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SKEW Ω-SETS COINCIDE WITH Ω-POSETS

by Francis BORCEUX and Rosanna CRUCIANI*

Résumé

Lorsque $\mathcal{Q}$ est un quantale non commutatif, les $\mathcal{Q}$-ensembles correspondants sont munis d'une égalité nòn symétrique à valeurs dans $\mathcal{Q}$. Dans le cas classique d'un locale Ω, nous prouvons que la catégorie des Ω-ensembles non symétriques est équivalente à la catégorie des Ω-faisceaux ordonnés.

Mots clés français: Ω-ensemble, faisceau, topos.
Mots clés anglais: Ω-sets, sheaf, topos.
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Introduction

Every set A is provided with an equality and a membership relation: given elements a, b of A, the formula $a = b$ takes the truth value *true* or *false* according to the case; analogously, the formula $a \in A$ takes the truth value *true* exactly for all elements a of A.

Logicians have first replaced the truth values *true*, *false* by a (complete) boolean algebra of truth values (see [4]), and more generally by a locale of truth values (see [5]). The corresponding logics are intuitionistic.

Given a locale Ω, an Ω-set is then defined as a set A provided with an "equality"

$$A \times A \longrightarrow \Omega; \quad (a, a') \mapsto [a = a']$$

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where $[a = a'] \in \Omega$ is interpreted as the "truth value" of the formula $a = a'$, and $[a = a]$ is interpreted as the truth value of the formula $a \in A$. Such an "equality" is requested to satisfy the "symmetry" and "transitivity" axioms

$$\begin{aligned} [a = a'] &= [a' = a] \\ [a = a'] \wedge [a' = a''] &\leq [a = a''] \end{aligned}$$

where a, a', a'' are elements of A. In the same spirit, a morphism $f\colon A \longrightarrow B$ is a mapping

$$A \times B \longrightarrow \Omega; \quad (a, b) \mapsto \big[f(a) = b\big]$$

where $\big[f(a) = b\big] \in \Omega$ is interpreted as the "truth value" of the formula $f(a) = b$ and the axioms express the functional requirements on f (see [3], section 2.8).

The locale Ω itself, provided with the multiplication $[u = v] = u \wedge v$, is the terminal Ω-set. More generally, every sheaf F on Ω determines an Ω-set: the corresponding Ω-set A is the disjoint union $\amalg_{u\in\Omega} F(u)$ of all "elements" of F at all levels and $[a = a']$ is the biggest level $u \in \Omega$ where the restrictions of a and a' concidc. In particular if $a \in F(u)$, one gets $u = [a = a]$ as truth value of the formula $a \in A$. In fact, it is well known that the category of Ω-sets is equivalent to the category of sheaves on Ω (see [5]).

Those last years, multiplicative lattices have been intensively studied from a logical point of view, under the name *quantales* (see [6] and [7]). This is a complete lattice provided with an associative multiplication which distributes over joins in each variable. Locales are just the case where multiplication is the meet operation. Canonical examples are given by lattices of ideals of rings or algebras, with the multiplication of the quantale induced by that of ideals.

The theory of Ω-sets can be carried over to the case of a quantale $\mathcal{Q}$, replacing everywhere the meet operation $\wedge$ by the multiplication $\&$ of the quantale (see [1] and [2]). Since the applications of quantale theory are essentially found in the study of non commutative rings or algebras, the multiplication $\&$ is generally not commutative. If we want the quantale $\mathcal{Q}$, provided with the equality $[u = v] = u \,\&\, v$, to be a canonical $\mathcal{Q}$-set, we are forced to omit the symmetry axiom $[a = a'] = [a' = a]$ in the definition of $\mathcal{Q}$-sets. All authors studying non commutative quantales agree on this point.

