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PULLBACKS EQUIVALENT TO PSEUDOPULLBACKS

by André JOYAL and Ross STREET

RESUME. Le but de cette note est de caractériser les foncteurs le long desquels le "pullback" est équivalent à un "pseudopullback", de généraliser ce résultat à des 2-catégories arbitraires, et d'en déduire un résultat sur les "pushouts" de foncteurs.

Introduction

The difference between pullbacks and pseudopullbacks (or iso-comma objects) is emphasised in those works where the latter are introduced. It is certainly true that they are not equivalent in general, although some particular cases of equivalence have been noted [2] in the opposite of the 2-category of categories. The purpose of this note is to:

(a) characterise those functors pullback along which is canonically equivalent to pseudopullback,

(b) lift this result up to arbitrary 2-categories, and

(c) deduce a result about pushouts of functors.

The *pseudopullback* of a pair of functors $f : A \longrightarrow C$, $g : X \longrightarrow C$ is the category H whose objects are triples (a, θ, x) consisting of objects $a \in A$, $x \in X$ and an isomorphism $\theta : fa \xrightarrow{\cong} gx$ in C, and whose arrows $(\alpha, \xi) : (a, \theta, x) \longrightarrow (b, \phi, y)$ consist of arrows α in A, ξ in X such that

$$g\xi \circ \theta = \phi \circ f\alpha .$$

Let P denote the pullback of f, g. There is a canonical comparison functor

$$n : P \longrightarrow H, \qquad n(a, x) = (a, 1, x), \qquad n(\alpha, \xi) = (\alpha, \xi).$$

Clearly n is fully faithful.

Let us say that a functor $g : X \longrightarrow C$ *has invertible-path lifting (ipl)* when, for all invertible arrows $\gamma : gx \longrightarrow c$ in C, there exists an invertible arrow $\phi : x \longrightarrow x'$ in X such that $gx' = c$ and $g\phi = \gamma$. This is reminiscent of the homotopy lifting property in algebraic topology [1 ; p.66]; for us, the chaotic category I with two objects 0, 1 plays the role of the unit interval.

Theorem 1 *Given a functor* $g : X \longrightarrow C$, *the canonical comparison functor* n *is an equivalence for all functors* $f : A \longrightarrow C$ *if and only if* g *has invertible-path lifting.*

Proof For sufficiency it remains to see that the condition implies that n is essentially surjective on objects. Take $(a, \theta, x) \in H$ and apply the ipl condition to $\theta^{-1} : gx \longrightarrow fa$ to obtain $\phi : x \longrightarrow x'$ with $gx' = fa$ and $g\phi = \theta^{-1}$. This gives an isomorphism $(1, \phi) : (a, \theta, x) \longrightarrow n(a, x)$.

Conversely, use the fact that n is essentially surjective when f is the constant functor $x : \mathbf{1} \longrightarrow C$.$_{\text{qed}}$

Fibrations and opfibrations provide examples of functors having ipl. Also, any functor which is surjective on objects and is full on isomorphisms has ipl. A more interesting class of examples (perhaps suggested by [1 ; p.57 Exercise E1]) is provided by the next result in which [X, T] denotes the usual category of functors from X to T and natural transformations between them.

Theorem 2 *The functor* $m : C \longrightarrow X$ *is injective on objects if and only if, for all categories* T, *the functor*

$$[m, T] : [X, T] \longrightarrow [C, T],$$

given by composition with m, *has invertible-path lifting.*

Proof Suppose m is injective on objects. Take an invertible natural transformation $\gamma : km \longrightarrow h$ where $k : X \longrightarrow T$, $h : C \longrightarrow T$ are functors. We define $k' : X \longrightarrow T$ on objects by

$$k'x = \begin{cases} hc & \text{when } mc = x \text{ for some } c \in C, \\ kx & \text{otherwise}. \end{cases}$$

This is valid since m is injective on objects. Also put

$$(\phi_x : kx \to k'x) = \begin{cases} \gamma_c & \text{when } mc = x, \\ 1_{kx} & \text{otherwise}. \end{cases}$$

Then $k' : X \longrightarrow T$ becomes a functor by defining it on arrows in the unique way such that $k'm = h$ and ϕ is natural. Then $\phi m = \gamma$, as required.

