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HEEGARD AND REGULAR GENUS AGREE FOR COMPACT 3-MANIFOLDS

by Paola CRISTOFORI

Rèsumé. Le genre de Heegaard et le genre régulier sont deux invariants pour les 3-variétés. On sait déjà qu'ils coïncident pour les 3-variétés orientables avec frontière; on sait aussi que le genre régulier d'une 3-variété non orientable fermée est exactement le double de son genre de Heegaard. Dans cet article, nous prouvons que ces résultats s'étendent au cas général des 3-variétés compactes.

1. Introduction.

Throughout this paper we consider only compact, connected, PL-manifolds and PL-maps.

The Heegaard genus is a well-known topological invariant, introduced in [12], for closed 3-manifolds and it has been often used in the study of 3-manifolds.

More recently Montesinos extended the definition of Heegaard genus to compact 3-manifolds with boundary (see [14]).

The regular genus is an invariant for compact n-manifolds, which was introduced in the combinatorial setting, by using the theory of representation of manifolds by coloured graphs ([9] and [10]).

Many results are already known about this invariant: for example the characterization of the n-dimensional sphere and disk as the only n-manifolds (with either empty or connected boundary) having genus zero ([11]) and some interesting results in dimension four and five with the regard to the classification of manifolds according to their regular genus (see [3] and [4]).

It is well-known that for compact surfaces the regular genus equals the genus.

Furthermore, for orientable 3-manifolds with boundary, the regular genus coincides with the Heegaard one (see [5]). The coincidence is proved by using

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we have, according to [9], that:
- $\mathcal{G}(M) = \mathcal{H}(M)$ if M is orientable;
- $\mathcal{G}(M) = 2\mathcal{H}(M)$ if M is non-orientable.

In this paper we prove that the above result extends to 3-manifolds with boundary, i.e. that the second equality holds also when M has non-empty boundary. The proof uses combinatorial techniques and constructions, which are not in the least affected by the orientability or non-orientability of M. Therefore, it is also an alternative proof of the coincidence of the two invariants in the orientable case.

As a consequence of our result the regular genus can be thought as a "natural" extension to dimension n of the classical concepts of genus of a surface and Heegaard genus of a 3-manifold.

2. Coloured graphs and the regular genus of a manifold.

A *(n+1)-coloured graph (with boundary)* is a pair (Γ, γ), where $\Gamma = (V(\Gamma), E(\Gamma))$ is a multigraph (i.e. loops are forbidden) and $\gamma : E(\Gamma) \rightarrow \Delta_n = \{0, 1, \dots, n\}$ a map, which is injective on each pair of adjacent edges of Γ.

For each $B \subseteq \Delta_n$, we call *B-residues* the connected components of the multigraph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$; we set $\hat{\imath} = \Delta_n \setminus \{i\}$ for each $i \in \Delta_n$; furthermore we write Γ_i, Γ_{ij} instead of $\Gamma_{\{i\}}, \Gamma_{\{i,j\}}$.

The vertices of Γ whose degree is strictly less than $n+1$ are called *boundary vertices*; if (Γ, γ) has no boundary vertices, it is called *without boundary*.

In the following we shall always consider only $(n+1)$-coloured graphs which are *regular with respect to the colour n*, i.e. $\Gamma_{\hat{n}}$ is regular of degree n.

The *boundary graph* of (Γ, γ) is a n-coloured graph (without boundary) $(\partial\Gamma, {}^{\partial}\gamma)$ whose vertices are the boundary vertices of (Γ, γ) and, for each $i \in \Delta_{n-1}$ two vertices of $(\partial\Gamma, {}^{\partial}\gamma)$ are joined by an i-coloured edge iff they belong to the same $\{n, i\}$-residue of (Γ, γ).

If K is an n-dimensional homogeneous pseudocomplex [13], and $V(K)$ its set of vertices, we call *coloured n-complex* the pair (K, ξ) where $\xi : V(K) \rightarrow \Delta_n$ is a map which is injective on every simplex of K.

If σ^h is an h-simplex of K then the *disjoint star* $std(\sigma^h, K)$ *of* σ^h *in* K is the pseudocomplex obtained by taking the disjoint union of the h-simplexes of K containing σ^h and identifying the $(n-1)$-simplexes containing σ^h together with all their faces.

The *disjoint link* $lkd(\sigma^h, K)$ *of* σ^h *in* K is the pseudocomplex:

$$lkd(\sigma^h, K) = \{\tau \in std(\sigma^h, K) \mid \sigma^h \cap \tau = \emptyset\} \subset std(\sigma^h, K).$$

A *quasi-manifold* is a polyhedra $|K|$, where K is a pseudocomplex, such that:

- each $(n-1)$-simplex is face of exactly two n-simplexes of K;
- for each simplex σ of K, $std(\sigma, K)$ is strongly connected.

