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DEFORMATIONS OF (BI)TENSOR CATEGORIES

by L. CRANE & D.N. YETTER

RESUME. Motivés par des problèmes soulevés par la théorie topologique des champs quantiques, les Auteurs introduisent une théorie de déformation pour des catégories tensorielles et des catégories bitensorielles.

1 Introduction

In [5], a new approach was suggested to the construction of four dimensional Topological Quantum Field Theories (TQFTs), proceeding from a new algebraic structure called a Hopf category. In [7], it was argued that a well behaved 4d TQFT would in fact contain such a category at least formally.

An approach to construction of Hopf categories was also outlined in [5], making use of the canonical bases of Lusztig et al [15]. This proceeded nicely enough, except that there was no natural truncation of the category corresponding to the case where the deformation parameter was a root of unity, so all the sums in the tornado formula of [5] were divergent.

This situation is similar to what would have resulted if somebody had attempted to construct a 3d TQFT before the discovery of quantum groups. Formal state sums could be written using representations of a Lie algebra (or its universal enveloping algebra) but they would diverge. In fact such sums were written in a different context, as evaluations of spin networks [17]. The discovery of quantum groups made it possible to obtain finite TQFTs by setting the deformation parameter equal to a root of unity [3].

The key to this progress is the theory of the deformation of Hopf algebras, as applied to the universal enveloping algebras (UEAs) of simple Lie algebras. Infinitesimal deformations can be classified in terms of a double complex analogous to the complex which computes the Hochschild cohomology of an algebra. Certain interesting examples, which lead to global deformations, correspond to Poisson- Lie algebras, or equivalently to Lie bialgebras, or Manin triples. Once the interesting infinitesimal deformations of the UEAs

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were known, it turned out to be straightforward to extend them to find the quantum groups, whence the 3d TQFTs followed.

The purpose of this paper is to attempt an analogous procedure for Hopf categories. We begin by defining a double complex for a "bialgebra category," whose 3rd cohomology classifies infinitesimal deformations of the category. Next we apply this complex to the cases of finite groups and the categorifications of quantum groups produced by Lusztig [15]. We obtain suggestive preliminary results.

Of course, an infinitesimal deformation is not yet a finite one. Still less is it a truncation. However, contrary to the folk adage, lightning tends to strike the same places over and over. The double complex we construct can also be used to compute the obstructions to extension of any infinitesimal deformation to a formal series deformation, so that at least a plausible avenue of research is opened.

Let us remind the reader of the suggestion that 4d TQFTs may be the basis for a formulation of the quantum theory of gravity [4]. If this physical idea is correct, then 4d TQFTs from state sums should exist, and the program begun in this paper has a good chance of finding them.

In any case, the deformation theory of categories introduced here is natural, and of intrinsic interest.

The contents of this paper are as follows: chapter 2 describes the complex which defines the cohomology of a tensor category, and relates infinitesimal deformations to the third cohomology. Chapter 3 describes the double complex for a bitensor category, and relates it to infinitesimal deformations of bialgebra categories. Chapter 4 explores the construction of infinitesimal deformations in the most interesting cases.

Let us emphasize that this paper has the purpose of opening a new direction for research. We pose many more questions than we answer.

2 Cohomology and Deformations of Tensor Categories

The deformation theory developed here is very similar in abstract form to the deformation theory of algebras and bialgebras. Perhaps the not so categorical reader would do well to study the treatment of that theory in [3] before reading this chapter. The main formal difference is that deformations appear in H^3 rather than H^2. This is because in a category we deform associators, rather than products, and similarly for the rest of the structure.

Unfortunately, it will not be practical to make this discussion self contained. The category theoretic ideas can be found in [12], while the definition of a bialgebra category is in [5, 7, 4].

In the following we consider the question of deforming the structure maps of a tensor category, that is an abelian category $\mathcal{C}$ equipped with a biexact functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (or equivalently an exact functor $\otimes : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$, where $\boxtimes$ denotes the universal target category for biexact functors) which is associative up to a specified natural isomorphism $\alpha : \otimes(\otimes \times 1) \Longrightarrow \otimes(1 \times \otimes)$ satisfying the usual Stasheff pentagon. A tensor category is unital if it is equipped with an object I, and natural isomorphisms $\rho : - \otimes I \Longrightarrow Id_{\mathcal{C}}$ and $\lambda : I \otimes - \Longrightarrow Id_{\mathcal{C}}$ satisfying the usual triangle relation with α.

We will consider the case in which the category is K-linear for K some field, usually $\mathbb{C}$. We denote the category of finite-dimensional vector-spaces over K by **VECT**.

