu \to GF(v\bullet u)$.

By horizontal duality (1.2), a *colax double functor* $F\colon \mathbb{A} \to \mathbb{B}$ has special cells $\varphi A\colon F(1^\bullet_A) \to 1^\bullet_{FA}$ and $\varphi(u, v)\colon F(v\bullet u) \to Fv\bullet Fu$ forming a lax double functor between the horizontal opposites, $F^h\colon \mathbb{A}^h \to \mathbb{B}^h$. A *pseudo double functor* (between pseudo double categories) is a lax one in which the comparison cells are horizontally invertible; the co-notion is equivalent.

A *unitary* (co)lax double functor (between unitary structures) has $F(1^\bullet_A) = 1^\bullet_{FA}$ and identity cells φA. In the previous Sections we always restrict to this case, as motivated in 4.3.

7.3. Horizontal transformations. Let $\mathbb{A}$ and $\mathbb{B}$ be pseudo double categories. Proceeding as in 1.4, a *horizontal transformation of lax double functors* $H\colon F \to G\colon \mathbb{A} \to \mathbb{B}$ can be defined as a lax double functor $H\colon \mathbb{H}2\times\mathbb{A} \to \mathbb{B}$, with $F = H.(\delta^-\times\mathbb{A})$, $G = H.(\delta^+\times\mathbb{A})$.

H amounts thus to giving two lax double functors (F, φ), (G, ψ), with additional data (as in the strict case, 1.4):

a) for each object A in $\mathbb{A}$, a horizontal map $HA\colon FA \to GA$,

b) for each vertical map $u\colon A \twoheadrightarrow B$ in $\mathbb{A}$, a cell $Hu\colon (Fu \,{}^{HA}_{HB}\, Gu)$,

satisfying the following relaxed conditions (the third does not include φ, ψ and coincides with (ht.3) in 1.4):

(lht.1) (*coherence with identity comparison*) for all A, $(\varphi A \mid H1^\bullet_A) = (1^\bullet_{HA} \mid \psi A)$

(lht.2) (*coherence with composition comparison*) for $w = v\bullet u$ in $\mathbb{A}$,

$$(\varphi(u,v) \mid Hw) = \left(\frac{Hu}{Hv} \mid \psi(u,v)\right);$$

(lht.3) (*naturality*) for $a\colon (u \,{}^{f}_{g}\, v)$ in $\mathbb{A}$, $(Fa \mid Hv) = (Hu \mid Ga)$.

The horizontal composition of horizontal transformations is strictly categorical. In the unitary case, one takes $H(1^\bullet_A) = 1^\bullet_{HA}$.

7.4. Strong vertical transformations and modifications. Let us restrict to the unitary case. A general vertical transformation $U\colon F \twoheadrightarrow G\colon \mathbb{A} \to \mathbb{B}$ of *lax* double functors should be defined as a lax double functor $\mathbb{V}2{\times}\mathbb{A} \to \mathbb{B}$. This makes a complicated structure (with *two* systems of comparison *cells* for naturality), of which we have few examples and no present use. We prefer therefore to restrict to a simpler notion (with *one* system of *isocells*), which is needed for sesqui-isomorphisms of lax double functors (2.5, 4.5).

A *strong vertical transformation* $U\colon F \twoheadrightarrow G$ of (unitary) lax double functors consists of vertical maps $UA\colon FA \twoheadrightarrow GA$, cells $Uf\colon (UA \, {}^{Ff}_{Gf} \, UA')$ and special *iso*cells $Uu\colon UB\bullet Fu \to Gu\bullet UA\colon FA \twoheadrightarrow GB$ (*naturality comparison*), under the axioms:

(svt.1) for A in $\mathbb{A}$, $\quad U(1_A) = 1_{UA} = U(1^\bullet_A)$,

(svt.2) for f, g horizontal in $\mathbb{A}$, $\quad U(gf) = Ug \circ Uf$,

(svt.3) for $a\colon (u \, {}^f_g \, v)$ in $\mathbb{A}$

$$\begin{tikzcd}
\bullet \arrow[r, "Ff"] \arrow[d, "Fu"'] \arrow[dr, phantom, "Fa"] & \bullet \arrow[r, equal] \arrow[d, "Fv"] \arrow[ddr, phantom, "Uv"] & \bullet \arrow[d, "UA'"] \\
\bullet \arrow[r, "Fg"] \arrow[d, "UB"'] \arrow[dr, phantom, "Ug"] & \bullet \arrow[d, "UB'"] & \bullet \arrow[d, "Gv"] \\
\bullet \arrow[r, "Gg"'] & \bullet \arrow[r, equal] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "Fu"'] \arrow[ddr, phantom, "Uu"] & \bullet \arrow[r, "Ff"] \arrow[d, "UA"] \arrow[dr, phantom, "Uf"] & \bullet \arrow[d, "UA'"] \\
\bullet \arrow[d, "UB"'] & \bullet \arrow[r, "Gf"] \arrow[d, "Gu"] \arrow[dr, phantom, "Ga"] & \bullet \arrow[d, "Gv"] \\
\bullet \arrow[r, equal] & \bullet \arrow[r, "Gg"'] & \bullet
\end{tikzcd}$$

(svt.4) for $w = v\bullet u$ in $\mathbb{A}$, the following pasting

$$\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[dd, "Fv\bullet Fu"'] & \bullet \arrow[r, equal] \arrow[d, "Fu"'] \arrow[dr, phantom, "1"] & \bullet \arrow[r, equal] \arrow[d, "Fu"'] & \bullet \arrow[r, equal] \arrow[d, "Fu"'] \arrow[dr, phantom, "Uu"] & \bullet \arrow[r, equal] \arrow[d, "UA"] & \bullet \arrow[r, equal] \arrow[d, "UA"] \arrow[dr, phantom, "1"] & \bullet \arrow[d, "UA"] \\
& \bullet \arrow[r, equal] \arrow[d, "Fv"'] & \bullet \arrow[d, "UB"'] & \bullet \arrow[d, "UB"'] & \bullet \arrow[d, "Gu"] & \bullet \arrow[r, equal] \arrow[dd, "Gv\bullet Gu"] \arrow[ddr, phantom, "\psi"] & \bullet \arrow[dd, "G(v\bullet u)"] \\
\bullet \arrow[r, phantom, "\alpha"] \arrow[d, "UC"'] & \bullet \arrow[r, phantom, "Uv"] \arrow[d, "UC"'] & \bullet \arrow[r, phantom, "\alpha^{-1}"] \arrow[d, "Gv"'] & \bullet \arrow[r, equal, "\bullet"'] \arrow[d, "Gv"'] \arrow[dr, phantom, "1"] & \bullet \arrow[r, phantom, "\alpha"] \arrow[d, "Gv"] & & \\
\bullet \arrow[r, equal] & \bullet \arrow[r, equal] & \bullet \arrow[r, equal] & \bullet \arrow[r, equal] & \bullet \arrow[r, equal] & \bullet \arrow[r, equal] & \bullet
\end{tikzcd}$$

coincides with $(1_{UC}\bullet\varphi(u, v)) \mid Uw)$.

The vertical composition $W = V\bullet U$, for $V\colon G \to S$, has naturality cells Wu obtained by pasting the ones of U and V, and correcting the result by the associativity isocells α

(1) $WA = VA\bullet UA\colon FA \twoheadrightarrow SA$, $\qquad Wf = Vf\bullet Uf\colon (WA \,{}^{Ff}_{Sf}\, WA')$

$\quad Wu\colon WB\bullet Fu \to Su\bullet WA\colon FA \twoheadrightarrow GB$

$$\begin{array}{ccccccccccc}
\bullet & = & \bullet & = & \bullet & = & \bullet & = & \bullet & = & \bullet \\
Fu\downarrow & & Fu\downarrow & & UA\downarrow & & \downarrow UA & 1 & \downarrow UA & & \\
\bullet & \alpha^{-1} & \bullet & Uu & \bullet & & \bullet & = & \bullet & & \downarrow WA \\
 & & UB\downarrow & & Gu\downarrow & \alpha & \downarrow Gu & & \downarrow VA & & \\
WB\downarrow & & \bullet & = & \bullet & & \bullet & Vu & \bullet & \alpha^{-1} & \bullet \\
 & & VB\downarrow & 1 & VB\downarrow & & \downarrow VB & & \downarrow Su & & \downarrow Su \\
\bullet & = & \bullet & = & \bullet & = & \bullet & = & \bullet & = & \bullet
\end{array}$$

A *strong modification* $\mu\colon (U \,{}^{H}_{K}\, V)\colon ({}^{F}_{F'}\,{}^{G}_{G'})$ of lax double functors is defined as in the strict case (1.6), up to inserting, in the diagrams of the coherence conditions, the comparison cells of the (strong) vertical transformations. Thus, the first axiom (md.1) stays unchanged, while (md.2) is replaced with