From now on we shall restrict our attention to the coloured complexes which triangulate quasi-manifolds (the reason of this restriction will become clear in the following).

Given a $(n+1)$-coloured graph (Γ, γ), consider the coloured complex $(K(\Gamma), \xi(\Gamma))$, constructed as follows:

- take one n-simplex $\sigma(a)$ for each $a \in V(\Gamma)$;
- for each $i \in \Delta_n$ and each pair a, b of i-adjacent vertices of Γ, identify the $(n-1)$-faces of $\sigma(a)$ and $\sigma(b)$ opposite to the i-coloured vertices, taking care to identify vertices of the same colour.

It is clear that the construction can be easily reversed. Our previous restriction to the class of quasi-manifolds assures that the two constructions are really inverse to each other (see [7]).

Note that, by the above construction, there is a bijective correspondence between the h-simplexes $(0 \leqslant h \leqslant dim K(\Gamma))$ of $K(\Gamma)$ and the $(n-h)$-residues of Γ, in the sense that, if σ^h is an h-simplex of $K(\Gamma)$, whose vertices are labelled by $\{i_0, ..., i_h\}$, then there is a unique $(n-h)$-residue Ξ of Γ whose edges are coloured by $\Delta_n \setminus \{i_0, ... i_h\}$ and such that $K(\Xi) = lkd(\sigma^h, K)$.

A direct way to see this correspondence is to think (Γ, γ) imbedded in $K(\Gamma)$ as its dual 1-skeleton, i.e. the vertices of Γ are the barycenters of the n-simplexes of $K(\Gamma)$ and the edges of Γ are the 1-cells dual of the $(n-1)$-simplexes of $K(\Gamma)$. Of course the $(n-1)$-simplex dual to an edge e with $\gamma(e) = i$ has its vertices labelled by $\hat{\imath}$.

If M is a manifold (with boundary) of dimension n and (Γ, γ) a $(n+1)$-coloured graph (with boundary) such that $|K(\Gamma)| \cong M$, we say that M is *represented* by (Γ, γ). In this case we have that ∂M is represented by the boundary graph of (Γ, γ) and that M is orientable iff (Γ, γ) is bipartite.

A $(n+1)$-coloured graph with non-empty boundary (resp. without boundary) (Γ, γ) is *contracted* iff $\Gamma_{\hat{n}}$ is connected and, for each $i \in \Delta_{n-1}$, the number of connected components of $\Gamma_{\hat{\imath}}$ equals the number of connected components of $\partial\Gamma$ (resp. is 1).

A contracted $(n+1)$-coloured graph representing a n-manifold M is called a *crystallization* of M.

For a general survey on coloured graphs and crystallizations see [6].

A *(n-1)-pondered structure* is a triple $(\bar{\Gamma},\bar{\gamma},\omega)$ where $\bar{\Gamma}$ is an oriented pseudograph, regular of degree $2n$, $\bar{\gamma}: E(\bar{\Gamma}) \to \Delta_n$ is a map and $\omega : E(\bar{\Gamma}) \to \Delta_2$ is another map, called *weight on* $\bar{\Gamma}$, such that:

(1) for each $i \in \hat{1}$, the connected components of $\bar{\gamma}^{-1}(i)$ are elementary cycles (possibly of length one);

(2) for each $e \in \bar{\gamma}^{-1}(i), i \neq 1$, we have $\omega(e) = 1$;

(3) if e, f are adjacent 1-coloured (oriented) edges of $\bar{\Gamma}$, let us denote by $e(0), f(0)$ (resp. $e(1), f(1)$) the first (resp. the second) endpoints of e and f. Then, with regard to the weights of e and f, we have the following possibilities:

$$- \ e(1) = f(0) \Rightarrow \begin{cases} \omega(e) = \omega(f) = 1 \\ \omega(e) = 1 \quad \omega(f) = 0 \\ \omega(e) = 2 \quad \omega(f) = 0 \\ \omega(e) = 0 \quad \omega(f) = 2 \\ \omega(e) = 2 \quad \omega(f) = 1 \end{cases}$$

$$- \ e(1) = f(1) \Rightarrow \begin{cases} \omega(e) = 1 \quad \omega(f) = 0 \\ \omega(e) = 0 \quad \omega(f) = 2 \end{cases}$$

$$- \ e(0) = f(0) \Rightarrow \begin{cases} \omega(e) = 1 \quad \omega(f) = 2 \\ \omega(e) = 0 \quad \omega(f) = 2 \end{cases}.$$

