CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

GAVIN C. WRAITH

Using the generic interval

Cahiers de topologie et géométrie différentielle catégoriques, tome 34, nº 4 (1993), p. 259-266

<http://www.numdam.org/item?id=CTGDC_1993__34_4_259_0>

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

VOL. XXXIV-4 (1993)

USING THE GENERIC INTERVAL

by Gavin C. WRAITH

Résumé. Tout treillis distributif peut se compléter librement en algebre Booléenne. Pour réaliser le complété de l'intervalle [0,1] comme homéomorphe d'une sphère de dimension infinie, il est pratique d'utiliser le résultat qui identifie le topos classifiant des intervalles avec la catégorie des ensembles simpliciaux, et l'intervalle générique avec le 1-simplexe standard. Nous démontrons á partir de là que l'espace des partitions finies d'un intervalle est homéomorphe à l'espace projectif réel de dimension infinie.

Introduction

The key idea behind "fuzzy sets" is the replacement of the unit interval I for $\{0,1\}$ as the codomain for characteristic functions of subsets. The difficulty with this is that I is not a Boolean algebra. By analogy with probability, the assumption is usually made that $1-p$ is the complement of p, but this interpretation means that the usual rules of logic have to go out of the window.

Of course I is a distributive lattice, so the *and* and *or* operations behave correctly. Every distributive lattice D embeds in a Boolean algebra, $B(D)$, its Boolean completion, by a homomorphism of lattices that is universal for homomorphisms from D to Boolean algebras. One might ask "what does $B(D)$ look like?".

As I is a topological distributive lattice, there are general reasons, which will be explained shortly, why we can expect $B(I)$ to be a topological Boolean algebra. The canonical embedding $i : I \to B(I)$ defines a path joining the bottom element $i(0)$ to the top element $i(1)$ so we get a homotopy between a constant function and the identity function. In consequence $B(I)$ is contractible. This argument shows that for topological lattices, path connectedness is equivalent to contractibility. Furthermore, negation is a fixedpoint-free involution on $B(I)$. So $B(I)$ is the total space for a universal double covering. The double covering given by the projection from the infinite dimensional sphere to infinite dimensional real projective space immediately suggests itself, as the standard contractible double-covering.

This result, by itself is not remarkable, but a method of establishing it, using the generic interval, in the sense of classifying toposes, seems to me to have some interest. This method was suggested to me by F.W.Lawvere over a coffee table at Murten in Switzerland many years ago.
I am grateful to the referee for reminding me that it was A.Joyal who first pointed out, in 1974 at the Isle of Thorns, the fact that the category of simplicial sets is the classifying topos for *intervals*.

The earliest reference in which the details of this are spelled out in print appears to be [3], and the first in book form to be Chapter 6 of [6]. It seemed to me appropriate to publish this note as a coda.

Relatively Free Algebras

Recall ([5],[4]) that a finitary algebraic theory T is a category with finite products, in which all the objects are finite products of a single *base* object. The category *T-Mod* of T-models is the category of product preserving set-valued functors on T, and natural transformations.
A map $f : T \to T'$ of theories is defined to be a product preserving functor that takes the base object of T to that of T'. Composition with f defines the functor $f^{\#}$, *forget along f*, from *T'-Mod* to *T-Mod*. This functor has a right adjoint $f_{\#}$, *free along f*, from *T-Mod* to *T'-Mod*.
We are concerned with the case

$$T = \textit{Distributive Lattices}$$

$$T' = \textit{Boolean Algebras}$$

$$f = \textit{the standard inclusion}$$

$$f^{\#} = \textit{forget negation}$$

$$f_{\#} = \textit{Booleanization}$$

If X is a T-model, we can write down the construction of (the underlying set of) $f_\#(X)$ quite generally as a colimit of finite products of (the underlying set of) X. The formula for $f_\#(X)$ is the coequalizer of the pair of functions

$$u, v \ : \ \sum_{n,m} T'(n, m) \times T(m, n) \times X^m \rightarrow \sum_{n} T'(n,1) \times X^n$$

where $T(m, n)$ denotes the set of n-ples of m-ary operations of T, u is given by the T-action $T(m, n) \times X^m \rightarrow X^n$, and v is given by composing $f : T(m, n) \rightarrow T'(m, n)$ with composition of operations in T'.

We can draw two lessons from the form of this construction. The first is that if X is a topological T-algebra, we can interpret this construction in the category of topological spaces and continuous maps, with $T(m, n)$ and $T'(m, n)$ taken to be discrete spaces, so that $f_\#(X)$ acquires a topology. The forgetful functor from topological spaces and continuous maps to sets and functions has both left and right adjoints, and so preserves both limits and colimits. This explains why we may expect $B(I)$ to be a topological Boolean algebra.