Definition 2.1 *An* **infinitesimal deformation** *of a K-linear tensor category $\mathcal{C}$ over an Artinian local K-algebra R is an R-linear tensor category $\tilde{\mathcal{C}}$ with the same objects as $\mathcal{C}$, but with $Hom_{\tilde{\mathcal{C}}}(a,b) = Hom_{\mathcal{C}}(a,b) \otimes_K R$, and composition extended by bilinearity, and for which the structure map(s) α (ρ and λ) reduce mod $\mathfrak{m}$ to the structure maps for $\mathcal{C}$, where $\mathfrak{m}$ is the maximal ideal of R. A deformation over $K[\epsilon]/ < \epsilon^{n+1} >$ is an n^{th}* **order deformation**.

Similarly an $\mathfrak{m}$**-adic deformation** *of $\mathcal{C}$ over an $\mathfrak{m}$-adically complete local K-algebra R is an R-linear tensor category $\tilde{\mathcal{C}}$ with the same objects as $\mathcal{C}$, but with $Hom_{\tilde{\mathcal{C}}}(a,b) = Hom_{\mathcal{C}}(a,b)\widehat{\otimes_K}R$, and composition extended by bilinearity and continuity, and for which the structure map(s) α (ρ and λ) reduce mod $\mathfrak{m}$ to the structure maps for $\mathcal{C}$, where $\mathfrak{m}$ is the maximal ideal of R. (Here $\widehat{\otimes_K}$ is the $\mathfrak{m}$-adic completion of the ordinary tensor product.) An $\mathfrak{m}$-adic deformation over $K[[x]]$ is* **formal series deformation**.

Two deformations (in any of the above senses) are **equivalent** *if there exists a monoidal functor, whose underlying functor is the identity, and whose structure maps reduce mod $\mathfrak{m}$ to identity maps from one to the other.*

Finally, if $K = \mathbb{C}$ (or $\mathbb{R}$), and all hom-spaces in $\mathcal{C}$ are finite dimensional, a **finite deformation** *of $\mathcal{C}$ is a K-linear tensor category with the same and maps as $\mathcal{C}$, but with structure maps given by the structure maps of a formal series deformation evaluated at $x = \xi$ for some $\xi \in K$ such that the formal series defining all of the structure maps converge at ξ.*

Ultimately our interest is in finite deformations, but their study and construction in general is beyond our present capabilities. In some particularly simple cases finite deformations can be constructed directly (cf. Crane/Yetter [6]).

In the present work, we will confine ourselves to the classification of first order deformations, and consideration of the obstructions to their extensions to higher order and formal series deformations.

To accomplish this classification, it is convenient to introduce a cochain complex (over K) associated to any K-linear tensor category:

First, we fix notation for the totally left and right parenthesized iterates of $\otimes$ as follows:

$$\otimes^n = \otimes(1 \boxtimes \otimes)(1 \boxtimes 1 \boxtimes \otimes)...(1 \boxtimes ...1 \boxtimes \otimes)$$

$$^n\otimes = \otimes(\otimes \boxtimes 1)(\otimes \boxtimes 1 \boxtimes 1)...(\otimes \boxtimes 1 \boxtimes ...1)$$

letting $\otimes^0 =^0 \otimes = Id_{\mathcal{C}}$.

Now, observe that by the K-bilinearity of composition, the collection of natural transformations between any two functors targetted at a K-linear category forms a K-vector space $Nat[F, G]$.

We now define the K-vector spaces in our complex by

$$X^n = Nat[^n\otimes, \otimes^n]$$

Thus elements of X^n have components of the form

$$f_{A_1,A_2,...,A_n} : (...(A_1 \otimes A_2)... \otimes A_n) \longrightarrow (A_1 \otimes (A_2 \otimes (... \otimes A_n)...))$$

In order to define the coboundary maps, and in much of what follows, it will be very convenient to have a notation for a sort of generalized composition of maps. To be precise, given some maps $f_1, f_2, ..., f_k$ all of whose sources and targets are variously parenthesized tensor products of the same word of objects, we will denote the composite

$$a_0 \circ f_1 \circ a_1 \circ f_2 \circ ... \circ f_k \circ a_k$$

by $\lceil f_1 f_2 ... f_k \rceil$ where a_0 is the generalized associator from the fully left-parenthesized tensor product to the source of f_1, a_i (for $i = 1, ..., k-1$) is the generalized associator from the target of f_i to the source of f_{i+1}, and a_k is the generalized associator from the target of f_k to the fully right-parenthesized tensor product.

