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A NOTE ON CATEGORIES ENRICHED IN QUANTALOIDS AND MODAL AND TEMPORAL LOGIC

by Kimmo I. ROSENTHAL *

RESUME. Dans cet article, l'auteur observe que la sémantique des préfaisceaux relationnels pour la logique des prédicats modale et temporelle, développée par Ghilardi et Meloni, peut être généralisée en utilisant la théorie des catégories enrichies dans un quantaloïde. Cette généralisation exploite l'équivalence entre les préfaisceaux relationnels sur une catégorie localement petite **A** et les catégories enrichies dans le quantaloïde libre **P(A)** établie dans un article antérieur.

In [5,6], Ghilardi and Meloni develop a semantics for modal and temporal predicate logic using the concept of a relational presheaf on a small category **A** (more generally, they consider graphs). By a relational presheaf, we mean a lax functor $F: \mathbf{A}^{op} \rightarrow \mathbf{Rel}$, where **Rel** is the category of sets and relations. A morphism of relational presheaves F and G is a lax natural transformation $R: F \rightarrow G$, whose component relations are actually functions.

Relational presheaves have arisen independently in another context in the theory of quantaloids [14,17]. A quantaloid **Q** is a category enriched in the category of sup-lattices **SL**. Thus, **Q** is locally small with each hom-set $\mathbf{Q}(a,b)$ a complete lattice, such that composition preserves sups in each variable. In [14], Rosenthal showed that for every locally small category **A**, one can construct the free quantaloid **P(A)**, with this construction providing the left adjoint to the forgetful functor from the category of quantaloids to the category of locally small categories.

Furthermore, viewing **P(A)** as a bicategory and using the well-established theory of categories enriched in a bicategory, one can show that categories enriched in **P(A)** correspond precisely to relational presheaves on **A**.

This correspondence leads to the observation central to this paper, namely that the relational presheaf semantics for

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modal and temporal predicate logic, developed by Ghilardi and Meloni in [6], can be generalized so as to be interpreted for categories enriched in any quantaloid. The case where the base bicategory is a free quantaloid **P(A)** reduces to the case of relational presheaves on **A**.

After some preliminaries on quantaloids, we describe the work of Ghilardi and Meloni in the setting of categories enriched in an arbitrary quantaloid. This is done in a natural way by reinterpreting the relevant idea for relational presheaves in the language of enriched category theory.

We then proceed to consider examples. First, we consider the case of relational presheaves and describe the connection between them and categories enriched in free quantaloids. Thus, the investigation in [6] for relational presheaves becomes a special case of our theory. We observe then that Goguen's notion of non-deterministic flow diagram programs [7,8] is described by relational presheaves and thus fits readily into this framework. We conclude our examples by examining the connection with the theory of modules over a quantaloid. Finally, we pose some suggestions for further areas of research.

1. Modal and temporal logic for categories enriched in a quantaloid.

A. Quantaloids are a natural categorical generalization of quantales, which are partially ordered algebraic structures which have received much interest recently in many quarters. (For a thorough introduction to quantales, see Rosenthal [13].) In addition to a series of articles by Rosenthal examining various aspects of quantaloids [14,15,16], quantaloids have been utilized in [1] to keep track of typing on processes, in [2] to study tree automata categorically (also see [16]) and in [11] to investigate distributive categories of relations.

We begin with the definition of quantaloid.

DEFINITION 1.1. A *quantaloid* is a locally small category **Q** such that:

(1) for $a,b \in \mathbf{Q}$, the hom-set $\mathbf{Q}(a,b)$ is a complete lattice.

(2) composition of morphisms in **Q** preserves sups in both variables.

From another perspective, this says that **Q** is enriched in the symmetric, monoidal, closed category **SL** of sup-lattices.

From (2) above, it follows that there are left and right residuations (adjoints to composition) analogous to those of quantale theory. (For details, see [14,16 or 17]).

If **Q** and **S** are quantaloids, then we can define a *quantaloid homomorphism* to be a functor $F: \mathbf{Q} \to \mathbf{S}$ such that on hom-sets it induces a sup-lattice morphism $\mathbf{Q}(a,b) \to \mathbf{S}(Fa,Fb)$; in other words it is just an **SL**-enriched functor.