Given a $(n-1)$-pondered structure $(\bar{\Gamma},\bar{\gamma},\omega)$, the *bijoin over* $(\bar{\Gamma},\bar{\gamma},\omega)$ is a $(n+1)$-coloured graph (without boundary) $B(\bar{\Gamma},\bar{\gamma},\omega) = (\Gamma,\gamma)$, constructed as follows:

1) $V(\Gamma) = V(\bar{\Gamma}) \times \{0,1\}$;

2) for each $a \in V(\bar{\Gamma})$ $(a,0)$ and $(a,1)$ are 0-adjacent in (Γ,γ);

3) for each $e \in E(\bar{\Gamma})$ $(e(0),h)$ and $(e(1),k)$ $(h,k \in \{0,1\})$ are $\bar{\gamma}(e)$-adjacent in (Γ,γ) iff $h \leqslant k$ and $h + k = \omega(e)$.

Remark 1. If $(\bar{\Gamma}',\bar{\gamma}',\omega')$ is a pondered structure obtained from $(\bar{\Gamma},\bar{\gamma},\omega)$ by inverting the orientations on some of the edges of $\omega^{-1}(\{0,2\})$, then the bijoins $(\Gamma',\gamma') = B(\bar{\Gamma}',\bar{\gamma}',\omega')$ and $(\Gamma,\gamma) = B(\bar{\Gamma},\bar{\gamma},\omega)$ are graph-isomorphic by a isomorphism which preserves the colours, i.e. there exists a graph-isomorphism $\Phi : \Gamma \to \Gamma'$ such that $a, b \in V(\Gamma)$ are i-adjacent in (Γ,γ) iff $\Phi(a)$ and $\Phi(b)$ are i-adjacent in (Γ',γ') (for each $i \in \Delta_n$) (see [2, Lemma 7]).

Hence, as far as we are only interested in the resulting $(n+1)$-coloured graph $B(\bar{\Gamma},\bar{\gamma},\omega)$, we can drop the orientations on the edges belonging to

$\omega^{-1}(\{0,2\})$, since they are not essential (up to graph-isomorphisms). Therefore in the following, whenever defining a pondered structure, we shall give the orientations only to the edges of weight 1.

The definitions of pondered structure and bijoin can be found in [2]; with respect to the original definitions, we have only changed the roles of the colours 0,1 and n because it turns out to be useful in our proofs.

Let (Γ,γ) be an $(n+1)$-coloured graph, we call *extended graph associated to* (Γ,γ) the $(n+1)$-coloured graph (Γ^*,γ^*) obtained in the following way:

- add to $V(\Gamma)$ a set V^* in bijective correspondence with the set of the boundary-vertices of (Γ,γ);
- add to $E(\Gamma)$ the set of all possible n-coloured edges having as endpoints a boundary-vertex of (Γ,γ) and its correspondent vertex in V^*.

Now we describe a particular type of imbeddings of a coloured graph into a surface ([8],[10]).

A *regular imbedding* of (Γ,γ) into a surface (with boundary) F, is a cellular imbedding of (Γ^*,γ^*) into F, such that:

(a) the image of a vertex of Γ^* lies in ∂F iff the vertex belongs to V^*;
(b) the boundary of any region of the imbedding is either the image of a cycle of (Γ^*,γ^*) (*internal region*) or the union of the image α of a path in (Γ^*,γ^*) and an arc of ∂F, the intersection consisting of the images of two vertices belonging to V^* (*boundary region*);
(c) there exists a cyclic permutation $\varepsilon=(\varepsilon_0,\dots,\varepsilon_{n-1},n)$ of Δ_n such that for each internal region (resp. boundary region), the edges of its boundary (resp. of α) are alternatively coloured ε_i and ε_{i+1}, $i\in\mathbb{Z}_{n+1}$.

Proposition 4 of [10] assures that for each cyclic permutation $\varepsilon=(\varepsilon_0,\dots,\varepsilon_{n-1},n)$ of Δ_n, there exists a regular imbedding of (Γ,γ) into a surface (with boundary) F_ε which is orientable iff (Γ,γ) is bipartite and is called the *regular surface* associated to (Γ,γ) and ε.

Let us define $\chi_\varepsilon(\Gamma)=\chi(F_\varepsilon)$ and $\rho_\varepsilon(\Gamma)=genus(F_\varepsilon)$.

Denote by $g_{ij}(\Gamma)$ (resp. ${}^{\partial}g_{ij}(\Gamma)$) the number of cycles of Γ_{ij} (resp. $\partial\Gamma_{ij}$) and by $p(\Gamma)$ (resp. $\bar{p}(\Gamma)$) the order of (Γ,γ) (resp. of $(\partial\Gamma,{}^{\partial}\gamma)$); set $\dot{p}(\Gamma)=p(\Gamma)-\bar{p}(\Gamma)$.