Except the fact that we need to include $\lceil \ \rceil$ to obtain well-defined formulas familiar formulas define the coboundary maps for our complexes:

If $\phi \in X^n$, then $\delta(\phi) \in X^{n+1}$ is defined by

$$\delta(\phi)_{A_0,\ldots,A_n} =$$
$$\lceil A_0 \otimes \phi_{A_1,\ldots,A_n} \rceil + [\sum_{i=1}^{n-1} (-1)^i \lceil \phi_{A_1,\ldots,A_i \otimes A_{i+1},\ldots,A_n} \rceil] + \lceil \phi_{A_0,\ldots,A_{n-1}} \otimes A_n \rceil$$

It follows from the coherence theorem of Mac Lane and the same argument which show the coboundary in the bar resolution satisfies $\delta^2 = 0$ that these coboundaries satisfy $\delta^2 = 0$. Thus we have a cochain complex associated to any K-linear tensor category. We denote the cohomology groups of the complex by $\mathfrak{H}^\bullet(\mathcal{C})$, where the tensor structure on $\mathcal{C}$ is understood. [2]

The significance of this may be found by considering the pentagon relation for infinitesimal deformations of the associator.

If in the Stasheff pentagon giving the coherence condition on α, we replace all occurences of α with occurences of $\alpha + \epsilon a^{(1)}$ (where $\epsilon^2 = 0$), we find that the condition that the new Stasheff pentagon to commute reduces to

$$\delta(a^{(1)}) = 0$$

where $a^{(1)}$ is considered as an element of X^3.

Thus, first order deformations correspond to 3-cocycles in our complex.

More, however, is true: consider now equivalences of first order deformations. The main (only in the non-unital case) structure map has components of the form

$$1_{A \otimes B} + \phi_{A,B}\epsilon : A \otimes B \longrightarrow A \otimes B$$

where ϕ is some natural endomorphism of $\otimes$.

Now, suppose such a natural transformation defines a monoidal functor from a first order deformation with associator $\alpha + a\epsilon$ to another with associator $\alpha + b\epsilon$. Writing out the hexagon coherence condition for a monoidal functor, and evaluating the legs then gives

[2] In cases where more than one K-linear monoidal structure is being considered on the same category, it would be necessary to use the notation $\mathfrak{H}^\bullet(\mathcal{C}, \otimes, \alpha)$ to distinguish the structures, since the group depends on the monoidal structure.

$$\alpha_{A,B,C} + \{a_{A,B,C} + \phi_{A,B\otimes C}(\alpha_{A,B,C}) + (1_A \otimes \phi_{B,C})(\alpha_{A,B,C})\}\epsilon =$$
$$\alpha_{A,B,C} + \{\alpha_{A,B,C}(\phi_{A\otimes B,C}) + \alpha_{A,B,C}(\phi_{A,B} \otimes 1_C) + b_{A,B,C}\}\epsilon$$

Cancelling equal terms, solving for $b_{A,B,C}$, and observing that the compositions with α are describing the operation of $\lceil \quad \rceil$, we find that this is precisely the condition that

$$b = a + \delta(\phi)$$

Thus, we have shown:

Theorem 2.2 *First order deformations of a tensor category $\mathcal{C}, \otimes, \alpha$ are described by 3-cocycles in the complex $\{X^n, \delta\}$., and they are classified up to equivalence by the cohomology group $\mathfrak{H}^3(\mathcal{C})$.*

Let us now examine the obstructions to extending a first order deformation to a higher order deformation.

Once again we begin with a commutative Stasheff pentagon, this time with legs given by components of $\alpha + a^{(1)}\epsilon$.

Replacing these with corresponding components of $\alpha + a^{(1)}\eta + a^{(2)}\eta^2$ (where $\eta^3 = 0$), and calculating as before gives us the condition that

$$\begin{aligned}
&\delta(a^{(2)}) = \\
&\quad \lceil a^{(1)}_{A\otimes B,C,D} a^{(1)}_{A,B,C\otimes D}\rceil - \lceil a^{(1)}_{A,B\otimes C,D}(1_A \otimes a^{(1)}_{B,C,D})\rceil \\
&\quad - \lceil (a^{(1)}_{A,B,C} \otimes 1_D)(1_A \otimes a^{(1)}_{B,C,D})\rceil - \lceil (a^{(1)}_{A,B,C} \otimes 1_D) a^{(1)}_{A,B\otimes C,D}\rceil
\end{aligned}$$

Thus, the cochain on the right can be regarded as an obstruction to the extension to a second order deformation. It is unclear at this writing whether (or under what circumstances) this cochain is closed.