It is amazing that up to now, nobody had tried to investigate what those non-symmetric $\mathcal{Q}$-sets are, in the classical case of a locale Ω. The purpose of this paper is to prove that the category of non symmetric Ω-sets is equivalent to the category of partially ordered Ω-sheaves, that is, of posets in the category of Ω-sheaves. For that reason, we suggest that the notation $[a \leq a']$ should rather be used to denote the non-symmetric "equality" $[a = a']$ of a non-symmetric $\mathcal{Q}$-set over a quantale $\mathcal{Q}$.

1 Ω-sets

Let Ω be a fixed locale. We introduce first some definitions and notation; the reader is invited to consult [3], section 2.8, for more details on the intuition underlying those definitions. To avoid any ambiguity, we use the terminology "symmetric Ω-set" and "skew Ω-set" to distinguish the symmetric case from the non-symmetric one; we use the notation Ω_{sy}-Set and Ω_{sk}-Set to indicate the corresponding categories.

First of all the symmetric Ω-sets, which are just called Ω-sets in [3].

Definition 1.1. *Let Ω be a locale. A symmetric Ω-set is defined as a pair $\big(A, [\cdot = \cdot]\big)$ where A is a set and $[\cdot = \cdot]\colon A \times A \longrightarrow \Omega$ is a mapping satisfying the following axioms:*

(s1) $[a = a'] = [a' = a]$,

(s2) $[a = a'] \wedge [a' = a''] \leq [a = a'']$,

where a, a', a'' are elements of A.

Definition 1.2. *Let Ω be a locale and A, B two symmetric Ω-sets. A morphism of Ω-sets $f\colon A \longrightarrow B$ is a mapping $[f\cdot = \cdot]\colon A \times B \longrightarrow \Omega$ satisfying*

(m1) $[a' = a] \wedge [fa = b] \leq [fa' = b]$,

(m2) $[fa = b] \wedge [b = b'] \leq [fa = b']$,

(m3) $[fa = b] \wedge [fa = b'] \leq [b = b']$,

(m4) $\bigvee_{b \in B}[fa = b] = [a = a]$,

for all elements a, a' *in* A *and* b, b' *in* B.

The symmetric Ω-sets and their morphisms constitute a category, which we denote Ω_{sy}-Set. This category is extensively studied in [3], section 2.8; in particular the following result holds.

Lemma 1.3. *Let* Ω *be a locale and* $f: A \longrightarrow B$ *a morphism of symmetric* Ω*-sets. The following relations hold:*

1. $[a = a'] \leq [a = a]$,
2. $[a = a'] \leq [a' = a']$,
3. $\bigvee_{a \in A}[a' = a] \wedge [fa = b] = [fa' = b]$,
4. $\bigvee_{b \in B}[fa = b] \wedge [b = b'] = [fa = b']$,
5. $[a = a'] \leq \bigvee_{b \in B}[fa = b] \wedge [fa' = b]$,

for all elements a, a' *of* A *and* b, b' *of* B. ∎

Now the case of skew Ω-sets.

Definition 1.4. *Let* Ω *be a locale. A skew* Ω*-set is a pair* $\big(A, [\cdot = \cdot]\big)$ *where* A *is a set and* $[\cdot = \cdot]: A \times A \longrightarrow \Omega$ *is a mapping satisfying the following axioms:*

(S1) $[a = a'] \leq [a = a]$,

(S2) $[a = a'] \leq [a' = a']$,

(S3) $[a = a'] \wedge [a' = a''] \leq [a = a'']$,

for all elements a, a', a'' *of* A.

Following the intuition recalled in the introduction, $[a = a]$ should be thought as $[a \in A]$. The following proposition follows at once from definition 1.4.

Lemma 1.5. *Let Ω be a locale and A a skew Ω-set. The following relations hold:*

1. $\bigvee_{a\in A}[a = a'] = [a' = a'] = \bigvee_{a\in A}[a' = a]$,
2. $\bigvee_{a'\in A}[a = a'] \wedge [a' = a''] = [a = a'']$,

for all elements a, a', a'' of A. ■

The unsymmetry of the equality, together with lemma 1.3, makes sensible the following definition of morphisms.