According to [10, Proposition 4], we have formulas to compute $\chi_\varepsilon(\Gamma)$ and $\rho_\varepsilon(\Gamma)$:

$$\chi_\varepsilon(\Gamma)=\sum_{i\in\mathbb{Z}_{n+1}} g_{\varepsilon_i\varepsilon_{i+1}}(\Gamma)+\frac{(1-n)\dot{p}(\Gamma)}{2}+\frac{(2-n)\bar{p}(\Gamma)}{2}$$

$$\rho_\varepsilon(\Gamma) = \begin{cases} 1 - \frac{1}{2}(\chi_\varepsilon(\Gamma) + {}^\partial g_{\varepsilon_0\varepsilon_{n-1}}(\Gamma)) & \text{if } \Gamma \text{ is bipartite} \\ 2 - \chi_\varepsilon(\Gamma) - {}^\partial g_{\varepsilon_0\varepsilon_{n-1}}(\Gamma) & \text{if } \Gamma \text{ is not bipartite.} \end{cases}$$

The *regular genus* $\rho(\Gamma)$ of (Γ, γ) is defined as the minimum $\rho_\varepsilon(\Gamma)$ among all cyclic permutations ε of Δ_n.

Following [9], we call *regular genus* of a n-manifold M the non-negative integer:

$$\mathcal{G}(M) = \min\{\rho(\Gamma) | (\Gamma, \gamma) \text{ represents } M\}.$$

Finally, we recall that, by [1, Theorem 1], given a 3-manifold with boundary M, there always exists a crystallization of M, whose genus is $\mathcal{G}(M)$. We shall use this result in our proofs.

3. Heegaard splittings and diagrams.

In the following we shall always denote by F_g the closed orientable (resp. non-orientable) surface of genus g (resp. of genus $2g$).

All the definitions of this section are extensions of those given for the orientable case in [14] and [5].

A *hollow handlebody of genus* g is a 3-manifold with boundary X_g, which is obtained from $F_g \times [0,1]$ by attaching 2- and 3-handles along $F_g \times \{1\}$. We call $F_g \times \{0\}$ the *free boundary* of X_g.

Note that X_g is orientable iff F_g is orientable.

We call X_g a *proper* handlebody of genus g iff $\partial X_g = F_g \times \{0\}$.

Lemma 1. *X_g is a proper handlebody iff it is a handlebody in the usual sense, i.e. it is obtained by attaching g 1-handles along $\partial \mathbb{D}^3$.*

The proof is the same as in [5, Remark 1], since it does not depend on the orientability of the surface F_g.

Given a 3-manifold with boundary M, a *generalized Heegaard splitting of genus g* of M is a pair (X_g, Y_g) of hollow handlebodies of genus g, such that $X_g \cup Y_g = M$ and $X_g \cap Y_g = F_g \times \{0\}$ is the free boundary of both.

If at least one of the two hollow handlebodies (say X_g) is proper, the pair (X_g, Y_g) is called a *proper Heegaard splitting of genus g* of M.

We define the *Heegaard genus* of a 3-manifold with boundary M as:

$$\mathcal{H}(M) = \min\{g | \text{ there exists a proper Heegaard splitting of genus } g \text{ of } M\}.$$

By Lemma 1, it is clear that this definition is a generalization of the classical one for closed 3-manifolds [12].

Clearly from the above definition arises the problem of the existence of proper Heegaard splittings. As in the case of closed manifolds we can prove that every 3-manifold M with boundary admits a proper Heegaard splitting, starting from a triangulation K of M. The argument is exactly the same as sketched in [5, Proposition 1] for the orientable case.

A *generalized Heegaard diagram* is a triple $(F_g; v, w)$ where v and w are systems of simple, closed, disjoint curves on F_g.

As in the orientable case (see [14] and [5]), given a generalized Heegaard splitting of genus g, (X_g, Y_g), we can consider the generalized Heegaard diagram $(F_g; v, w)$, where $v = \{v_1, \dots, v_r\}$ (resp. $w = \{w_1, \dots, w_s\}$) is formed by the attacching spheres of the 2-handles of X_g (resp. of Y_g).

Conversely, from a generalized Heegaard diagram $(F_g; v, w)$, we can obtain a hollow handlebody X (resp. Y) by considering $F_g \times [0,1]$ (resp. $F_g \times [-1,0]$) with 2-handles attached along $F_g \times \{1\}$ (resp. $F_g \times \{-1\}$) according to v (resp. to w) and possibly by capping off some of the resulting spherical boundary components by 3-handles. If M is the 3-manifold (with boundary) obtained from $X \cup Y$ by identifying their free boundaries, then (X, Y) is a generalized Heegaard splitting of the 3-manifold M. In this case we say that $(F_g; v, w)$ *represents* M.