In as similar way, the condition needed to extend an n-th order deformation

$$\alpha^{(n)} = \alpha + a^{(1)}\epsilon + \ldots + a^{(n-1)}\epsilon^{n-1} \quad (\epsilon^n = 0)$$

to an $n+1$-st order deformation

$$\alpha^{(n+1)} = \alpha + a^{(1)}\eta + \ldots + a^{(n)}\eta^n \quad (\eta^{n+1} = 0)$$

is given by

$$\delta(a^{(n+1)}) =$$

$$\sum_{\substack{i+j=n+1 \\ 1\leq i,j\leq n}} \lceil a^{(i)}_{A\otimes B,C,D} a^{(j)}_{A,B,C\otimes D}\rceil$$

$$- \sum_{\substack{i+j=n+1 \\ 1\leq i,j\leq n}} \Big[\lceil a^{(i)}_{A,B\otimes C,D}(1_A \otimes a^{(j)}_{B,C,D})\rceil + \lceil (a^{(i)}_{A,B,C} \otimes 1_D)(1_A \otimes a^{(j)}_{B,C,D})\rceil$$

$$+ \lceil (a^{(i)}_{A,B,C} \otimes 1_D) a^{(j)}_{A,B\otimes C,D}\rceil\Big]$$

$$- \sum_{\substack{i+j+k = n+1 \\ 1\leq i,j,k\leq n}} \lceil (a^{(i)}_{A,B,C} \otimes 1_D) a^{(j)}_{A,B\otimes C,D}(1_A \otimes a^{(k)}_{B,C,D})\rceil$$

3 Cohomology and Deformations of Bitensor Categories

A bitensor category is a K-linear abelian category C with two fundamental structures, a (biexact) tensor product $\otimes$ (equivalently, an exact functor $C \boxtimes C \to C$), which is associative up to a natural isomorphism α which satisfies the usual Stasheff pentagon, and a tensor coproduct Δ which is an exact functor $C \to C \boxtimes C$ which is coassociative up to a natural isomorphism β which satisfies a dual Stasheff pentagon, and moreover satisfies the condition that Δ is a monoidal functor, and $\otimes$ is a cotensor functor (the dual condition), and the structural transformations are inverse to each other. To be more precise , Δ is equipped with a natural transformation with typical component

$$\kappa_{A,B} : \Delta(A \otimes B) \to \Delta(A) \otimes \Delta(B),$$

where the second $\otimes$ is the tensor product in $C \boxtimes C$, and κ satisfies the usual hexagonal coherence condition for monoidal functors. Moreover, κ^{-1} is the structural natural transformation for $\otimes$ as a cotensor functor, and as such, satisfies the dual coherence condition. We shall refer to the natural isomorphism κ as the "coherer". A bitensor category is biunital when it is equipped with a unit functor $1 : \mathbf{VECT} \to C$ and a counit functor $\epsilon : C \to \mathbf{VECT}$ satisfying the usual triangle, dual triangle and conditions that they respect the cotensor and tensor structures, up to mutually inverse natural transformations. In the biunital case, we denote the counit tranformations by r and l, and the remaining stuctural transformations by δ (counit preserves $\otimes$), τ (coproduct preserves I), and η (counit preserves I).

A Hopf category is a biunital bitensor category equipped, moreover, with an operation on objects, S, generalizing dual objects in a suitable sense.

As in the case of a tensor category, it is the structural isomorphisms which we deform, subject to the coherence axioms. The isomorphisms are natural transformations between combinations of structural functors, so the terms in finite order (or formal series) deformations will live in collections of natural transformations between the functors, which are vector spaces in the case of K-linear categories categories and exact functors.

As we had done for $\otimes$, we fix notation for the totally left and right parenthesized iterates of Δ as follows:

$$\Delta^n = (1 \boxtimes ...1 \boxtimes \Delta)...(1 \boxtimes \Delta)\Delta$$

$$^n\Delta = (\Delta \boxtimes 1 \boxtimes ...1)...(\Delta \boxtimes 1)\Delta$$

In order for us to place our deformation theory in a cohomological setting, it will be necessary first to examine the coherence theorem for bitensor categories (whether biunital or not).

Fortunately, the structure is given in terms of notions for which coherence theorems are well known (monoidal categories and monoidal functors) or their duals.