Conversely, let Z_m denote the "mapping cylinder" of m; that is, Z_m is the pushout of $\partial_0 : C \longrightarrow C \times I$ and m. Since the objects of the pushout of functors are the elements of the set theoretical pushout of the object functions, $\partial_1 : C \longrightarrow Z_m$ is injective on objects. Apply ipl of the functor $[m, Z_m]$ to see that $mc = mc'$ implies $\partial_1 c = \partial_1 c'$; hence, $c = c'$.$_{\text{qed}}$

We now obtain some implications of Theorem 1 for any 2-category $\mathcal{K}$, but first we need to review some standard definitions. A *bipullback* of a pair of arrows $f : A \longrightarrow C$, $g : X \longrightarrow C$ in $\mathcal{K}$ is an object H together with an equivalence of categories between $\mathcal{K}(K, H)$ and the pseudopullback of the functors $\mathcal{K}(K, f)$, $\mathcal{K}(K, g)$ such that the equivalence is pseudonatural in objects $K \in \mathcal{K}$.

An arrow $u : U \longrightarrow V$ in $\mathcal{K}$ is an *equivalence* when there exist an arrow $v : V \longrightarrow U$ and invertible 2-cells $u \circ v \cong 1_V$, $1_V \cong v \circ u$. By changing one of them if necessary, we can also suppose that these invertible 2-cells satisfy the conditions for an adjunction $u \dashv v$. It follows that $u : U \longrightarrow V$ is an equivalence in $\mathcal{K}$ if and only if each of the functors

$$\mathcal{K}(K, u) : \mathcal{K}(K, U) \longrightarrow \mathcal{K}(K, V)$$

is an equivalence.

Clearly the bipullback of a given pair of arrows is unique up to equivalence, and any object equivalent to a bipullback is also a bipullback of the pair.

An arrow $g : X \longrightarrow C$ in $\mathcal{K}$ has *invertible-path lifting* when each invertible 2-cell

$$\begin{tikzcd} & K \arrow[dl, "x"'] \arrow[dr, "c"] \arrow[d, phantom, "\cong"] & \\ X \arrow[rr, "g"'] & {} & C \end{tikzcd}$$

is, for some invertible 2-cell as indicated, equal to

$$\begin{tikzcd} & K \arrow[dl, "x"'] \arrow[dl, bend left, "x'"] \arrow[dr, "c"] & \\ X \arrow[rr, "g"'] & {} & C \end{tikzcd} \quad \text{with } x \cong x' .$$

This is equivalent to the condition that each of the functors $\mathcal{K}(K, g)$ have ipl. Hence, Theorem 2 says precisely that functors which are injective on objects are those having ipl in the opposite of the 2-category of categories.

Corollary 1 *In any 2-category, a pullback of a pair of arrows is also a bipullback if one of the arrows has ipl.*

Corollary 2 *The pushout of functors* $m : C \longrightarrow X$, $t : C \longrightarrow A$ *is canonically equivalent to their pseudopushout if* m *is injective on objects.*

Since the bipullback of an equivalence is an equivalence, we also have:

Corollary 3 *The pushout of an equivalence of categories, along a functor which is injective on objects, is an equivalence of categories.*

Remark Using a "co-mapping cylinder", we see that any functor $g : X \longrightarrow C$ can be written as a composite $g = p \circ i$ where $i : X \longrightarrow G$ is an equivalence and $p : G \longrightarrow C$ has ipl. Here G is the pseudopullback of g and 1_C, and i is the canonical comparison from the pullback X of g and 1_C. This can be used to see that any cospan f, g of functors can be replaced by an equivalent cospan whose pullback gives a bipullback for f, g. [This carries over to 2-categories too in the obvious way.]

References

[1] E.H. Spanier, *Algebraic Topology* (McGraw-Hill, 1966).

[2] R.H. Street, Fibrations in bicategories, *Cahiers topologie et géométrie différentielle* 21 (1980) 111-160; 28 (1987) 53-56.

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