(smd.2) for every vertical arrow $u\colon A \twoheadrightarrow B$ in $\mathbb{A}$

$$\begin{array}{ccccc}
\bullet & = & \bullet & \xrightarrow{HA} & \bullet \\
Fu\downarrow & & UA\downarrow & \mu A & \downarrow VA \\
\bullet & Uu & \bullet & -KA\to & \bullet \\
UB\downarrow & & Fu\downarrow & Ku & \downarrow G'u \\
\bullet & = & \bullet & \xrightarrow[KB]{} & \bullet
\end{array}
\quad = \quad
\begin{array}{ccccc}
\bullet & \xrightarrow{HA} & \bullet & = & \bullet \\
Fu\downarrow & Hu & \downarrow Gu & & \downarrow VA \\
\bullet & -HB\to & \bullet & Vu & \bullet \\
UB\downarrow & \mu B & \downarrow VB & & \downarrow G'u \\
\bullet & \xrightarrow[KB]{} & \bullet & = & \bullet
\end{array}$$

The vertical composition of strong vertical transformations is associative up to special modifications $\hat{\alpha}(U, V, W)\colon W\bullet(V\bullet U) \to (W\bullet V)\bullet U$, whose component at the object A is the special isocell $\alpha(UA, VA, WA)$. This completes the definition of the pseudo double category $\mathrm{Lax}(\mathbb{A}, \mathbb{B})$ of (unitary) lax double functors between pseudo double categories (with their horizontal transformations, strong vertical transformations and strong modifications).

A vertical transformation of (unitary) *pseudo* double functors will be, by definition, a strong one. (This notion is essentially equivalent to a pseudo double functor $\mathbb{V}2\times\mathbb{A} \to \mathbb{B}$, which comes equipped with *two* systems of comparison *isocells*). Similarly, a modification is meant to be strong. We have now the pseudo double category $\mathrm{Ps}(\mathbb{A}, \mathbb{B})$ of pseudo double functors, their horizontal and vertical transformations, their modifications.

7.5. Strictification Theorem. *Every pseudo-double category* $\mathbb{P}$ *has an associated double category* $\mathbb{A}$, *equivalent to it by means of pseudo double functors* $F\colon \mathbb{P} \to \mathbb{A}$, $G\colon \mathbb{A} \to \mathbb{P}$ *with* $GF = 1$, $FG \cong 1$ *(horizontally isomorphic). Every pseudo double functor* $S\colon \mathbb{P} \to \mathbb{Q}$ *can be similarly replaced by an ordinary double functor* $S'\colon \mathbb{A} \to \mathbb{B}$. *(But* lax *double functors cannot be similarly transferred, even keeping them lax.)*

Proof. Recall that the pseudo-double category $\mathbb{P}$ has a *vertical bicategory* $\mathbf{V}\mathbb{P}$, whose cells $a\colon u \to v$ are provided by the *special* cells $a\colon (u \,{}^1_1\, v)$ of $\mathbb{P}$; its associativity and unit comparisons α, λ, ρ are the ones of $\mathbb{P}$. We shall repeatedly use the coherence theorem for bicategories (Mac Lane-Paré [19]), which –loosely speaking– says that any "natural" diagram of $\mathbf{V}\mathbb{P}$ made up of instances of the comparisons α, λ, ρ commutes (natural means that "accidental" composites must not occur).

We want to replace $\mathbf{V}\mathbb{P}$ with the free category $\mathbf{V}$ on the (graph of the) old vertical arrows. A new vertical arrow $\overline{u} = (u_1, \ldots u_n)\colon A \twoheadrightarrow A'$ is thus a string of old vertical arrows $A = A_0 \twoheadrightarrow A_1 \twoheadrightarrow \ldots \twoheadrightarrow A_n = A'$, including an empty string $e_A\colon A \twoheadrightarrow A$ for each object; their composition is concatenation.