To state it properly, however, we require some preliminaries. First, we will restrict our attention to the case where all of our categories are equivalent as categories without additional structure to a category $A - \mathbf{mod}$ for A a finite-dimensional K-algebra. In this case $\mathcal{C} \boxtimes \mathcal{D}$ is given by $A \otimes_K B - mod$ when $\mathcal{C}$ (resp. $\mathcal{D}$) is equivalent to $A - \mathbf{mod}$ (resp. $B - \mathbf{mod}$). In this setting, the monoidal bicategory structure given by $\boxtimes$ has pentagons and triangles which commute exactly (the structural modifications are identities), so the 1-categorical coherence theorem of Mac Lane applies, and we may disregard the parenthesization of iterated $\boxtimes$, and the intervention of associator and unit functors. Thus we may use the notation $\mathcal{C}^{\boxtimes n}$ without fear of ambiguity.

Second, we must note that if $\mathcal{C}$ is a bitensor category, so is $\mathcal{C}^{\boxtimes n}$. The structure functors are given by applying a "shuffle" functor before or after the $\boxtimes$-power of the corresponding structure functor for $\mathcal{C}$.

Theorem 3.1 (Coherence Theorem for Bitensor Categories) *Given two expressions for functors Φ, Φ' from an n-fold $\boxtimes$-power of a bitensor category to an m-fold $\boxtimes$-power of the same category, given in terms of $Id, \otimes, \Delta, I, \epsilon, \boxtimes$, and composition of functors, (where the structural functors may lie in any $\boxtimes$-power of C), and given two expressions for natural isomorphisms between these functors in terms of the structural transformations for the categories, identity transformations, $\boxtimes$, and the 1- and 2-dimensional compositions of natural transformations, then in any instantiation of these expressions by the structures from a particular bitensor category, the natural isomorphisms named by the two expressions are equal.*

proof: First, observe that this is a coherence theorem of the very classical "all diagrams of a certain form commute." The proof of the theorem will consist in piecing together in an appropriate way the two classical coherence theorems of this form–Mac Lane's coherence theorem for monoidal categories [16] and Epstein's coherence theorem for (strong) monoidal functors [9]–and their duals. For the same reasons as in those classical theorems, we must deal with formal expressions for functors and natural transformations to avoid "coincidental" compositions.

The proof is reasonably standard: for any expression for a functor of the given form, we construct a particular "canonical" expression for a natural isomorphism to another such expression for a functor. Then, given two expressions for functors related by a natural transformation named by a *single instance* of a structural natural isomorphism, identity transformations, $\boxtimes$, and 1-dimensional composition of natural transformations, we show that the diagram of natural transformations formed by this "prolongation" of the structure map and the two "canonical" expressions closes and commutes.

(The "canonical" has quotation marks, since it is only once the theorem is established that we will know that the map named by the composite is, in fact, canonical. *A priori* it is dependent upon the construction given.)

Note that this suffices, since

1. by the middle-four-interchange law, any expression for a natural isomorphism of the sort described in the theorem will factor into a 2-dimensional composition of expressions of this restricted sort, and

2. any composite of such expressions is then seen to be equal to the composite of the "canonical" expression for the source, followed by the inverse of the "canonical" expression for the target.

Our "canonical" expressions consists of a composite c_1 of instances of the structure maps κ, δ, τ, and η to move all occurences of Δ and ϵ "inside" all occurences of $\otimes$, and remove all applications of Δ or ϵ to I; followed by a composite c_2 of instances of β, r, and l to remove all occurences of ϵ applied to a cofactor of Δ and to completely right coassociate all iterated Δ's; followed by a composite c_3 of instances of α, ρ and λ to remove all instances of I tensored with other objects, and completely right associate all iterated $\otimes$'s.

Note that we have chosen an order to compose the three constituent composites, but have not specified the order within each composite. This is possible for c_3 (resp. c_2) by the coherence theorem of Mac Lane, (resp. its dual), and the functoriality properties of $\boxtimes$ and the 1-dimensional composition of natural transformations. For c_1, the order of application is constrained by the nesting of the various functors, but within those constraints, the resulting composite is independent of the order of application by virtue of the functoriality properties of $\boxtimes$ and the 1-dimensional composition of natural transformations.

In the circumstances of the theorem, we will let c_i (resp. c_i') $i = 1, 2, 3$ denote the components of the "canonical" map from Φ (resp. Φ').

We now have three cases

1 The natural isomorphism f from Φ to Φ' is a prolongation of κ, δ, τ, or η.

2 The natural isomorphism f from Φ to Φ' is a prolongation of β, r, or l.

3 The natural isomorphism f from Φ to Φ' is a prolongation of α, ρ, or λ.

In Case 1, it follows from the same argument that shows that c_1 is well-defined that the targets of c_1 and c_1' coincide, and that $c_1'(f) = c_1$.