The second draws on the fact that only finite products are used in the construction. By taking $T(m, n)$ and $T'(m, n)$ to be constant objects, we can interpret the construction $f_\#(X)$ when X is an object in a topos (see [4]), and this construction is preserved by inverse image parts of geometric morphisms, which preserve arbitrary colimits and finite limits.

If U is the generic T-model, and X is classified by a geometric morphism x (so that X is $x^*(U)$, up to isomorphism) then we may take $f_\#(X)$ to be $x^*(f_\#(U))$. This leads us on from asking what $B(I)$ looks like to wider questions.

Dualities

The category *FBOOL*, of finite Boolean algebras and homomorphisms, is the dual (that is, equivalent to the opposite category) of *FSET*, the category of finite sets and functions. The most elegant way of expressing this is to denote by *2* the *schizophrenic object* which is both the two-element set $\{0,1\}$ in *FSET* and the Boolean algebra $\{0,1\}$ in *FBOOL*. Then the contravariant functors

$$Hom(\_,2) : FBOOL \to FSET \qquad Hom(\_,2) : FSET \to FBOOL$$

are adjoint equivalences. This is the finite form of Stone duality [2].

Let *FDLAT* denote the category of finite distributive lattices and homomorphisms, and let *FPO* denote the category of finite partially ordered sets and monotone functions. Again, denote by 2 the partially ordered set $\{0,1\}$, with $0 < 1$, as an object of *FPO*, and also the finite distributive lattice $\{0,1\}$. The category *FDLAT* is dual to *FPO*, and again, the contravariant functors

$$Hom(\_,2) : FDLAT \to FPO \quad . \; Hom(\_,2) : FPO \to FDLAT$$

are adjoint equivalences.

Dual to the functor $disc : FSET \to FPO$, which makes discrete partially ordered sets, we have the functor $FBOOL \to FDLAT$ which forgets negation, and remembers only the lattice structure. Right adjoint to $disc : FSET \to FPO$ we have the functor $forget : FPO \to FSET$ which forgets the partial ordering, whose dual is the Booleanization functor $B : FDLAT \to FBOOL$, which is left adjoint to the functor $FBOOL \to FDLAT$ mentioned above. So we have that

$$B(D) = Hom(forget(Hom(D,2)),2)$$

i.e. the Boolean algebra of sets of lattice homomorphisms $D \to 2$, describes the Booleanization of a finite distributive lattice D. Every distributive lattice is the union of its finite sublattices, so in general we can describe the Booleanization $B(L)$ of a (not necessarily finite) distributive lattice L as the filtered colimit of the $B(D)$'s as D ranges over the finite sublattices of L.

We can specialize the duality between *FDLAT* and *FPO* by restricting attention to linearly ordered objects. Let *FLLAT* denote the category of finite linearly ordered distributive lattices and homomorphisms. Equivalently we can describe the objects of *FLLAT* as finite linear orders that have maximum and minimum elements (which need not be distinct), because linear orders automatically have binary meets and joins. Similarly, we can describe the maps of *FLLAT* as monotone maps that preserve maximum elements and preserve minimum elements, because the preservation of meets and joins is automatic for linear orders.

Let *FLO* denote the category of finite linear orders and monotone functions. Then the duality between *FDLAT* and *FPO* restricts to a duality between *FLLAT* and *FLO*. The ordinal sum $X+Y$ of two linear orders - the disjoint union of X and Y with $x < y$ for all x in X and y in Y - makes *FLO* into a monoidal category with the empty linear order as unit object. We obtain a dual monoidal structure $A\&B$ on *FLLAT* got by identifying the maximum element 1 of A with the minimum element 0 of B, with the one element object of *FFLAT* as unit object.

We denote by *FINT* the full subcategory of *FLLAT* given by the objects whose maximum and minimum elements are distinct. We call a linear order with distinct maximum and minimum elements an *interval*, so *FINT* is the category of finite intervals and homomorphisms. Dual to *FINT* is the full subcategory of *FLO* of nonempty finite linear orders, which we call Δ - it is usually called the *simplicial category*. We can summarise these dualities by the following diagram:

$$\begin{array}{ccccccc} \Delta & \rightarrow & FLO & \rightarrow & FPO & \rightarrow & FSET \\ | & & | & & | & & | \\ FINT & \rightarrow & FLLAT & \rightarrow & FDLAT & \rightarrow & FBOOL \end{array}$$

The generic interval

The classifying topos for intervals can be described equivalently as the category of *SET*-valued functors on *FINT*, or as $\tilde{\Delta}$, presheaves on Δ, i.e. the category *SSET* of simplicial sets. See sections 7 and 8 of chapter VIII of [1] for a complete treatment, including details of a geometric presentation of the theory of intervals. If J is an interval in a topos E, then its classifying geometric morphism has for its inverse image functor "geometric realization of simplicial sets using J".