We now list a few examples of quantaloids to help the reader understand them better.

EXAMPLES. (1) A quantaloid with one object is precisely what is called a unital quantale. Note that if **Q** is a quantaloid, then the hom-sets $\mathbf{Q}(a,a)$ are unital quantales for all $a \in \mathbf{Q}$. (See [13] for a detailed introduction to quantales.)

(2) Both **SL**, the category of sup-lattices, and **Rel**, the category of sets and relations, are quantaloids.

(3) *Free quantaloids.* Let **A** be a locally small category. Define a quantaloid **P(A)** as follows. The objects of **P(A)** are precisely those of **A**. Given $a, b \in \mathbf{A}$, then $\mathbf{P(A)}(a,b) = \mathbf{P}(\mathbf{A}(a,b))$, the power set of the hom-set $\mathbf{A}(a,b)$. If $S: a \to b$ and $T: b \to c$ are sets of morphisms of **A**, let

$$TS = \{g \circ f \mid g \in T, f \in S\}.$$

This operation preserves unions in each variable and thus we have a quantaloid. **P(A)** is called the *free quantaloid on* **A**. This is because **P** defines a monad on the category of locally small categories and functors and quantaloids are precisely the **P**-algebras. (For details, see [14].)

Our interest is in discussing categories enriched in quantaloids. A quantaloid is a locally partially ordered bicategory and we can utilize the theory of enriched categories and functors for bicategories. We recall for the reader the relevant definitions for the special case of quantaloids. (For an excellent introduction to enriched categories and functors, see [10]. See [3] and the references therein for the bicategory case.)

DEFINITION 1.2. Let **Q** be a quantaloid. A set X is a **Q**-*category* iff it comes equipped with the following data:
(1) a function $\rho: X \to \mathrm{Obj}(\mathbf{Q})$ assigning to $x \in X$ an object $\rho(x) \in \mathbf{Q}$;
(2) an enrichment, which assigns to every pair $x,y \in X$ a morphism $X(x,y): \rho(x) \to \rho(y)$ in **Q** such that:
 (a) $i_{\rho(x)} \leq X(x,x)$ for all $x \in X$,
 (b) $X(y,z) \circ X(x,y) \leq X(x,z)$ for all $x,y,z \in X$.
 If $r \in \mathbf{Q}$, we shall denote $\rho^{-1}(r)$ by $X(r)$.

DEFINITION 1.3. Let X and Y be **Q**-categories. A **Q**-*functor* is a function $\varphi: X \to Y$ such that:

(1) $\rho_X(x) = \rho_Y(\varphi(x))$,

(2) $X(x,a) \leq Y(\varphi(x),\varphi(a))$ for all $x,a \in X$.

B. In work begun in [5] and then further developed in [6], Ghilardi and Meloni describe a relational semantics for modal and temporal predicate logic using categories of relational presheaves. Motivated by the connection between relational presheaves and enriched categories established in [14], we shall show that their ideas can be easily transcribed to the setting of categories enriched in a quantaloid **Q**. We shall concentrate on the definition of the operators of "future and past necessity" and "future and past possibility" in the enriched case. We shall not reproduce all of their results nor present a formal description of their notion of *temporal doctrine.*

DEFINITION 2.1. Let **Q** be a quantaloid and X be a **Q**-category. An *attribute* A of X is a collection of sets $A = \{A(r)\}_{r \in \mathbf{Q}}$, where $A(r) \subseteq X(r)$ for all r.

Let **D**(X) denote the set of all attributes of X. We can now define the relevant modal operators on attributes as follows.

DEFINITION 2.2. Let A be an attribute of a **Q**-category X.

(1) "past possibility" $\Diamond$. Let $x \in X(r)$ with $r \in \mathbf{Q}$. $x \in (\Diamond A)(r)$ iff there exists a morphism $f: s \to r$ in **Q** and a $y \in A(s)$ such that $f \leq X(y,x)$.

(2) "future necessity" $\Box$. Let $x \in X(r)$ with $r \in \mathbf{Q}$. $x \in (\Box A)(r)$ iff for every morphism $g: r \to u$ in **Q** and every $z \in X(u)$, we have that $g \leq X(x,z)$ implies $z \in A(u)$.