Definition 1.6. *Let Ω be a locale and A, B skew Ω-sets. A morphism of skew Ω-sets $f\colon A \longrightarrow B$ is a pair of mappings*

$$[f\cdot = \cdot]\colon A \times B \longrightarrow \Omega \quad ; \quad (a,b) \mapsto [fa = b]$$
$$[\cdot = f\cdot]\colon B \times A \longrightarrow \Omega \quad ; \quad (b,a) \mapsto [b = fa]$$

satisfying

(M1) $\bigvee_{a\in A}[a' = a] \wedge [fa = b] = [fa' = b]$,

(M2) $\bigvee_{a\in A}[b = fa] \wedge [a = a'] = [b = fa']$,

(M3) $\bigvee_{b\in B}[fa = b] \wedge [b = b'] = [fa = b']$,

(M4) $\bigvee_{b\in B}[b' = b] \wedge [b = fa] = [b' = fa]$,

(M5) $[b = fa] \wedge [fa = b'] \leq [b = b']$,

(M6) $[a = a'] \leq \bigvee_{b\in B}[fa = b] \wedge [b = fa']$,

for all elements a, a' in A and b, b' in B.

Straightforward computations, using axioms (M1)–(M6), yield the the following result:

Lemma 1.7. *Let Ω be a locale and $f\colon A \longrightarrow B$ a morphism of skew Ω-sets. The following relations hold:*

1. $\bigvee_{b\in B}[fa = b] = [a = a] = \bigvee_{b\in B}[b = fa]$,
2. $[fa = b] \leq [b = b]$,

3. $[b = fa] \leq [b = b]$,

4. $\bigvee_{a \in A}[a = a] \leq \bigvee_{b \in B}[b = b]$,

for all elements a in A and b in B. ■

We are now ready to define the composite of two morphisms of skew Ω-sets; straightforward computations yield the proof of the following proposition.

Proposition 1.8. *Let Ω be a locale. Given two morphims of skew Ω-sets $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$, the formulae*

$$\Big[(g \circ f)a = c\Big] = \bigvee_{b \in B} [fa = b] \wedge [gb = c],$$

$$\Big[c = (g \circ f)a\Big] = \bigvee_{b \in B} [c = gb] \wedge [b = fa],$$

for all $a \in A$, $c \in C$, define a composite morphism of skew Ω-sets $g \circ f\colon A \longrightarrow C$. With this composition law, the skew Ω-sets and their morphisms become a category where the identity on a skew Ω-set A is given by $[1_A a = a'] = [a = a'] = [a = 1_A a']$, for all $a, a' \in A$. ■

We write Ω_{sk}-Set for the category of skew Ω-sets and their morphisms. Proposition 1.3 indicates at once that, putting $[b = fa] = [fa = b]$ in definition 1.2, the category Ω_{sy}-Set of symmetric Ω-sets is a subcategory of that Ω_{sk}-Set of skew Ω-sets. Our purpose in this section is to prove that Ω_{sy}-Set is in fact a full coreflective subcategoy of Ω_{sk}-Set.

Lemma 1.9. *Let Ω be a locale. Givezn two morphims of skew Ω-sets $f, g\colon A \longrightarrow B$, the following equivalence holds*

$$[fa = b] \leq [ga = b] \iff [b = ga] \leq [b = fa]$$

for all elements a in A and b in B.

Proof: For all a in A and b in B,

$$[b = ga] \quad = \quad \bigvee_{a' \in A} [b = ga'] \wedge [a' = a]$$

$$\begin{aligned}
&\leq \bigvee_{a'\in A}\bigvee_{b'\in B}[b=ga']\wedge[fa'=b']\wedge[b'=fa]\\
&\leq \bigvee_{a'\in A}\bigvee_{b'\in B}[b=ga']\wedge[ga'=b']\wedge[b'=fa]\\
&\leq \bigvee_{b'\in B}[b=b']\wedge[b'=fa]\\
&= [b=fa]
\end{aligned}$$

which proves one implication; the other one is analogous. ■

Corollary 1.10. *Let Ω be a locale. Given two morphims of skew Ω-sets $f, g: A \longrightarrow B$, the relations $[fa=b] \leq [ga=b]$ and $[b=fa] \leq [b=ga]$, for all a in A and b in B, imply $f=g$.* ■

