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Representations of modules and Cauchy completeness


<http://www.numdam.org/item?id=CTGDC_1994__35_3_263_0>
Soit $R$ un anneau, non nécessairement commutatif, et soit $M$ un $R$-module à droite. Soit $\mathcal{V}$ la catégorie monoidale des idéaux faibles à droite de $R$. Nous décrivons une $\mathcal{V}$-catégorie Cauchy-complète construite à partir de $M$, dont les objets sont les éléments de $M$.

1. Introduction

Recently, Borceux and Van den Bossche presented in [1] an interesting ‘generic’ sheaf representation of commutative rings from a quantale theoretic point of view. But it does not seem to be very clear how this is connected with classical sheaves.

On the other hand, F.W. Lawvere introduced in [3] the notion of cauchy complete $\mathcal{V}$-category, a notion which generalizes that of cauchy complete metric space. In [4], [3], R.F.C. Walters showed that sheaves on a space could be regarded as cauchy complete categories on the distributive bicategory of relations of the space.

The authors believe that this language of cauchy complete categories based on a distributive bicategory provides a flexible and precise language for analysing sheaf-like representations of algebraic structures. The general methods of enriched category theory provide an excellent guide to this particular situation.

We intend a series of investigations, of which this is the first, into different bicategories suitable for analysing different aspects of the representation of $R$-mod, $R$ a non-commutative ring.

Firstly, if we choose the base bicategory to be the monoidal category $\text{Id}(R)$, whose objects are ideals, whose arrows are inclusions, and whose tensor product is the product of ideals (i.e., the bicategory is a quantale), then it is not too hard to show that each $R$-module $M$ occurs as a symmetric skeletal cauchy complete $\mathcal{V}$-category $L(M)$, whose objects are the elements of $M$ (more details will be given in Section 2).

However, in the light of papers [3] and [4], to obtain representations which are more sheaf-like, we prefer to choose the bicategory to be a locale or a category having the form $\text{Rel}(C)$ (the definition see [3]) with a small category $C$.

For this reason we introduce the notion of weak right ideals of a non-commutative ring $R$: weak right ideals are subsets of $R$ closed under right-multiplication by elements of the ring and “weak addition”. The poset of weak right ideals is then a locale. We consider in this paper the monoidal category $\mathcal{V}$ whose objects are weak right ideals of a non-commutative ring $R$, whose arrows are inclusions, and whose tensor products is intersection. Given a right-$R$ module (or left-$R$ module) $M$ we construct a symmetric skeletal cauchy complete $\mathcal{V}$-category $L(M)$, whose objects are the elements of $M$. 

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In our future paper, we will consider such representations whose base bicategory has the form \( \text{Rel}(C) \) with \( C \) a small category.

The notation of this paper is that of [3] and [4], with the simplification that the base category is a monoidal category \( \mathcal{V} \) rather than a bicategory \( \mathcal{B} \).

After this paper was written we became aware of further work of Borceux, with Cruciani, extending [1] to the non-commutative case. Their main contribution is to present a notion of sheaf over a quantale, but again without our enriched-categorical setting.

2. The base bicategory is a quantale

In this short section, we sketch how the results in [1] are connected with the theory of enriched category, in particular, with the theory of cauchy completion. Our main results will be given in next two sections.

Let \( R \) be a commutative ring with an identity and \( \mathcal{V} = \text{Id}(R) \) the distributive monoidal category of ideals of \( R \).

**Definition 2.1.** If \( M \) is a right-\( R \)-module, then \( \mathcal{V} \)-category \( L(M) \) is defined by

(i) the set of objects of \( L(M) \) is \( M \);

(ii) if \( m, m' \in M \), then the hom \( d(m, m') \) is \( \text{Ann}(m - m') \),

where \( \text{Ann}(m) = \{ r \in R \mid 0 = mr \} \).

**Lemma 2.1.** Each \( L(M) \) is a symmetric skeletal \( \mathcal{V} \)-category.

**Proof.** See section 4.

**Lemma 2.2.** For any adjoint pair of \( \mathcal{V} \)-modules

\[
\phi : \{\ast\} \longrightarrow M \quad \text{and} \quad \psi : M \longrightarrow \{\ast\},
\]

it is the case that \( \phi(\ast, m) = \psi(m, \ast) \) for all \( m \in M \).

**Proof.** Note that \( R \) is commutative.

**Lemma 2.3.** Each \( L(M) \) is cauchy complete.

**Proof.** For the details see section 4.

Now let \( f : M_1 \rightarrow M_2 \) be a right \( R \)-module morphism and let \( Lf = f \). Then we have,

**Lemma 2.4.** \( Lf \) is a \( \mathcal{V} \)-functor from \( \mathcal{V} \)-category \( \mathcal{V}_1 \) to the \( \mathcal{V} \)-category \( \mathcal{V}_2 \).