In Case 2, by using the functoriality properties of $\boxtimes$ and the 1-dimensional composition of natural transformations, and the dual of the coherence theorem for monoidal functors, we can construct a natural isomorphism $f^!$ from the target of c_1 to the target of c_1' such that $f^!$ is a composition of prolongations of β's, r's, and l's, and $c_1'(f) = f^!(c_1)$. It then follows from the same argument that shows c_2 is well-defined that the targets of c_2 and c_2' coincide, and $c_2'(f^!) = c_2$.

Finally, for Case 3, by using the functoriality properties of $\boxtimes$ and the 1-dimensional composition, and the coherence theorem for monoidal functors, we can construct a natural isomorphism $f^!$ from the target of c_1 to the target of c_1' such that $f^!$ is a composition of prolongations of α's, ρ's, and λ's, and $c_1'(f) = f^!(c_1)$. By using the functoriality properties of $\boxtimes$ and the 1-dimensional composition, and the naturality properties of prolongations of β, r, and l, we can construct a natural isomorphism $f^{!!}$ from the target of c_2 to the target of c_2' such that $f^{!!}$ is a composition of prolongations of α's, ρ's, and λ's, and $c_2'(f^!) = f^{!!}(c_2)$. It follows by the same argument that shows c_3 is well-defined that the targets of c_3 and c_3' coincide, and $c_3'(f^{!!}) = c_3$. □

We shall call an instantiation of expressions of the type given in the previous theorem a pair of *commensurable functors*, and the unique natural isomorphism obtained by instantiating an expression of the type in the theorem the *commensuration*. Given commensurable functors F and G, we will denote the commensuration by $\gamma^{F,G}$.

Now, observe that $\Delta^n({}^n\otimes)$ and $[\otimes^i]^{\boxtimes j} sh [{}^j\Delta]^{\boxtimes i} = \otimes^i_j [{}^j\Delta]^{\boxtimes i} = [\otimes^i]^{\boxtimes j}\Delta_i$ are commensurable functors, where sh is the "shuffle functor" from $[C^{\boxtimes j}]^{\boxtimes i}$ to $[C^{\boxtimes i}]^{\boxtimes j}$. Given a sequence of natural transformations $f_1, ..., f_n$ such that the source of f_1 is commensurable with $\Delta^n({}^n\otimes)$, and the target of f_i is commensurable with the source of f_{i+1}, and the target of f_n is commensurable with $[\otimes^i]^{\boxtimes j} sh [{}^j\Delta]^{\boxtimes i}$ let

$$\lceil f_1, ..., f_n \rceil : \Delta^j({}^i\otimes) \Rightarrow [\otimes^i]^{\boxtimes j} sh [{}^j\Delta]^{\boxtimes i}$$

denote the composition of the given natural transformations alternated with the appropriate commensurations.

For any bitensor category, we can now define a double complex of vector spaces

$$(X^{\bullet,*}, d^{\bullet,*} : X^{\bullet,*} \to X^{\bullet+1,*}, \delta^{\bullet,*} : X^{\bullet,*} \to X^{\bullet,*+1}),$$

where X^{ij} is the space of natural transformations between the two (commensurable) functors $\Delta^j\ {}^i\otimes$ and $[\otimes^i]^{\boxtimes j} sh [{}^j\Delta]^{\boxtimes i}$ from the i-fold to the j-fold tensor power of $\mathbf{C}$ to itself. (Notice that because our category if k-linear, these collections of natural transformations are k-vector spaces.) And

$$\begin{aligned} d(s) \quad &= \quad \lceil \otimes^n \boxtimes 1(1 \boxtimes s)_{sh(\Delta^{\boxtimes n}(-)} \rceil \\ &+ \left[\sum_{i=1}^{m} (-1)^i \lceil 1^{i-1} \boxtimes \Delta \boxtimes 1^{m-i} \rceil \right] + (-1)^{m+1} \lceil 1 \boxtimes \otimes^n (s \boxtimes 1)_{sh(\Delta^{\boxtimes n}(-)} \rceil \end{aligned}$$

and

$$\begin{aligned}\delta(s) &= \lceil \otimes^{\boxtimes m}(sh(1 \boxtimes s)_{{}^n\Delta\boxtimes 1(-)}) \rceil \\ &\quad + \left[\sum_{i=1}^{n} (-1)^i \lceil s_{1^{i-1}\boxtimes\otimes\boxtimes 1^{n-i}(-)} \rceil\right] + (-1)^{n+1} \lceil \otimes^{\boxtimes m}(sh(s \boxtimes 1)_{1\boxtimes^n\Delta(-)}) \rceil\end{aligned}$$

in each case for $s \in X^{n,m}$.