The definitions of "future possibility" $\Diamond$ and "past necessity" $\Box$ are dual to the above two definitions. We shall include them here for the completeness and then state the basic relationships that are satisfied.

(3) "future possibility" $\Diamond$. Let $x \in X(r)$ with $r \in \mathbf{Q}$. $x \in (\Diamond A)(r)$ iff there exists a morphism $g: r \to u$ in **Q** and $z \in A(u)$ such that $g \leq X(x,z)$.

(4) "past necessity" $\Box$. Let $x \in X(r)$ with $r \in \mathbf{Q}$. $x \in (\Box A)(r)$ iff for every morphism $f: s \to r$ in **Q** and every $y \in X(s)$, we have that $f \leq X(y,x)$ implies $y \in A(s)$.

The following proposition can easily be verified using the definition of enriched category and we omit the proof.

PROPOSITION 2.1. *Let* **Q** *be a quantaloid and let* A *be an attribute of* X*, where* X *is a* **Q**-*category. Then:*

(Adjointness) $\Diamond \vdash \Box$, $\Diamond \vdash \Box$.

(Interdefinability) $\neg(\Diamond A) = (\Box\neg A)$, $\neg(\Diamond A) = (\Box\neg A)$.

(Closure and coclosure properties)

$$A \subseteq \Diamond A \text{ and } \Diamond\Diamond A = \Diamond A,\ A \subseteq \Diamond A \text{ and } \Diamond\Diamond A = \Diamond A,$$
$$\Box A \subseteq A \text{ and } \Box\Box A = \Box A,\ \Box A \subseteq A \text{ and } \Box\Box A = \Box A.$$

(Note: $\neg$ denotes set-theoretic complement.)

An important consideration in Ghilardi and Meloni [6] is the behavior of these operators with regard to inverse image (or *substitution* in logical terms). All of their results carry over. Let $\varphi: X \to Y$ be a **Q**-functor with X, Y **Q**-categories. If A is an attribute of Y, then $\varphi^{-1}(A)$ is the attribute of X with $\varphi^{-1}(A)(r) = \varphi^{-1}(A(r))$ for all $r \in \mathbf{Q}$. The following holds.

PROPOSITION 2.2 *Let* **Q** *be a quantaloid,* X, Y *be* **Q**-*categories and* $\varphi: X \to Y$ *a* **Q**-*functor. Let* $A \in \mathbf{D}(Y)$. *Then*

(1) $$\Diamond(\varphi^{-1}(A)) \subseteq \varphi^{-1}(\Diamond A),$$

(2) $$\Diamond(\varphi^{-1}(A)) \subseteq \varphi^{-1}(\Diamond A),$$

(3) $$\varphi^{-1}(\Box A) \subseteq \Box(\varphi^{-1}(A)),$$

(4) $$\varphi^{-1}(\Box A) \subseteq \Box(\varphi^{-1}(A)).$$

PROOF. These all follow easily from the fact that φ is a **Q**-functor. We shall prove the first one and leave the rest for the reader. $x \in \Diamond(\varphi^{-1}(A))(r)$ iff there exist

$$f: s \to r \text{ in } \mathbf{Q} \text{ and } z \in \varphi^{-1}(A)(s) \text{ with } f \leq X(z,x).$$

Since φ is a **Q**-functor, we have $X(z,x) \leq Y(\varphi(z),\varphi(x))$ and since $\varphi(z) \in A(s)$, we have that $\varphi(x) \in \Diamond A$, proving that $x \in \varphi^{-1}(\Diamond A)$.

If $\varphi: X \to Y$ is a **Q**-functor, then $\varphi^{-1}: \mathbf{D}(Y) \to \mathbf{D}(X)$ has left and right adjoints

$$\exists\varphi: \mathbf{D}(X) \to \mathbf{D}(Y) \text{ and } \forall\varphi: \mathbf{D}(X) \to \mathbf{D}(Y).$$

The various relationships that these satisfy in [6] with regard to attributes still hold in this setting utilizing the definition of **Q**-functor. We shall not go into the details here.

Ghilardhi and Meloni also dicuss various notions such as open maps and discrete and étale objects, all of which require the existence of products $X \times Y$. We shall now discuss products briefly in the context of **Q**-categories. We need a preliminary definition.