Proposition 1.11. *Let Ω be a locale. The category Ω_{sy}-Set of symmetric Ω-sets is a full subcategory of the category Ω_{sk}-Set of skew Ω-sets.*

Proof: By lemma 1.3, every symmetric Ω-set is a skew Ω-set and putting $[b=fa]=[fa=b]$ in definition 1.2 yields the inclusion of Ω_{sy}-Set in Ω_{sk}-Set.

To prove the fullness of the inclusion, consider symmetric Ω-sets A and B, together with a morphism $f: A \longrightarrow B$ of skew Ω-sets. We must prove that $[fa=b]=[b=fa]$ for all elements a in A and b in B. By proposition 1.7

$$\begin{aligned}
[fa=b]\wedge[fa=b'] &= [a=a]\wedge[fa=b]\wedge[fa=b']\\
&= \bigvee_{b''\in B}[b''=fa]\wedge[fa=b]\wedge[fa=b']\\
&= \bigvee_{b''\in B}[b''=fa]\wedge[fa=b]\wedge[b''=fa]\wedge[fa=b']\\
&\leq \bigvee_{b''\in B}[b''=b]\wedge[b''=b']\\
&= [b=b']
\end{aligned}$$

for all a in A and b in B. The corresponding relation for $[b=fa]$ is proved in the same way.

This yields the inequalities

$$[fa=b] \;=\; [a=a]\wedge[fa=b]$$

$$\begin{aligned} &\leq \bigvee_{b' \in B} [fa = b'] \wedge [b' = fa] \wedge [fa = b] \\ &\leq \bigvee_{b' \in B} [b = b'] \wedge [b' = fa] \\ &= [b = fa] \end{aligned}$$

for all a in A and b in B. Analogously $[b = fa] \leq [fa = b]$, from which the result. ∎

Theorem 1.12. *Let Ω be a locale. The category Ω_{sy}-*Set *of symmetric Ω-sets is a full coreflective subcategory of the category Ω_{sk}-*Set *of skew Ω-sets.*

Proof: Given an Ω-set $A = \big(A, [\cdot = \cdot]\big)$, the formula

$$[[a = a']] = [a = a'] \wedge [a' = a]$$

for all a, a' in A defines clearly a symmetric Ω-set $A_{sy} = \big(A, [[\cdot = \cdot]]\big)$. Moreover putting

$$[ha = a'] = [a = a'], \qquad [a' = ha] = [a' = a]$$

for all a in A and a' in A_{sy} yields immediately a morphism $h\colon A_{sy} \longrightarrow A$. Let us prove that the pair (A_{sy}, h) is the coreflection of A in Ω_{sy}-Set.

For this consider a symmetric Ω-set B and a morphism of skew Ω-sets $f\colon B \longrightarrow A$. The relation

$$[[f_{sy} b = a]] = [fb = a] \wedge [a = fb]$$

for all a in A and b in B, defines a morphism $f_{sy}\colon B \longrightarrow A_{sy}$ of symmetric Ω-sets. One gets at once

$$[hf_{sy} b = a] = \bigvee_{a' \in A} [fb = a'] \wedge [a' = fb] \wedge [a' = a] \leq [fb = a],$$

$$[a = hf_{sy} b] = \bigvee_{a' \in A} [a = a'] \wedge [fb = a'] \wedge [a' = fb] \leq [a = fb].$$

Corollary 1.10 implies then $h \circ f_{sy} = f$.