The free category $\mathbf{V}$ comes with an evaluation morphism of reflexive graphs $(-)^\wedge\colon \mathbf{V} \to \mathbf{V}\mathbb{P}$, taking the string $\overline{u}$ to $\hat{u} = (\ldots(u_3 \bullet (u_2 \bullet u_1))\ldots)\colon A \twoheadrightarrow A'$ (and e_A to $1^\bullet_A$); it is actually a unitary morphism of bicategories, with comparison

(1) $\psi(\overline{u}, \overline{v})\colon \hat{v} \bullet \hat{u} \to (\overline{v} \bullet \overline{u})^\wedge\colon A \twoheadrightarrow A''$

obtained by composing instances of α (or from λ, ρ, if $\overline{u}$ or $\overline{v}$ is a new vertical identity). The coherence theorem for $\mathbf{V}\mathbb{P}$ says that we get the same result, no matter how this is done, and that (1) is indeed coherent with the associativity isocells α of $\mathbf{V}\mathbb{P}$ (and the trivial ones for $\mathbf{V}$)

$$(2)\qquad \begin{tikzcd} \hat w\bullet(\hat v\bullet\hat u) \arrow[r, "1\bullet\psi"] \arrow[d, "\alpha"'] & \hat w\bullet(v\bullet u)^\wedge \arrow[r, "\psi"] & (w\bullet(v\bullet u))^\wedge \arrow[d, equal] \\ (\hat w\bullet\hat v)\bullet\hat u \arrow[r, "\psi\bullet 1"] & (w\bullet v)^\wedge\bullet\hat u \arrow[r, "\psi"] & ((w\bullet v)\bullet u)^\wedge \end{tikzcd}$$

Now, the objects and horizontal arrows of $\mathbb{A}$ are the same as those of $\mathbb{P}$, as is horizontal composition of arrows. The new vertical arrows are the previous strings, and $\mathrm{ver}_1\mathbb{A} = \mathbf{V}$. A new double cell a is represented by an old double cell $\hat a\colon (\hat u\,{}^{f}_{g}\,\hat v)$

$$(3)\qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\bar u"'] \arrow[dr, phantom, "a"] & B \arrow[d, "\bar v"] \\ A' \arrow[r, "g"'] & B' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, equal] \arrow[d, "\bar u"'] \arrow[dr, phantom, "1"] & A \arrow[d, "\bar u"] \\ A' \arrow[r, equal] & A' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "e_f"] & B \arrow[d, "e"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

in particular, the horizontal identity $1_{\bar u}$ of a new vertical arrow is represented by $1_{\hat u}$, and the vertical identity e_f of a horizontal arrow is represented by $1^\bullet_f$.

Horizontal composition of double cells in $\mathbb{A}$ is like in $\mathbb{P}$, and forms a category. The vertical composition $b\bullet a$ of new double cells is expressed by the following old cell $(b\bullet a)^\wedge$

$$(4)\qquad \begin{tikzcd} \bullet \arrow[r, equal] \arrow[dd, "(\bar u'\bullet\bar u)^\wedge"'] & \bullet \arrow[r] \arrow[d, "\hat u"'] \arrow[dr, phantom, "\hat a"] & \bullet \arrow[r, equal] \arrow[d, "\hat v"] & \bullet \arrow[dd, "(\bar v'\bullet\bar v)^\wedge"] \\ \psi^{-1} & \bullet \arrow[r] \arrow[d, "\hat u'"'] \arrow[dr, phantom, "\hat b"] & \bullet \arrow[d, "\hat v'"] & \psi \\ \bullet \arrow[r, equal] & \bullet \arrow[r] & \bullet \arrow[r, equal] & \bullet \end{tikzcd}$$

To prove that $\mathbb{A}$ is a double category, the main point is vertical associativity, $c\bullet(b\bullet a) = (c\bullet b)\bullet a$. These new cells are expressed in $\mathbb{P}$ as the composites of the following diagrams (where the boldface characters denote the cells which are to be vertically composed *first*), with $u^\# = (\bar u''\bullet\bar u'\bullet\bar u)^\wedge$ and $v^\# = (\bar u''\bullet\bar u'\bullet\bar u)$

(5)
$$
\begin{array}{ccccccccccc}
\bullet & = & \bullet & = & \bullet & \longrightarrow & \bullet & = & \bullet & = & \bullet \\
 & & & & \downarrow & \hat{a} & \downarrow & & & & \\
 & & \downarrow & \psi^{-1} & \bullet & \longrightarrow & \bullet & \psi & \downarrow & & \\
u^{\#}\downarrow & \psi^{-1} & & & \downarrow & \hat{b} & \downarrow & & & \psi & \downarrow v^{\#} \\
 & & \bullet & = & \bullet & \longrightarrow & \bullet & = & \bullet & & \\
 & & \downarrow & 1 & \downarrow & \hat{c} & \downarrow & 1 & \downarrow & & \\
\bullet & = & \bullet & = & \bullet & \longrightarrow & \bullet & = & \bullet & = & \bullet
\end{array}
$$