In the case of a biunital bitensor category, we can easily extend our complex to include the values of 0 for i and j, interpreting $\mathcal{C}^{\boxtimes 0}$ as **VECT**, $\otimes^0$ and ${}^0\otimes$ as the functor I, and Δ^0 and ${}^0\Delta$ as the functor ϵ.

It follows by a diagram chase from the coherence of the bialgebra category that $d^2 = \delta^2 = d\delta + \delta d = 0$. Thus we have a bicomplex whose cohomology can be defined in the usual manner.

Definition 3.2 *The bicomplex described above is the* **basic bicomplex** *of the bitensor category. The total complex of the basic bicomplex, indexed by $X^n = \oplus_{i+j=n+1} X^{i,j}$ is the* **basic complex** *of the bitensor category.*

Definition 3.3 *The larger bicomplex described above is the* **extended bicomplex** *of a biunital bitensor category.*

Now we note that the three structural natural transformations of a bialgebra category live in the third diagonal of the basic bicomplex. Specifically, the associator α for the tensor product lives in $X^{3,1}$ the coassociator β for the coproduct lives in $X^{1,3}$, and the "coherer" κ lives in $X^{2,2}$.

By an infinitesimal deformation of a bialgebra category we mean an infinitesimal deformation of its structural natural transformations which satisfies the coherence axioms to first order in the infinitesimal parameter. This makes sense because natural transformations are combinations of morphisms, and all the spaces of morphisms for our spaces are vector spaces.

Concretely we let $\kappa' = \kappa + k\epsilon, \alpha' = \alpha + a\epsilon, \beta' = \beta + b\epsilon$, for $\epsilon^2 = 0$. When we write out the coherence axioms for the new maps, we find the only new conditions beyond the coherence of the triple α, κ, β are

$$d(a) = \delta(a) + d(k) = d(k) + \delta(b) = \delta(b) = 0.$$

The deformations of our category (as a bitensor category) correspond to cocycles of the basic complex. Similarly, the equivalence classes of deformations under infinitesimal monoidal equivalence correspond to cohomology classes.

Theorem 3.4 *The equivalence classes of infinitesimal deformations of a bialgebra category correspond to classes in the third cohomology of its basic complex.*

proof: Once it is observed that the structural maps for a bitensor functor are elements of $X^{1,2}$ and $X^{2,1}$, it is easy to check (by writing out the hexagon coherence conditions for monoidal and dual monoidal functors) that a bitensor functor structure for the identity functor given over $K[\epsilon]/<\epsilon^2>$ is described by a total 2-cochain which cobounds the difference between the two bitensor structures (as 3-cochains).

(The fact that the third cohomology group appears here, rather than the second *á lá* Hochschild, is suggestive in relation to the categorical ladder picture in TQFT. We know that a TQFT can be constructed from a finite group plus a cocycle of the group. The cocycle of the group must be chosen to match the dimension of the TQFT. Thus if a 2-cocycle of a bialgebra gives rise to a 3d theory, it is plausible that a 3-cocycle of a bialgebra category would generate a 4d theory. All this raises the question whether there is a classifying space of some sort for a bialgebra category whose cohomology is related to the cohomology of our bicomplex.)

This theorem is not very useful in itself, since it does not suggest a way to find interesting examples of cocycles. However, for a biunital bitensor category, we can embed the basic bicomplex into the extended bicomplex. Any element of $X^{0,3}$ on which δ gives 0, or any element of $X^{3,0}$ on which d gives 0 can be pushed back into the basic bicomplex to give a candidate for a deformation. This is analogous to the process which led to the quantum groups: the classical r matrix lives in an extended bicomplex, and the vanishing of the analog of the Steenrod square of its differential is precisely the classical Yang-Baxter equation. See [3]. (Of course, the classical Yang- Baxter equation was not for any element of the complex associated to the Hopf algebra, but only to one of a very special form related to the Lie algebra. At the moment we do not know an analogous *ansatz* for the categorified situation.)

Thus, we now have an a pair of interesting new equations to investigate for Hopf categories :

$d(s) = 0, \quad s \in X^{3,0} \ (D1)$

or

$\delta(t) = 0, \quad t \in X^{0,3} \ (D2)$

In addition, we can ask about the equation which says that the infinitesimal defomration contructed from a solution to (D1) or (D2) can be extended to a second order deformation.