To prove the uniqueness of f_{sy}, consider a morphism $g\colon B \longrightarrow A_{sy}$ of symmetric Ω-sets such that $h \circ g = f$, that is

$$\bigvee_{a' \in A} [gb = a'] \wedge [a' = a] = [fb = a] \ ; \ \bigvee_{a' \in A} [a = a'] \wedge [gb = a'] = [a = fb]$$

for all a in A and b in B. In those conditions

$$\begin{aligned}
[[f_{sy}b = a]] &= [fb = a] \wedge [a = fb] \\
&= \bigvee_{a' \in A} [gb = a'] \wedge [a' = a] \wedge \bigvee_{a'' \in A} [a = a''] \wedge [gb = a''] \\
&\geq \bigvee_{a' \in A} [gb = a'] \wedge [a' = a] \wedge [a = a'] \\
&= [gb = a]
\end{aligned}$$

for all a in A and b in B. By symmetry and corollary 1.10, we conclude that $f_{sy} = g$. ∎

2 Complete Ω-sets

The purpose of this section is to generalize, to the case of skew Ω-sets, the theory of complete Ω-sets. It is proved in [3], section 2.9, that the category Ω_{sy}^{c}-Set of complete symmetric Ω-sets is equivalent to the category Ω_{sy}-Set of symmetric Ω-sets. Analogously, we prove that the category Ω_{sk}^{c}-Set of complete skew Ω-sets is equivalent to the category Ω_{sk}-Set of skew Ω-sets.

Definition 2.1. *Let Ω be a locale and A a skew Ω-set. A singleton s of A is a pair of mappings*

$$[s = \cdot]\colon A \longrightarrow \Omega, \qquad [\cdot = s]\colon A \longrightarrow \Omega$$

satisfying the axioms

(σ1) $\bigvee_{a \in A}[s = a] \wedge [a = a'] = [s = a']$

(σ2) $\bigvee_{a \in A}[a' = a] \wedge [a = s] = [a' = s]$

(σ3) $[a' = s] \wedge [s = a] \leq [a' = a]$

(σ4) $\bigvee_{a \in A}[s = a] = \bigvee_{a \in A}\big([s = a] \wedge [a = s]\big) = \bigvee_{a \in A}[a = s]$

for all elements a, a' of A. The quantity in ($\sigma 4$) is called the "support" of the singleton.

Obviously, for each a in A, the pair $\big([a = \cdot], [\cdot = a]\big)$ is a singleton of A.

Definition 2.2. *Let Ω be a locale. A skew Ω-set A is complete when every singleton of A has the form $\big([a = \cdot], [\cdot = a]\big)$ for a unique element $a \in A$.*

Given a locale Ω and an element $u \in \Omega$, let us write $\mathbf{1}_u$ for the symmetric Ω-set obtained by providing the singleton $\{*\}$ with the Ω-equality $[* = *] = u$. Comparing definitions 1.6 and 2.1, we get at once the following result.

Proposition 2.3. *Let Ω be a locale and A a skew Ω-set. There is a bijection between*

1. *the singletons of A with support u,*

2. *the morphisms of skew Ω-sets $\mathbf{1}_u \longrightarrow A$,*

for every element $u \in \Omega$. ∎

Together with proposition 1.11, this result indicates in particular that, in the case of symmetric Ω-sets, definition 2.1 agrees with the classical definition of a singleton (see [3]). In particular, a symmetric Ω-set is complete in the sense of our definition 2.2 if and only if it is complete in the classical sense.

Lemma 2.4. *Consider a skew Ω-set A and a singleton s of A. The following relations hold, where $\widehat{A}$ indicates the set of singletons of A:*

1. $[s = a] \leq [a = a]$,

2. $[a = s] \leq [a = a]$,

3. $\bigvee_{s \in \widehat{A}} [a = s] \wedge [s = a'] = [a = a']$

for all elements a, a' in A.