**Proof.** It suffices to show that \( d_1(m, m') \leq d_2(fm, fm') \) but this is obvious.

We have thus proved the following

**Theorem 2.1.** \( L \) is a functor from the category \( \text{MOD-}R \) to the category \( \mathcal{V} - \text{Cat}_{cc} \) of symmetric skeletal cauchy complete \( \mathcal{V} \)-categories and \( \mathcal{V} \)-functors.
After this paper was written we became aware of further work of Borceux, with Cruciani, extending [1] to the non-commutative case. Their proof and ours in the next sections suggest that the above statements remain true if a commutative ring $R$ is replaced by a non-commutative one.

However, our main contributions are contained in the next two sections.

3. The locale of weak right ideals

Let $R$ be a not-necessarily commutative ring with identity.

**Definition 3.1.** A subset $I$ of $R$ is called a weak right ideal of $R$ if

(i) $IR \subseteq I$;

(ii) for all $a \in R$ and central elements $u_1, u_2, \ldots, u_n \in R$ with $\sum_{i=1}^{n} u_i = 1$, if $au_i \in I$ for $i = 1, \ldots, n$, then $a \in I$.

For convenience, if $I$ satisfies (ii), we will say that $I$ is closed under weak addition. It is clear that each right ideal is a weak right ideal.

**Example 3.1.** Let $R$ be a (not necessarily commutative) ring without central elements except 0 and invertible elements. Then any union of right ideals of $R$ is a weak right ideal. The free ring $R$ with more than one generator, over a skew-field, is such an example. Moreover, any simple ring (i.e., a ring with no non-trivial two sided ideals) is such a ring. A particular example is the ring of $n \times n$ matrices over a skew-field.

**Example 3.2.** On the other hand, however, in the Euclidean domain, $\mathbb{Z}$, the fact that $I$ is closed under weak addition implies that $I$ is closed under addition; that is that $I$ is an ideal. The proof is as follows. If $I$ is a weak ideal and $m, n$ lie in $I$ and $d = \text{HCF}(m, n)$ then write $m = dm', n = dn'$ and solve $u_1 + u_2 = m'x + n'y = 1$. Since $du_1 = mx \in I$ and $du_2 = ny \in I$ then closure under weak addition implies that $d \in I$. It is then trivial that $m + n = d(m' + n') \in I$.

**Lemma 3.1.** The set $\text{WRId}(R)$ of all weak right ideals of $R$ is closed under intersections and hence is a complete lattice.

**Proof.** Easy

**Lemma 3.2.** The suprema $\bigvee J_i$ of $J_i \in \text{WRId}(R)$ is calculated as follows: $\bigvee J_i = \{a \in R| \exists \text{ central elements } u_i \text{ with } \sum u_i = 1, \text{ such that } au_i \in J_i(i) \text{ for some } J_i(i)\}$

**Proof.** First check that $\bigvee J_i$ is a weak right ideal:

1. If $a \in \bigvee J_i$, then there are central elements $\{u_k\}$ with $\sum u_k = 1$ and $u_k a \in J_i(k)$. Thus for any $x \in R$, one has $u_k ax \in J_i(k)$ and hence $ax \in \bigvee J_i$.

2. If there are central elements $z_k$ with $\sum z_k = 1$ so that $z_k a \in \bigvee J_i$, then there are central elements $u_{k,i}$ with $\sum u_{k,i} = 1$ such that $u_{k,i}z_k a \in J_i(k,i)$. Note that $u_{k,i}z_k$ are central elements with $\sum_{k,i} u_{k,i}z_k = 1$, so that $a \in \bigvee J_i$.

Then we note that $\bigcup J_i \subseteq \bigvee J_i$ and it is easy to check $\bigvee J_i$ is the least upper bound of $J_i$'s.

**Theorem 3.1.** $\text{WRId}(R)$ is a locale.
Proof. It suffices to prove that \( J \cap \bigvee I_i \subseteq \bigvee (J \cap I_i) \) for any \( J, I_i \in \text{WId}(R) \). Let \( x \in J \cap \bigvee I_i \). Then \( x \in J \) and there are centrals \( u_k \) with \( \sum u_k = 1 \) such that \( u_k x \in J_{i(k)} \). Moreover we have \( u_i x = x u_i \in J \) since \( u_i \) are central and \( J R \subseteq J \). Thus \( u_k x \in J \cap I_{i(k)} \) and hence \( x \in \bigvee J \cap I_i \).