DEFINITION 2.3. Let **Q** be a quantaloid. **Q** is called *weakly distributive* iff:

(1) $f\circ(g\wedge h)\leq(f\circ g)\wedge(f\circ h)$, whenever f,g,h are morphisms of **Q** with f composable with g and h.

(2) $(g\wedge h)\circ k\leq(g\wedge k)\circ(h\wedge k)$, whenever g,h,k are morphisms of **Q** with k composable with g and h.

Let X and Y be **Q**-categories with **Q** weakly distributive. We can define X×Y as follows:

If $a\in\mathbf{Q}$, let $(X\times Y)(a) = X(a)\times Y(a)$.

If $(x,y)\in(X\times Y)(a)$ and $(x',y')\in(X\times Y)(b)$ define the enrichment

$$(X\times Y)((x,y)(x',y')) = X(x,x')\wedge Y(y,y').$$

Weak distributivity must be used to verify Definition 1.2 (2b).

PROPOSITION 2.3. *If* **Q** *is a weakly distributive quantaloid, then* X×Y *is a* **Q**-*category.*

Thus, if **Q** is weakly distributive, we can discuss products in this setting. One can easily see that free quantaloids **P(A)** are weakly distributive.

2. Examples.

A. *Relational presheaves.*

We are interested in the special case where **Q** is of the form **P(A)**. **P(A)**-categories turn out to be related to lax relation-valued functors $\mathbf{A}^{op}\to\mathbf{Rel}$, which were called "non-deterministic functors" in [14] and "relational presheaves" in [5,6]. We shall define relational presheaves and their morphisms and then state the main result, which served as the motivation for the general considerations of §1.

DEFINITION 3.1. Let **A** be a locally small category. A *relational presheaf* on **A** is a lax functor $F:\mathbf{A}^{op}\to\mathbf{Rel}$, where laxity means:

(1) $F(f)\circ F(g)\subseteq F(g\circ f)$ for all composable morphisms f,g of **A**.

(2) $\Delta_{F(a)}\subseteq F(1_a)$, where Δ is the diagonal relation and $a\in\mathbf{A}$.

DEFINITION 3.2. Let $F:\mathbf{A}^{op}\to\mathbf{Rel}$ and $G:\mathbf{A}^{op}\to\mathbf{Rel}$ be relational presheaves on **A**. A *relational presheaf morphism* (or *rp-morphism*) $R:F\to G$ is a lax natural transformation such that R_a is a function from $F(a)$ to $G(a)$ for all $a\in\mathbf{A}$.

Composition of rp-morphisms $R:F\to G$ and $S:G\to H$ is defined by $(S \circ R)_a = S_a \circ R_a$. Relational presheaves together with rp-morphisms form a category denoted **R(A)**. If we let **P(A)-Cat** denote the category of **P(A)**-categories and **P(A)**-functors, then in [14], it was proved that these two categories are equivalent. We proceed as follows.

Let $F: \mathbf{A}^{op} \to \mathbf{Rel}$ be a relational presheaf. Define a **P(A)**-category X_F by $X_F(a) = F(a)$ and, if $x \in F(a)$, $y \in F(b)$, then

$$X_F(x,y) = \{f: a \to b \mid (y,x) \in F(f)\}.$$

Let $R: F \to G$ be an rp-morphism of relational presheaves on **A**. Define $\partial_R: X_F \to X_G$ as follows: If $x \in X_F(a) = F(a)$, let $\partial_R(x) = R_a(x)$. Then, $\partial_R: X_F \to X_G$ is a **P(A)**-functor.

We can now define a functor $\Phi: \mathbf{R(A)} \to \mathbf{P(A)\text{-}Cat}$ by taking $\Phi(F) = X_F$, where $F \in \mathbf{R(A)}$ and $\Phi(R) = \partial_R$ for $R: F \to G$ an rp-morphism.

THEOREM 3.1. $\Phi: \mathbf{R(A)} \to \mathbf{P(A)\text{-}Cat}$ *is an equivalence of categories.*

Details of all the above can be found in [14] or [17]. We now describe the transition from **P(A)**-categories to relational presheaves.

If X is a **P(A)**-category, define a relational presheaf F_X by $F_X(a) = X(a)$ and if $f: a \to b$ is a morphism in **A**, then the relation $F(f)$ is defined by $(y,x) \in F(f)$ iff $f \in X(x,y)$.