(6)
$$
\begin{array}{ccccccccccc}
\bullet & = & \bullet & = & \bullet & \longrightarrow & \bullet & = & \bullet & = & \bullet \\
 & & \downarrow & 1 & \downarrow & \hat{a} & \downarrow & 1 & \downarrow & & \\
 & & \bullet & = & \bullet & \longrightarrow & \bullet & = & \bullet & & \\
u^{\#}\downarrow & \psi^{-1} & & & \downarrow & \hat{b} & \downarrow & & & \psi & \downarrow v^{\#} \\
 & & \downarrow & \psi^{-1} & \bullet & \longrightarrow & \bullet & \psi & \downarrow & & \\
 & & & & \downarrow & \hat{c} & \downarrow & & & & \\
\bullet & = & \bullet & = & \bullet & \longrightarrow & \bullet & = & \bullet & = & \bullet
\end{array}
$$

and these cells coincide. Actually, form a solid diagram inserting two associativity isocells of $\mathbb{P}$,

(7) $\alpha\colon \hat{u}''\bullet(\hat{u}'\bullet\hat{u}) \to (\hat{u}''\bullet\hat{u}')\bullet\hat{u}$, $\qquad$ $\alpha\colon \hat{v}''\bullet(\hat{v}'\bullet\hat{v}) \to (\hat{v}''\bullet\hat{v}')\bullet\hat{v}$,

so that the central prism commutes (by definition of pseudo double category); then, the right-hand part, involving the second instance of α and the right-hand part of the diagrams above, commutes by (2); and similarly the left-hand part.

The interchange law in $\mathbb{A}$ is obvious once it is written out (a cell ψ cancels with a ψ^{-1}). The identity laws are left to the reader.

The (unitary) pseudo double functor $G\colon \mathbb{A} \to \mathbb{P}$ has already been constructed: $G(\bar{u}) = \hat{u}$, $G(a) = \hat{a}$, with composition comparison $\psi(\bar{u}, \bar{v})\colon \hat{v}\bullet\hat{u} \to (\bar{v}\bullet\bar{u})^{\wedge}$. The embedding $F\colon \mathbb{P} \to \mathbb{A}$ is obvious (a vertical arrow is sent to the corresponding string of length 1). It preserves vertical identities and composition up to special new isocells $\varphi A\colon e_A \to 1^{\bullet}_A$ (represented by the double identity of A, $\Box_A$) and $\varphi(u, v)\colon Fv\bullet Fu \to F(v\bullet u)$ (represented by the horizontal identity of the old composite $v\bullet u$). Then, $GF = 1$ while FG is horizontally isomorphic to the identity on $\mathbb{A}$, by special isocells $H\bar{u}\colon (FG\bar{u} \overset{1}{\underset{1}{|}} \bar{u})$ represented by the horizontal identity of $\hat{u}$, for each string $\bar{u}$.

Finally, given a pseudo double functor $S\colon \mathbb{P} \to \mathbb{Q}$, with special isocells $\sigma A\colon 1^\bullet_{SA} \to S(1^\bullet_A)$ and $\sigma(u, v)\colon Sv{\bullet}Su \to S(v{\bullet}u)$, the strictified double functor $S'\colon \mathbb{A} \to \mathbb{B}$ has

(8) $S'(u_1,\dots u_n) = (Su_1,\dots Su_n)$, $\quad S'(e_A) = e_{SA},$ $\quad S'(a)\colon (S'\bar{u}\ {}^{Sf}_{Sg}\ S'\bar{v})$;

the cell $S'(a)$ is represented by a modification of $S(\hat{a})$

$$
(9)\qquad
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "(S'\bar{u})^\wedge"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[r] \arrow[d, "S\hat{u}"] \arrow[dr, phantom, "S\hat{a}"] & \bullet \arrow[r, equal] \arrow[d, "S\hat{v}"'] \arrow[dr, phantom, "\sigma^{-1}"] & \bullet \arrow[d, "(S'\bar{v})^\wedge"] \\
\bullet \arrow[r, equal] & \bullet \arrow[r] & \bullet \arrow[r, equal] & \bullet
\end{tikzcd}
$$

obtained from *generalised* isocells σ (well defined, by the coherence theorem of bicategories). This procedure needs an invertible comparison and does not work for a *lax* double functor.

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