Note that a principal right ideal \( aR \) is compact in the lattice \( \text{WRI}(R) \) in the usual sense and that each weak right ideal can be expressed as a join of \( aR \)'s. So \( \text{WRI}(R) \) is in fact an algebraic locale, and hence is spatial. Thus we have

**Theorem 3.2.** For a general ring \( R \) with identity, \( \text{WRI}(R) \) is a spatial locale.

**Remark 3.1.** Note that the lattice of all right ideals of a ring \( R \) is not necessarily a locale, even for commutative ring \( R \). For example, let \( R \) be the polynomial (commutative) ring \( k[X, Y] \), with two generators \( X \) and \( Y \), over a field \( k \). Then \( R \) is Noetherian, but is not Dedekind, since non-zero prime ideals are not necessarily maximal (for example, the prime ideal \( (X) \)). Thus \( \text{Id}(R) \) is not distributive by the fact that a Noetherian domain is Dedekind if \( \text{Id}(R) \) is distributive.

4. The construction

Let \( R \) be a not-necessarily commutative ring with an identity and \( V = \text{WRI}(R) \) the distributive monoidal category of weak right ideals of \( R \).

**Definition 4.1.** If \( M \) is a right-\( R \)-module, then \( V \)-category \( L(M) \) is defined by

1. the set of objects of \( L(M) \) is \( M \);
2. if \( m, m' \in M \) then the hom \( d(m, m') \) is \( \text{ann}_r(m - m') \),

where \( \text{ann}_r(m) = \{ r \in R | rm = 0 \} \) is a right ideal and hence is a weak right ideal.

**Lemma 4.1.** Each \( L(M) \) is a symmetric skeletal \( V \)-category.

**Proof.** The symmetry is obvious. If \( d(m, m') = R \), then trivially \( m = m' \) since \( R \) has an identity; so that \( L(M) \) is skeletal. It remains to show that

\[
\text{ann}_r(m' - m'') \cap \text{ann}_r(m - m') \subseteq \text{ann}_r(m - m'').
\]

Let \( x \in \text{ann}_r(m' - m'') \cap \text{ann}_r(m - m') \). Then

\[
(m' - m'')x = 0 = (m - m')x
\]

and hence \( (m - m'')x = 0 \); i.e., \( x \in \text{ann}_r(m - m'') \).

**Lemma 4.2.** For any adjoint pair of \( V \)-modules

\[
\phi : \{ * \} \rightarrow \rightarrow M \quad \text{and} \quad \psi : M \rightarrow \rightarrow \{ * \},
\]

it is the case that \( \phi(*, m) = \psi(m, *) \) for all \( m \in M \).

**Proof.** First for any \( m, m' \in M \), we have

\[
\phi(*, m) \cap \phi(*, m') = R \cap \phi(*, m) \cap \phi(*, m') = ( \bigvee_{m'' \in M} \psi(m'', *) ) \cap \phi(*, m) \cap \phi(*, m')
\]

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Then

\begin{equation*}
\text{Here we use Theorem 3.1. Similarly we have } \psi(m, *) \leq \phi(m, m).
\end{equation*}

Lemma 4.3. Each \( L(A, I) \) is cauchy complete.

**Proof.** Consider adjoint pair of \( \mathcal{V} \)-modules

\[
\phi : \{ * \} \rightarrow M \quad \text{and} \quad \psi : M \rightarrow \{ * \}.
\]

That is, \( \phi(*, m) \), \( \psi(m, *) \), \( m \in M \) are objects of \( \mathcal{V} \) satisfying the following:

1. \( d(m, m') \cap \phi(*, m') \leq \phi(*, m) \);
2. \( \psi(m', *) \cap d(m, m') \leq \psi(m, *) \);
3. \( R \leq \bigvee_{m} \psi(m, *) \cap \phi(*, m) \);
4. \( \phi(*, m') \cap \psi(m, *) \leq d(m, m') \).

By (3) we have

\[
R = \bigvee_{m} \psi(m, *) \cap \phi(*, m)
\]

So there are central elements \( \epsilon_{i} \in \phi(*, m_{i}) \cap \psi(m_{i}, *) \) such that \( \sum_{i=1}^{n} \epsilon_{i} = 1 \) by Lemma 3.2.

In particular, we have

\[
R = \bigvee_{i} \psi(m_{i}, *) \cap \phi(*, m_{i}).
\]

Let \( m_{0} = \sum_{i=1}^{n} m_{i} \epsilon_{i} \). We shall prove that

\[
\psi(m, *) \leq \text{ann}_{r}(m_{0} - m) \leq \phi(*, m), \text{for all } m \in M.
\]

To this end, we want to show

\[
\psi(m_{i}, *) \leq \text{ann}_{r}(m_{0} - m_{i}) \quad \text{for all } i.
\]