In much the same way, we can assert that the classifying topos for distributive lattices is $\widetilde{FPO}$, the category of presheaves on FPO, and the classifying topos for Boolean algebras is the category $\widetilde{FSET}$ of presheaves on $FSET$, with $Hom(\_,2)$, suitably interpreted, as the generic model, in either case. In $SSET$, the generic interval $Hom(\_,2)$ is the standard 1-simplex Δ_1.

Regarding an interval as a distributive lattice, we may enquire what object in $SSET$ is $B(\Delta_1)$, the Booleanization of the generic interval? From our discussion above, we can see that its classifying geometric morphism $\tilde{\Delta} \to \widetilde{FSET}$ is induced by the forgetful functor
$forget : \Delta \to FSET$.
In other words $B(\Delta_1)$ is the presheaf

$$Hom(forget(\_), forget(2))$$

with the canonical inclusion

$$\Delta_1 \to B(\Delta_1)$$

given by the application of *forget* to maps into *2*:

$$Hom(\_,2) \to Hom(forget(\_), forget(2))$$

Suppose that L is a finite nonempty linear order with k elements. Then $Hom(L,2)$ is the interval with $k+1$ elements, and so k *edges*. To find the smallest Boolean algebra this can sit inside, imagine the interval bent at right angles at each junction so as to form a path along the edges of a unit cube in k dimensions, from the origin $(0,0,...,0)$ to $(1,1,...,1)$. The vertices of the cube clearly form a Boolean algebra, which is just $Hom(forget(L), forget(2))$.

$$L : 0 \to x \to 1 \quad \Rightarrow \quad \begin{array}{ccc} 0 & \to & x \\ \downarrow & & \downarrow \\ x' & \to & 1 \end{array}$$

The unit interval I is the union of its finite subintervals. We get $B(I)$ as the filtered colimit of all the k-cube Boolean algebras we get from considering the $k+1$-element subintervals of I. This is rather hard to picture. Instead, we can realize $B(I)$ as the geometric realization of the simplicial set $B(\Delta_1)$. We have seen that this is just the presheaf $Hom(forget(\_), forget(2))$ on Δ. Its n-simplexes are therefore $n + 1$-vectors with 0's and 1's for components. An n-simplex belongs to the simplicial subset Δ_1 if the $n + 1$-vector is ordered, so that no 0 follows a 1. The face operators remove a component, and the degeneracy operators create a duplicate component following strictly after. We see immediately that $B(\Delta_1)$ has only two nondegenerate n-simplexes, namely (0,1,0 ...) and (1,0,1 ...), for each n. Furthermore, negation interchanges them. It is clear that $B(\Delta_1)$ is the standard simplicial decomposition of the infinite dimensional sphere, equivariant for the antipodal involution.

Partitions of I

We have an alternative description of $B(I)$. Consider the functions from (0,1] to $\{0,1\}$ whose graphs are finite unions of half-open horizontal intervals, open at the left, closed at the right. These have a Boolean algebra structure induced from that of $\{0,1\}$. We may identify the unit interval with the decreasing functions. For each t in I let f_t be the function given by

$$f_t(x) = 1 \text{ if } x < t$$
$$= 0 \text{ otherwise}$$

It is clear that we have another description of $B(I)$. Negation replaces a function f by $1 - f$. What remains if we identify f with $1 - f$? The answer is the finite set of points of discontinuity of f (or $1 - f$). These sets determine the finite partitions of I. The partitions of I into $n + 1$ pieces form a space homeomorphic to real-projective n-space.

I am grateful to R. Brown for bringing to my attention the functor defined in [1]. This is a product preserving endofunctor on the category of topological spaces and continuous maps, equipped with a natural embedding of the identity functor, whose values are all contractible.

The function space described above is just the value of this functor on $\{0,1\}$.

References

1. R. Brown & S. Morris , Embeddings in Contractible or Compact Objects, *Colloquium Mathematicum* 38, pp 213-222, 1978.

2. P.T.Johnstone , *Stone Spaces*, Cambridge University Press 1982.

3. P.T.Johnstone , On a topological topos, *Proc. London Math. Soc.* 38, 1979.

4. P.T.Johnstone & G.Wraith, Algebraic Theories in Toposes, Indexed Categories and Their Applications, *Lecture Notes in Mathematics*, No. 661, pp 141-242, Springer Verlag 1978.

5. F.Lawvere, Functorial Semantics of algebraic Theories, Ph.D. Thesis Columbia University 1963; summarized in *Proc. Nat. Acad. Sci. U.S.A.* Vol 50, pp 869- 872, 1963.

6. Saunders MacLane & Ieke Moerdijk, *Sheaves in Geometry and Logic,* Springer-Verlag 1992.

G.C.Wraith
Department of Mathematics,
Sussex University,
Falmer, East Sussex, BN1 9QH, UK

email: gavinw@syma.sussex.ac.uk