In the bialgebra situation, the combination of these two equations led to the classical Yang-Baxter equations, in the restricted *ansatz.*

4 Searching for Deformations in some Interesting Cases

A naive reader might suppose that the deformation equations D1 and D2 are rather disappointing, since they lead to a sort of cohomology of automorphisms of the identity or counit of the category. However, in the important cases, the unit and counit are not simple objects, so in fact we are led into interesting ground.

In the case of the quantum double of the group algebra of a finite group, the identity is a sum of one ordered pair of group elements for each group element. If we categorify in the natural way, so that each ordered pair of group elements is a simple object in the category, (see [6]) our equation D1 reduces to a cocycle on the group. In effect, we have reproduced Dijkgraf-Witten theory [8] in the language of deformed Hopf categories, since the group cocycle for Dijkgraf-Witten theory induces a finite (and thus) infinitesimal deformation of the Hopf category.

The other interesting case to apply our theory to is the categorification of the quantized UEA's constructed by Lusztig in his construction of the canonical bases [14]. In order to get a construction which worked for the entire QUEA, Lusztig was forced to replace the identity by a family of projectors corresponding to the weight lattice. (It must be cautioned that Lusztig only worked things out explicitly in the case of SL(2)). Thus the deformation equations translate into the coboundary equation for the complex for the group cohomology of the root lattice. This means that possible infinitesimal deformations of the bialgebra category correspond to 3-forms on the fundamental torus of the corresponding Lie group. We can see that even at the first order of deformation theory, our procedure seems to produce something only for certain Lie algebras- those of rank at least 3. Work is under way to examine the implications of the second order deformation equations in this situation.

It seems unlikely that a complete deformation can be found order by order. Such an approach is too difficult even for bialgebras. Let us simply cite the fact [11] that it is an open question whether the vanishing of the obstruction to a second order deformation is always enough to ensure a deformation to all orders for a bialgebra. Nevertheless, our preliminary results suggest that deformations may exist for the bialgebra categories associated to certain special Lie algebras only. Whether this could bear any relationship to the special choices of groups which appear in string theory and supergravity is not clear at the moment, but the possibility cannot be ruled out.

In order to clarify the situation, it will be necessary to find some global method for producing deformations. As of this writing, we have only begun to investigate the possibilities. Several lines of thought suggest themselves:

1. One could search for a categorified analog of Reshetikhin's proof that every Lie bialgebra produces a quantum group [18, 10]. In order to attempt this, we need to single out the part of the bialgebra category of Lusztig corresponding to the Lie algebra itself. This is rather delicate, since categories do not admit negative elements, but a way may be found.

2. It is possible to examine special 3-forms on the groups F_4 and E_6, related to their constructions from the triality of $SO(8)$. Perhaps the relationships of these 3-forms with the structure of the Lie algebras will make it possible to extend the corresponding cohomology classes of the root lattices to complete deformations of the corresponding bialgebra categories. If so, the special Lie algebras for which we can produce bialgebra category deformations will be physically interesting ones.

3. Lusztig constructed his categories as categories of perverse sheaves over flag varieties. The flag varieties are known to have q-deformations in the sense of non-commutative geometry [13]. Perhaps a suitable category of D-modules over the quantum flag algebras can be constructed.

5 Conclusions

Simple Lie groups and Lie algebras are very central constructions in mathematics. They appear in theoretical physics as the expressions of symmetry, which is a fundamental principle of that field. It has been a remarkable recent discovery that the universal enveloping algebras of Lie algebras, and the function algebras on Lie groups, admit deformations. This discovery came to mathematics by way of physics.

It is a further remarkable fact that the deformations of the universal enveloping algebras admit categorifications, i.e. are related to very special tensor categories.

There is no reason not to try to see if this process goes any farther. The question whether the categorifications of the deformations can themselves be deformed is a natural one.

The development in algebra we have outlined has had profound implications for topology, and at least curious ones for quantum field theory as well. Perhaps it is puzzling the the categories constructed by Lusztig do NOT seem to fit into the topological picture surrounding quantum groups. The direction of work begun in this paper has the potential of widening the topological picture to include Lusztig's categories as well.

Finally, it seems that the relationship between topological applications of algebraic structures and deformation theory can be direct. One of us [19] has recently discovered a brief proof of a theorem generalizing the well-known result of Birman and Lins [2] (cf. also [1] that the coefficients of the HOMFLY and Kauffman polynomials are Vassiliev invariants. The proof makes a direct connection between the stratification of the moduli space of embedded curves in R^3 and the deformation theory of *braided* tensor categories (cf. Yetter [20]). It is plausible to suggest that the deformations we are attempting to construct may play a similar role.

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