In fact, for any \( t \in \psi(m_{i}, *) \), we have

\[
(m_{0} - m_{i})t = (\sum_{j} m_{j} \epsilon_{j} - \sum_{j} m_{i} \epsilon_{j})t = \sum_{j=1}^{n} (m_{j} - m_{i}) \epsilon_{j} t = 0
\]

since \( \epsilon_{j} \) is central, and hence \( \epsilon_{j} t \in \phi(*, m_{j}) \cap \psi(m_{j}, *) \leq \text{ann}_{r}(m_{j} - m_{i}) \). So \( \psi(m_{i}, *) \leq \text{ann}_{r}(m_{0} - m_{i}) \).
Moreover, we have, by (2)
\[ \phi(\ast, m_0) \geq \bigvee_i \phi(\ast, m_i) \cap \text{ann}_r(m_0 - m_i) \geq \bigvee_i \phi(\ast, m_i) \cap \psi(m_i, \ast) = R. \]
Thus we have, by (2) again,
\[ \phi(\ast, m_0) \cap \text{ann}_r(m - m_0) \leq \phi(\ast, m) \]
and hence
\[ \text{ann}_r(m - m_0) \leq \phi(\ast, m). \]
On the other hand, by (4), we have
\[ \phi(\ast, m) \cap \psi(m_0, \ast) \leq \text{ann}_r(m_0 - m) \]
and hence
\[ \psi(m, \ast) \leq \text{ann}_r(m_0 - m) \leq \phi(\ast, m) \]
for all \( m \in M \).
Similarly, we have
\[ \phi(m, \ast) \leq \text{ann}_r(m_0 - m) \leq \psi(\ast, m) \]
(it also follows from Lemma 4.2). Thus we finally have
\[ \phi(\ast, m) = \text{ann}_r(m_0 - m) = \psi(m, \ast), \quad \text{for all } m \in M. \]
and so that this adjoint pair is induced from the point \( m_0 \in M \).

Now let \( f : M_1 \rightarrow M_2 \) for a left \( R \)- and right \( R \)-module morphism and let \( Lf = f \). Then we have,

**Lemma 4.4.** \( Lf \) is a \( \mathcal{V} \)-functor from \( \mathcal{V} \)-category \( \mathcal{V}_1 \) to the \( \mathcal{V} \)-category \( \mathcal{V}_2 \).

*Proof.* It suffices to show that \( d_1(m, m') \leq d_2(fm, fm') \) but this is obvious. \( \square \)

We have thus proved the following

**Theorem 4.1.** \( L \) is a functor from the category MOD-\( R \) to the category \( \mathcal{V} - \text{Cat}_{ec} \) of symmetric skeletal cauchy complete \( \mathcal{V} \)-categories and \( \mathcal{V} \)-functors.

5. Remarks

**Remark 5.1.** The questions of how characterize those symmetric skeletal cauchy complete \( \mathcal{V} \)-categories which is isomorphic to some \( L(M) \), and of the existence of a left adjoint of the functor \( L \), will be considered in the first author's forthcoming paper.

**Remark 5.2.** It is possible to establish a counterpart of the main results in §3 and §4 for the case that weak right ideals are replaced by weak ideals (i.e., those weak right ideals \( I \) satisfying \( RI \subseteq I \)). However, the proofs are similar to the previous sections and left to the readers.
Lemma 5.1. The set $\text{WId}(R)$ of all weak ideals of $R$ is closed under intersections and hence is a complete lattice.

Theorem 5.1. The supremum $\bigvee J_i$ of $J_i \in \text{WId}(R)$ is calculated as follows: $\bigvee J_i = \{ a \in R \mid \exists \text{ central elements } u_i \text{ with } \sum u_i = 1, \text{ such that } u_i a \in J_i(u_i) \}$ and hence $\text{WId}(R)$ is a locale.

Let $R$ be a not-necessarily commutative ring with an identity and $\mathcal{W} = \text{WId}R$ the distributive monoidal category of weak ideals of $R$.

Definition 5.1. If $M$ is a right-$R$-module, then $\mathcal{W}$-category $L(M)$ is defined by

1. the set of objects of $L(M)$ is $M$;
2. if $m, m' \in M$ then the hom $d(m, m')$ is $\text{Ann}_r(m - m')$, where $\text{Ann}_r(m) = \{ r \in R \mid \{ 0 \} = mRr \}$ is a two sided ideal and hence is a weak ideal.

Theorem 5.2. Each $L(M)$ is a symmetric skeletal cauchy complete $\mathcal{W}$-category.

Now let $f : M_1 \to M_2$ for a left $R$- and right $R$-module morphism and let $Lf = f$. Then we have,

Theorem 5.3. $L$ is a functor from the category MOD-$R$ to the category $\mathcal{W}$-$\text{Cat}_{cc}$ of symmetric skeletal cauchy complete $\mathcal{W}$-categories and $\mathcal{W}$-functors.

Acknowledgement. Both of the authors would like to thank Professor Ross Street, Max Kelly and all members of Sydney Category Theory Seminar for their helpful suggestions

References

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