A *proper Heegaard diagram* is a generalized Heegaard diagram whose corresponding splitting is proper.

Remark 2. If $(F_g; v = \{v_1, \dots, v_r\}, w = \{w_1, \dots, w_s\})$ is a proper Heegaard diagram representing a 3-manifold with boundary M, then $r \geqslant g$; moreover, we can always modify v in order to obtain a new proper Heegaard diagram $(F_g; v', w)$, representing M, such that $\#v' = g$ and v' is a complete system of meridian curves for F_g, i.e. $F_g \setminus v'$ is planar and connected.

Therefore, from now on, when considering a proper Heegaard diagram $(F_g; v, w)$, we can always suppose that v is a complete system of meridian curves for F_g.

Remark 3. Let $(F_g; v, w)$ be a proper Heegaard diagram of a 3-manifold with boundary M; by Lemma 1 and Remark 2, M can be constructed from $(F_g; v, w)$, in the following way:

- consider the proper (orientable or not, according to M) handlebody of genus g, X_g such that $\partial X_g = F_g$ and, for each $i = 1, \dots, g$, the curve v_i bounds a disk $\mathbb{D}_i$ in X_g such that $\{\mathbb{D}_1, \dots, \mathbb{D}_g\}$ is a complete system of meridian disks of X_g;

- for each $i = 1, \dots, s$ $(s = \# w)$, attach along ∂X_g a 2-handle according to w_i;
- attach some 3-handles in order to get the same number of spherical components as in ∂M.

Furthermore, if we cap off each boundary component of M, we obtain a singular 3-manifold $\hat{M}$ (i.e. each point x of $\hat{M}$ has a neighbourhood homeomorphic to a cone over a closed connected surface), which we shall call *associated to* M.

We briefly recall that, given a singular 3-manifold N, the *singular points* of N are those having neighbourhoods homeomorphic to cones over a surface which is not $\mathbb{S}^2$; note that, if we remove small open neighbourhoods of the singular points of N, we obtain a 3-manifold with boundary, which obviously has no spherical boundary components.

From now on we shall always consider only such 3-manifolds; in this way we make the correspondence between singular 3-manifolds and 3-manifolds bijective and we can properly define the 3-manifold with boundary *associated* to a given singular 3-manifold.

4. The relation between Heegaard and regular genus.

Throughout this section, in order to group together the orientable and non-orientable case, we shall use the following notations:

$$\ddot{\rho}_\varepsilon(\Gamma) = \begin{cases} \rho_\varepsilon(\Gamma) & \text{if } \Gamma \text{ is bipartite} \\ \frac{1}{2}\rho_\varepsilon(\Gamma) & \text{if } \Gamma \text{ is not bipartite.} \end{cases} \qquad \ddot{\rho}(\Gamma) = \min_{\varepsilon}\{\ddot{\rho}_\varepsilon(\Gamma)\}$$

$$\ddot{\mathcal{G}}(M) = \min\{\ddot{\rho}(\Gamma) | (\Gamma, \gamma) \text{ represents } M\} = \begin{cases} \mathcal{G}(M) & \text{if } M \text{ is orientable} \\ \frac{1}{2}\mathcal{G}(M) & \text{if } M \text{ is non-orientable.} \end{cases}$$

Proposition 1. *For every 3-manifold with boundary M, we have: $\ddot{\mathcal{G}}(M) = \mathcal{H}(M)$*

The proof is a consequence of some lemmas.

Lemma 2. *Given a crystallization (Γ, γ) of a 3-manifold with boundary M, for each cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, 3)$ of Δ_3, there exists a proper Heegaard splitting of M of genus $\ddot{\rho}_\varepsilon(\Gamma)$.*

The proof is hanalogous to that of the orientable case (see [5, Lemma 1]).

Before starting to complete the proof of Proposition 1, let us consider a Heegaard diagram $(F_g; v, w)$ of a 3-manifold with boundary M, such that $\#v = g$, $\#w = s$ and $w \neq \emptyset$. We can always consider a planar representation in $\mathbb{E}^2$ where each curve v_i corresponds to two circles v_i' and v_i'' such that, for each $i = 1, \dots, g$, v_i' (resp. v_i'') lies in the upper half-plane (resp. in the lower half-plane). Let us call v_0 the (closed) curve on F_g which corresponds to the x-axis in the planar representation; clearly $v_0 \cap v = \emptyset$, therefore $v \cup v_0$ is again a system of simple, closed, disjoint curves on F_g.