If $X, Y \in \mathbf{P(A)\text{-}Cat}$ and $\partial: X \to Y$ is a **P(A)**-functor, we can define $R_\partial: F_X \to F_Y$ by $R_{\partial,a}(x) = \partial(x)$ for $x \in F(a)$.

Using the functor Φ from Theorem 3.1, the description of the four operators $\cdot\!\diamond$, $\diamond\!\cdot$, $\cdot\!\Box$ and $\Box\!\cdot$ found in [6] transcribes exactly to the definitions of Definition 2.2.

The interpretation that can be assigned (see Ghilardi and Meloni [6]) is that an element of the category **A** is to be viewed as a "state" b and a morphism f: $b \to c$ is a "temporal development" from state b to state c; thus a morphism of **P(A)** is a set of temporal developments. If X is a **P(A)**-category, given $b \in \mathbf{A}$, $X(b)$ can be thought of as a set of events occuring at b and if $x \in X(b)$, $y \in X(c)$, then $X(x,y)$ is the set of all temporal developments, along which it is possible to relate event y to event x (or obtain y as a "descendant" of x).

Special case: Non-deterministic flow diagram programs. In [7] (and briefly in [8]), Goguen presents a way of describing flow diagram programs using essentially the theory of relational presheaves. Hence, again appealing to Theorem 3.1, these can be described using enriched category theory.

In this example, the elements of **A** are thought of as "states of control" and the morphisms of **A** are "transitions of control states". If X is a **P(A)**-category and $b \in$ **A**, then X(b) is a set of memory states of computations at state b, and letting x,y be as above, X(x,y) is the set of all transitions along which computation y may be obtained from computation x.

B. **Q**-*modules.*

Let **Q** be a quantaloid. In a very natural way, one can define the notion of a left **Q**-module as a lattice theoretic generalization from ring theory. Modules over a quantale (in particular, over a frame) were a key element in the descent theory for topoi developed by Joyal and Tierney [9] and modules are also central to the work of Abramsky and Vickers on process semantics [1]. (**Q**-modules are studied in some detail in [17].) We shall utilize the fact that modules can be viewed as **Q**-enriched categories.

We begin with the definition of a left **Q**-module for a quantaloid **Q**.

DEFINITION 3.3. Let **Q** be a quantaloid. A *left* **Q**-*module* M consists of the following data:

(1) for every $a \in \mathbf{Q}$, a sup-lattice M_a.

(2) for $a,b \in \mathbf{Q}$, we have an action $\mathbf{Q}(a,b)\times M_a \to M_b$, denoted by $(f,x) \mapsto f.x$ satisfying:

$i_a.x = x$ (for $a = b$),
$(f \circ g).x = f.(g.x)$ (for g composable with f),
$f.(\sup_\alpha x_\alpha) = \sup_\alpha (f.x_\alpha)$ (for all $\{x_\alpha\} \subset M_a$),
$(\sup_\alpha f_\alpha).x = \sup_\alpha (f_\alpha . x)$ (for all $\{f_\alpha\} \subset \mathbf{Q}(a,b)$).

DEFINITION 3.4. Let **Q** be a quantaloid and let M and N be left **Q**-modules. A **Q**-*module homomorphism from* M *to* N is given by the following data: For $a \in$ **Q**, we have a sup-lattice morphism ω_a: $M_a \to N_a$ satisfying $f.\omega_a(x) = \omega_b(f.x)$ for all morphisms $f \in \mathbf{Q}(a,b)$.

We thus obtain a category **Q**-MOD of left **Q**-modules, which is in fact a quantaloid. We shall just refer to modules, as they will all be understood to be left modules.

Equivalently, a left **Q**-module can be viewed as an **SL**-enriched functor $\mathbf{Q} \to \mathbf{SL}$, and a homomorphism is an **SL**-natural transformation $\omega: M \to N$. (Note: Because of how we have been writing composition, our notion of left module is a covariant functor, whereas in [1] it is a contravariant functor.)

If M is a **Q**-module, then it can be viewed as a **Q**-enriched category as follows. Let $M(a) = M_a$, and if $x \in M(a)$ and $y \in M(b)$, define the enrichment

$$M(x,y) = \sup\{f: a \to b \mid f.x \leq y\}.$$

It is easy to check that this makes M into a **Q**-category.