Proof: The two inequalities follow from $(\sigma 1)$ and $(\sigma 2)$ and the first two equalities in 1.5. The equality in the statement follows from $(\sigma 3)$ and the fact that every element a in A determines canonically a singleton of A. ∎

Proposition 2.5. *Let Ω be a locale and A, B two skew Ω-sets, with B complete. There exists a bijection between:*

1. *the morphisms of skew Ω-sets $f\colon A \longrightarrow B$;*
2. *the actual mappings $\varphi\colon A \longrightarrow B$ satisfying the two conditions*

$$[a=a]=[\varphi a=\varphi a], \qquad [a=a'] \leq [\varphi a=\varphi a']$$

for all elements a, a' in A.

The category Ω^c_{sk}-Set of complete skew Ω-sets and their morphisms as in 1.6 is equivalent to the category of complete skew Ω-sets and those mappings as in condition 2 of the statement, the composition of these mappings being the usual one.

Proof: Given f and an element a in A, $\big([fa=\cdot],[\cdot=fa]\big)$ is a singleton on B, thus is represented by a unique element of B which we choose as $\varphi(a)$. Conversely given φ, one defines f via the assignments

$$A \times B \longrightarrow \Omega \;\; ; \;\; (a,b) \mapsto [\varphi a = b]$$

$$B \times A \longrightarrow \Omega \;\; ; \;\; (b,a) \mapsto [b = \varphi a]$$

for all elements a in A and b in B. ∎

Theorem 2.6. *Let Ω be a locale. Every skew Ω-set is isomorphic to a complete skew Ω-set. Thus the category Ω_{sk}-Set of skew Ω-sets is equivalent to its full subcategory Ω^c_{sk}-Set of complete skew Ω-sets.*

Proof: For a given Ω-set A, the set $\widehat{A}$ of singletons of A is itself a skew Ω-set when provided with the equality

$$[s=s'] = \bigvee_{a\in A} [s=a] \wedge [a=s']$$

for all singletons s, s' of A. Indeed, by ($\sigma 4$), $\widehat{A}$ satisfies (S1) and (S2); moreover, for all $s, s', s'' \in \widehat{A}$, one has

$$\begin{aligned}
[s=s'] \wedge [s'=s''] &= \bigvee_{a\in A} [s=a] \wedge [a=s'] \wedge \bigvee_{a'\in A} [s'=a'] \wedge [a'=s''] \\
&\leq \bigvee_{a\in A} \bigvee_{a'\in A} [s=a] \wedge [a=a'] \wedge [a'=s''] \\
&= \bigvee_{a\in A} [s=a] \wedge [a=s''] \\
&= [s=s''],
\end{aligned}$$

from which axiom (S3) holds in $\widehat{A}$.

To prove that the skew Ω-set $\widehat{A}$ is complete, choose a singleton σ of $\widehat{A}$. For an element a in A, put

$$[s = a] = \bigvee_{s' \in \widehat{A}} \big([\sigma = s'] \wedge [s' = a]\big), \qquad [a = s] = \bigvee_{s' \in \widehat{A}} \big([a = s'] \wedge [s' = \sigma]\big).$$

It is routine to check that s verifies (σ1), (σ2), (σ3); let us prove it satisfies (σ4) as well, thus is a singleton of A.

$$\begin{aligned}
\bigvee_{a \in A} [s = a] &\leq \bigvee_{s' \in \widehat{A}} [\sigma = s'] \\
&= \bigvee_{s' \in \widehat{A}} [\sigma = s'] \wedge [s' = \sigma] \wedge [s' = s'] \\
&= \bigvee_{s' \in \widehat{A}} [\sigma = s'] \wedge [s' = \sigma] \wedge \bigvee_{a \in A} [s' = a] \wedge [a = s'] \\
&\leq \bigvee_{a \in A} \bigvee_{s' \in \widehat{A}} [\sigma = s'] \wedge [s' = a] \wedge \bigvee_{s'' \in \widehat{A}} [s'' = \sigma] \wedge [a = s''] \\
&= \bigvee_{a \in A} [s = a] \wedge [a = s].
\end{aligned}$$

and analogously for $[a = s]$. It follows at once that s is the unique singleton of A which represents the singleton σ of $\widehat{A}$.