Note that we can always suppose that (see, for example, [14]):

- each w_i intersects at least once the system $v \cup v_0$ (otherwise we can use isotopies to push w_i through one v_j $(j \in \{0, \dots, g\})$);
- each connected component of $F_g \setminus (v \cup v_0 \cup w)$ is an open disk (if there is one having more than one boundary component, we can push a curve w_i belonging to one boundary component through a curve v_j belonging to another boundary component $(i \in \{1, \dots, s\}$, $j \in \{0, \dots, g\})$).

All the above hypothesis assure that the planar representation of $(F_g; v \cup v_0, w)$ is a connected subset of the plane.

Furthermore we can give to all the circles v_i' $(i = 1, \dots, g)$ the same orientation; note that, as a consequence, each v_i'' $(i = 1, \dots, g)$ has the opposite (resp. the same) orientation as v_i' iff v_i corresponds to an "orientable" (resp. "non-orientable") 1-handle of the proper handlebody X_g, in the proper Heegaard splitting associated to the diagram. Hence we can divide the v_i'''s in two disjoint "orientation-classes" $\mathcal{O}$ and $\mathcal{O}'$ such that $v_i'' \in \mathcal{O}$ (resp. $v_i'' \in \mathcal{O}'$) iff it has the orientation opposite to that of v_i' (resp. the same orientation as v_i').

Let us construct a pondered structure $(\bar{\Gamma}, \bar{\gamma}, \omega)$ in the following way:

(i) $V(\bar{\Gamma})$ is the set of intersection points of the curves of $v \cup v_0$ with the curves of w;

(ii) for each $a, b \in V(\bar{\Gamma})$, join a and b by an edge iff they are joined by an arc of $v \cup v_0 \cup w$;

(iii) colour 2 the edges corresponding to an arc of $v \cup v_0$; colour 1 (resp. 3) the edges corresponding to arcs of w whose interiors lie in the lower (resp. upper) half-plane;

(iv) give to each 2-coloured edge the same orientation of the corresponding arc of $v \cup v_0$ and weight 1;

(v) double (i.e. draw an arc with the same endpoints and the same colour) each edge e which corresponds to an arc in w; if e is 3-coloured give to it and its "double" weight 1; if e is 1-coloured, then:

- if both the edges 2-adjacent to e correspond to arcs of v, then give to e and its "double" weight 1 (resp. give to e weight 0 and to its "double" weight 2) iff these arcs belong to the same orientation-class (resp. to different orientation-classes) of the v_i'''s;
- if only one of the edges 2-adjacent to e corresponds to an arc of v_0, then give to e and its "double" weight 1 (resp. give to e weight 0 and to its "double" weight 2) iff the other arc belong to $\mathcal{O}$ (resp. to $\mathcal{O}'$);
- if both the 2-coloured edges adjacent to e correspond to arcs of v_0, then give to e and its "double" weight 1;

- for each $e \in E(\bar{\Gamma})$ of weight 1, with $\bar{\gamma}(e) \in \hat{2}$, give to e and its "double" opposite orientations.

Let (Γ, γ) be the 4-coloured graph which is the bijoin over the pondered structure $(\bar{\Gamma}, \bar{\gamma}, \omega)$.

Let us state some results about (Γ, γ) :

Lemma 3. *Given the cyclic permutation* $\varepsilon = (0,1,2,3)$, *we have:* $\ddot{\rho}_\varepsilon(\Gamma) = g$.

Proof. Let h be the number of intersection points between the system of curves $v \cup v_0$ and the system of curves w. Then we have that $V(\Gamma) = 2h$.

For each $i, j \in \Delta_3$ let us denote by g_{ij} the number of connected components of $\Gamma_{\hat{i}\hat{j}}$.

Note that the Heegaard diagram induces on F_g a decomposition where:

- the vertices are the intersection points between the curves of $v \cup v_0$ and those of w;
- the edges are the arcs of $v \cup v_0 \cup w$, joining two vertices;
- the 2-cells are the connected components of $F_g \setminus (v \cup v_0 \cup w)$, which are in bijective correspondence with the {1,2}- and {2,3}-residues of Γ. Therefore we have:

$$2 - 2g = \chi(F_g) = h - 2h + (g_{12} + g_{23})$$

Hence

$$g_{12} + g_{23} = h + 2 - 2g.$$

Moreover note that:

$$g_{03} = g_{01} = \frac{h}{2}$$

since each {1,2}-coloured (resp. {2,3}-coloured) edge of $K(\Gamma)$ belongs, by construction, to exactly four tethraedra. Therefore we have:

$$\begin{aligned}\chi_\varepsilon(\Gamma) &= g_{01} + g_{12} + g_{23} + g_{03} - 2h = \\ &= \frac{h}{2} + h + 2 - 2g + \frac{h}{2} - 2h = 2 - 2g\end{aligned}$$