Now looking at our modal operators in this setting, we have the following interesting result.

PROPOSITION 3.1. *Let* **Q** *be a quantaloid,* M *be a* **Q***-module and let* A *be an attribute of* M. *Then*

(1) $\Diamond A$ *and* $\Diamond A$ *are both* **Q***-modules,*

(2) $\Box A$ *is a* **Q***-module.*

PROOF. (1) Let $x \in (\Diamond A)(r)$ and let $g: r \to s$ be a morphism of **Q**. We must show that $g.x \in (\Diamond A)(s)$. Since $x \in (\Diamond A)(r)$, there exist an $f: t \to r$ in **Q** and a $y \in A(t)$ such that $f \leq M(y,x)$, i.e., $f.y \leq x$. Then, we have that

$$(g \circ f).y = g.(f.y) \leq g.x, \text{ and thus } g \circ f \leq M(y, g.x).$$

Because $y \in A(t)$ and $g \circ f: t \to s$, it follows that $g.x \in (\Diamond A)(s)$, proving that $\Diamond A$ is a **Q**-module.

To prove that $\Diamond A$ is a module, let x and g be as above. We must show that $g.x \in (\Diamond A)(s)$. Since $x \in (\Diamond A)(r)$, there exist $h: r \to u$ and $z \in A(u)$ such that $h \leq M(x,z)$, i.e., $h.z \leq x$. Consider the morphism $(g \to_l h): s \to u$. This is the so-called left residuation in the quantaloid **Q** alluded to after Definition 1.1. (Given $k: s \to u$ in **Q**, then $k \leq (g \to_l h)$ iff $k \circ g \leq h$; in particular $(g \to_l h) \circ g \leq h$.) Thus

$$(g \to_l h).(g.x) = ((g \to_l h) \circ g).x \leq h.x \leq z,$$

thus $(g \to_l h) \leq M(g.x, z)$ proving that $g.x \in (\Diamond A)(s)$ and hence $\Diamond A$ is a **Q**-module.

(2) We shall now prove that $\Box A$ is a **Q**-module. Let x be in $(\Box A)(r)$ and let $g: r \to s$ be a morphism of **Q**. We know that for all morphisms $h: r \to u$ and for all $z \in M(u)$, if $h.x \leq z$ then we have $z \in A(u)$. Let $k: s \to u$ and let $z \in M(u)$. We must show that if $k.(g.x) \leq z$, then $z \in A(u)$. But, $k.(g.x) = (k \circ g).x$ and hence letting $h = k \circ g$ above, we can conclude that $g.x \in (\Box A)(s)$, as desired.

It does not seem that $\Box A$ has to be a **Q**-module; at least the obvious arguments to try do not work.

We shall call a **Q**-sub-module N of a **Q**-module M *upper closed* iff N_r is an upper set in the order it inherits from M_r for all r. The following is easily established.

COROLLARY 3.1. *Let* **Q** *be a quantaloid,* M *be a* **Q**-*module and let* A *be an attribute of* M. *Then*

(1) ◇A *is the smallest upper closed sub-module of* M *containing the attribute* A.

(2) □A *is the largest upper closed sub-module of* M *contained in the attribute* A.

This particular example of modules and modal operators seems to merit further study.

C. *Future possibilities.*

It seems that the temporal structures studied in [4] can be fruitfully generalized to the setting of quantaloids. They consider categories enriched in quantales and clearly one possibility is to generalize to quantaloids. The objects of the base quantaloid **Q** could be thought of as stages of some process and the morphisms as temporal developments between stages. If $a \in$ **Q**, the quantales **Q**(a,a) describe time while at stage a.

If X is a **Q**-category, another direction of investigation is to allow for the attributes **D**(X) of X to have a more general structure instead of the Boolean algebra structure on each component A(r) of an attribute A. We could allow A(r) to have the structure of a quantale for each $r \in$ **Q**, e.g. it could be a frame (perhaps there is a relation to the work of Reyes and Zolfaghari [12]), or if one were interested in linear logic, A(r) could be a Girard quantale for each r. One can then try to develop definitions of the operators ◇, ◇, □ and □ in this context.

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