It is now routine to observe that, for elements a in A and s in $\widehat{A}$, the relations

$$\begin{aligned}
[fa = s] &= [a = s] \qquad & [s = fa] &= [s = a] \\
[gs = a] &= [s = a] \qquad & [a = gs] &= [a = s]
\end{aligned}$$

define two inverse isomorphisms $f\colon A \longrightarrow \widehat{A}$, $g\colon \widehat{A} \longrightarrow A$ in the category of skew Ω-sets. ■

When A is a symmetric Ω-set, the Ω-set $\widehat{A}$ of singletons of A is itself a symmetric Ω-set and, via proposition 2.3, is exactly the Ω-set of singletons of A in the classical sense (see [3]).

3 The Ω-posets

By an Ω-poset we mean a poset object in the category of sheaves on Ω. To show the relation with skew Ω-sets, we need first the following lemma.

Lemma 3.1. *Let Ω be a locale and A a skew Ω-set. Composing with the coreflection morphism $h: A_{sy} \longrightarrow A$ of theorem 1.12 yields a bijection between the singletons of A_{sy} and those of A, when a singleton of support $u \in \Omega$ is viewed via 2.3 as a morphism with domain $\mathbf{1}_u$.*

Proof: By the coreflection property, since $\mathbf{1}_u$ is a symmetric Ω-set. ■

In particular, when A is a complete skew Ω-set, the symmetric Ω-set A_{sy} of theorem 1.12 is still complete, thus the data

$$A(u) = \Big\{a \in A \Big| [a = a] = u\Big\} = \Big\{a \in A_{sy} \Big| [[a = a]] = u\Big\}$$

define a sheaf on Ω, by a well-known result in the classical theory of Ω-sets (see [3], section 2.9). We shall keep writing this sheaf A.

Let us also recall that the subobject classifier in the category of sheaves on the locale Ω, still written Ω, is given by

$$\Omega(u) = \{v \mid v \leq u\}$$

for all elements $u \in \Omega$ (see [3]).

Proposition 3.2. *Let Ω be a locale and A a complete Ω-set. The data*

$$A(u) \times A(u) \longrightarrow \Omega(u), \qquad (a, a') \mapsto [a = a']$$

for all $u \in \Omega$, define a morphism of sheaves $A \longrightarrow \Omega$, which is the characteristic morphism of a reflexive, transitive and antisymmetric relation on the sheaf A.

Proof: Consider $v \leq u$ in Ω and write $a|_v$, $a'|_v$ for the restrictions of a, a' in $A(v)$. Viewing a, a' as singletons of A, this means

$$[a = x] \wedge v = \Big[a|_v = x\Big], \qquad \Big[x = a\Big] \wedge v = [x = a|_v],$$

$$[a' = x] \wedge v = \Big[a'|_v = x\Big], \qquad [x = a'] \wedge v = \Big[x = a'|_v\Big],$$

for all elements x of A. This implies immediately

$$\Big[a|_v = a'|_v\Big] = \Big[a = a'|_v\Big] \wedge v = [a = a'] \wedge v \wedge v = [a = a'] \wedge v$$

which is the required naturality condition.

Let us write $R \subseteq A \times A$ for the relation on A classified by the previous morphism of sheaves. That is, given $a, a' \in A(u)$,

$$(a, a') \in R(u) \text{ iff } [a = a'] = u.$$

This relation R is reflexive because

$$a \in A(u) \implies [a = a] = u \implies [a = a] = \text{top element of } \Omega(u).$$

It is transitive because given $a, a', a'' \in A(u)$

$$[a = a'] \wedge [a' = a''] \leq [a = a''],$$

thus

$$[a = a'] = u \text{ and } [a' = a''] = u \implies [a = a''] = u.$$

To prove that the relation R is also antisymmetric, choose $a, a' \in A(u)$ such that $[a = a'] = u = [a' = a]$. For each element x of A,

$$[a = x] \leq [a = a] \leq u,$$

thus

$$[a = x] = u \wedge [a = x] = [a' = a] \wedge [a = x] \leq [a' = x].$$

Analogously, $[a' = x] \leq [a = x]$ and thus $[a = x] = [a' = x]$. Since A is complete, this implies $a = a'$. ■