□

Lemma 4. *$|K(\Gamma)|$ is a singular 3-manifold whose singular vertices are all 0-coloured.*

Proof. We shall proof the lemma by directly computing the Euler characteristics of the disjoint links of the vertices of $K(\Gamma)$. For each $i \in \Delta_3$, let us denote by $g_{\hat{\imath}}$ the number of connected components of $\Gamma_{\hat{\imath}}$. Note that $g_{\hat{2}} = s$, i.e. the number of curves of the system w. Therefore, for each $i = 1, \dots, s$, let h_i be the number of intersection points of the curve $w_i \in w$ with the curves of the system $v \cup v_0$, $\Xi^{(i)}$ the connected component of $\Gamma_{\hat{2}}$ corresponding to w_i and $g_{hk}^{(i)}$ the number of connected components of the graph $\Xi_{hk}^{(i)}$. Then, for each $i = 1, \dots, s$, we have:

$$\chi(\Xi^{(i)}) = 2h_i - \frac{6h_i}{2} + (g_{01}^{(i)} + g_{03}^{(i)} + g_{13}^{(i)}) = -h_i + 2 + \frac{2h_i}{2} = 2$$

Since the planar representation of $(F_g; v \cup v_0, w)$ is a connected subset of the plane, we have:

$$g_{\hat{1}} = g_{\hat{3}} = 1$$

therefore:

$$\chi(\Gamma_{\hat{1}}) = 2h - 3h + (g_{02} + g_{23} + g_{03}) = -h + g + 1 + g_{23} + \frac{h}{2} = -\frac{h}{2} + g + 1 + g_{23}$$

$$\chi(\Gamma_{\hat{3}}) = 2h - 3h + (g_{01} + g_{12} + g_{02}) = -h + \frac{h}{2} + g_{12} + g + 1 = -\frac{h}{2} + g + 1 + g_{12}$$

and finally

$$\chi(\Gamma_{\hat{1}}) + \chi(\Gamma_{\hat{3}}) = -h + 2g + 2 + (g_{23} + g_{12}) = 4.$$

Since $\chi(\Gamma_{\hat{\imath}}) \leqslant 2$, for each $i = 1, 3$, it follows that:

$$\chi(\Gamma_{\hat{1}}) = \chi(\Gamma_{\hat{3}}) = 2.$$

□

Lemma 5. $|K(\Gamma)| \cong \hat{M}$, *where* $\hat{M}$ *is the singular 3-manifold associated to* M.

Proof. First note that, given a Heegaard diagram $(F_g; v, w)$ of a 3-manifold with boundary M, the singular 3-manifold $\hat{M}$ is obtained in the following way:

- let X_g be the handlebody of genus g such that $\partial X_g = F_g$ and $v_i = \partial \mathbb{D}_i$ $(i = 1, \dots, g)$, $\{\mathbb{D}_1, \dots, \mathbb{D}_g\}$ being a complete system of meridian disks for X_g;
- consider $N = X_g \cup H_1^{(2)} \cup \dots \cup H_s^{(2)}$ the 3-manifold with boundary obtained by adding, along X_g, the 2-handles $H_i^{(2)}$, whose attaching spheres are the curves w_i $(i = 1, \dots, s)$ (note that M can be constructed by adding some 3-handles on ∂N, i.e. by capping off all the spherical boundary components of N);
- finally $\hat{M}$ is obtained by capping off by a cone each component (spherical or not) of ∂N.

Let K_{13} and K_{02} be the 1-dimensional subcomplexes of $K(\Gamma)$ generated by the {1,3}- and {0,2}-coloured vertices respectively, and let H be the largest 2-dimensional subcomplex of the first barycentric subdivision of $K(\Gamma)$ disjoint from the first barycentric subdividisions of K_{13} and K_{02}. The surface $F = |H|$ splits $|K(\Gamma)|$ in two polyhedra $\mathcal{A}_{13}$ and $\mathcal{A}_{02}$ such that $\mathcal{A}_{13} \cap \mathcal{A}_{02} = F$ (see Lemma 2 above).

Let us show, now, that $\mathcal{A}_{13}$ is a proper (orientable or non-orientable) handlebody of genus $g = g_{02} - 1$.

In fact, we have the graph Γ imbedded in $F \subset |K(\Gamma)|$ and this imbedding is regular with respect to the cyclic permutation $\varepsilon = (0, 1, 2, 3)$. By Lemma 3, the genus of F is g or $2g$ according to its orientability, i.e. to Γ being bipartite or not.

Moreover we can think of $\mathcal{A}_{13}$ as constructed in the following way:

- consider a collar C of F in $\mathcal{A}_{13}$ and let C_1 be the component of ∂C corresponding to $F \times \{1\}$;
- add on C_1 the 2-handles $H_1^{(2)}, \dots, H_g^{(2)}$ whose attaching spheres are the {0,2}-residues of Γ except that corresponding to v_0.

We have now obtained a hollow handlebody of genus g with exactly two boundary components represented by the residues $\Gamma_{\hat{1}}$ and $\Gamma_{\hat{3}}$, which, by Lemma 4, are spheres; hence by adding two 3-handles we obtain a proper handlebody of genus g which is exactly $\mathcal{A}_{13}$.

Moreover, since the meridian curves of $\mathcal{A}_{13}$ are exactly the $\{0,2\}$-residues corresponding to $\{v_1, \dots, v_g\}$, then $\mathcal{A}_{13} = X_g$.

Let $\mathcal{S}$ be the set formed by the 0- and 2-coloured vertices of $K(\Gamma)$; let us consider the subcomplex of $K(\Gamma)$:

$$\tilde{K} = K(\Gamma) \setminus \Big(\bigcup_{a \in \mathcal{S}} \dot{s}td(a, K(\Gamma)) \Big)$$

where $\dot{s}td(a, K(\Gamma)) = std(a, K(\Gamma)) \setminus lkd(a, K(\Gamma))$.

Note that $\tilde{K}$ is obtained by adding on $\partial\mathcal{A}_{13} = \partial X_g$ the 2-handles:

$$\bar{H}_1^{(2)}, \bar{\bar{H}}_1^{(2)}, \dots, \bar{H}_s^{(2)}, \bar{\bar{H}}_s^{(2)}$$

such that the attacching spheres of $\bar{H}_i^{(2)}$ and $\bar{\bar{H}}_i^{(2)}$ are the two "parallel" $\{1,3\}$-residues corresponding to the same curve $w_i \in w$ $(i = 1, \dots, s)$.

Moreover, note that the 2-handles $\bar{H}_i^{(2)}$ and $\bar{\bar{H}}_i^{(2)}$ are attached to each other in such a way that, denoting by ϕ_i the "glueing map", for each $i = 1, \dots, s$, $\tilde{H}_i^{(2)} = \bar{H}_i^{(2)} \cup_{\phi_i} \bar{\bar{H}}_i^{(2)}$ is again a 2-handle. Therefore $|\tilde{K}| \cong X_g \cup \tilde{H}_1^{(2)} \cup \dots \cup \tilde{H}_s^{(2)}$ and we can obviously consider as attaching sphere for $\tilde{H}_i^{(2)}$ one of the two $\{1,3\}$- residues corresponding to w_i.

Hence $|\tilde{K}| \cong N$ and the components of ∂N are the disjoint links of the vertices of $\mathcal{S}$ (note that, if the vertex is 2-coloured then its disjoint link is a sphere by Lemma 4). $K(\Gamma)$ is therefore obtained from $\tilde{K}$ by attaching on $\partial\tilde{K}$ the disjoint stars (in $K(\Gamma)$) of the vertices belonging to $\mathcal{S}$, identifying the two copies of the disjoint link. Since the disjoint star is exactly the cone over the disjoint link, we have that $|K(\Gamma)| \cong \hat{M}$. □

Lemma 6. *Let (Γ, γ) be a 4-coloured graph without boundary representing a singular 3-manifold N and suppose that all the singular vertices of $K(\Gamma)$ are 0-coloured. There exists a 4-coloured graph with boundary $(\tilde{\Gamma}, \tilde{\gamma})$, which represents the 3-manifold with boundary associated to N, such that:*

$$\ddot{\rho}_\varepsilon(\tilde{\Gamma}) = \ddot{\rho}_\varepsilon(\Gamma) \quad with \quad \varepsilon = (0,1,2,3)$$

The proof of the above lemma is the same as in [5, Lemma 3], since it does not depend on the orientability of N.

Proof of Proposition 1. One inequality is the direct consequence of Lemma 2 and Theorem 1 of [1].

Suppose now that $(F_g; v, w)$ is a Heegaard diagram of M such that $\mathcal{H}(M) = g$.

If M is not a proper handlebody (i.e. $w \neq \emptyset$) then we can apply the construction described above to obtain a 4-coloured graph (Γ, γ); by applying Lemmas 3,4,5 and 6 to (Γ, γ) we obtain the required inequality.

If M is a proper handlebody of genus g then $\ddot{\mathcal{G}}(M) = g$ (see [10]).

Since $g = rank(M) \leqslant \mathcal{H}(M)$ again we have the required inequality. □

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