Theorem 3.3. *Let Ω be a locale. The category of skew Ω-sets is equivalent to the category of posets in the topos of sheaves on Ω.*

Proof: By theorem 2.6, it sufices to work with complete skew Ω-sets. For morphisms of complete skew Ω-sets, we use the description given in 2.5. Let us first construct a functor

$$P\colon \Omega\text{-Poset} \longrightarrow \Omega^c_{sk}\text{-Set}$$

from the category of Ω-poset to the category of complete skew Ω-sets.

Given a poset $(F, \leq)$ in the topos of sheaves on Ω, we provide the set $A = \amalg_{u \in \Omega} F(u)$ with the skew Ω-equality

$$[a = a'] = \bigvee \left\{ w \in \Omega \middle| a|_w \leq a'|_w \right\}$$

for all $a \in F(u)$, $a' \in F(v)$ and $w \leq u \wedge v$. It is straightforward to observe that we have so defined a skew Ω-set A. The symmetric Ω-set associated with A is the classical Ω-set associated with the sheaf F (see [3], section 2.9), thus it is complete. Since A and the associated symmetric Ω-set have the same singletons (see 3.1), it follows that A is a complets skew Ω-set.

A morphism of Ω-posets $\alpha: F \Longrightarrow G$ is mapped on $\amalg_{u\in\Omega}\alpha_u$; applying 2.5, this definition makes sense since each α_u is a poset morphism. This defines the functor P which, by construction, is faithful.

The functor P is also full. Indeed, write A and B for the complete skew Ω-sets associated with the Ω-posets F and G. A morphism $\varphi: A \longrightarrow B$ as in 2.5 is trivially also a morphism between the symmetric Ω-sets associated with A and B, thus is induced by a natural transformation $\alpha: F \Longrightarrow G$ (see [3], section 2.9). One has thus $\varphi = \amalg_{u\in\Omega}\alpha_u$. It remains to show that each α_u is a poset morphism. For this choose $a \leq a'$ in $F(u)$; this means $[a = a'] = u$ and therefore

$$\Big[\alpha_u(a) = \alpha_u(a')\Big] = \Big[\varphi a = \varphi a'\Big] \geq [a = a'] = u$$

from which $\Big[\alpha_u(a) = \alpha_u(a')\Big] = u$ since both elements are in $G(u)$. This means precisely $\alpha_u(a) \leq \alpha_u(a')$.

Finally it remains to observe that, by proposition 3.2, the full and faithful functor P is also essentially surjective on the objects. ■

Theorem 3.3 suggests that a skew Ω-set should rather be presented as a set A provided with an Ω-inequality

$$[\cdot = \cdot]: A \times A \longrightarrow \Omega, \qquad (a, a') \mapsto [a \leq a']$$

while a morphism $f: A \longrightarrow B$ of skew Ω-sets should be presented as a pair of mappings

$$\begin{aligned} [f\cdot \leq \cdot]&: A \times B \longrightarrow \Omega \quad ; \quad (a, b) \mapsto [fa \leq b], \\ [\cdot \leq f\cdot]&: B \times A \longrightarrow \Omega \quad ; \quad (b, a) \mapsto [b \leq fa]. \end{aligned}$$

The definitions remain thus exactly 1.4 and 1.6: just the notation has changed. This new notation improves largely the intuition; for example in theorem 1.12, the symmetric Ω-set A_{sy} has now the Ω-equality

$$[[a = a']] = [a \leq a'] \wedge [a' \leq a]$$

while the morphism f_{sy} of symmetric Ω-sets is given by

$$[[f_{sy}b = a]] = [fb \leq a] \wedge [a \leq fb].$$

In particular, we suggest to use this "inequality notation" instead of "equality" when working with $\mathcal{Q}$-sets on a quantale $\mathcal{Q}$.

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