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Note on a submonadicity

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RÉSUMÉ : Les catégories internes sont caractérisées comme certaines classes d'algèbres d'une monade

It is known [2] that the category *Simpl* $\mathbf{E}$ of simplicial objects in $\mathbf{E}$ is monadic above the category *Sp Simpl* $\mathbf{E}$ of split augmented simplicial objects in $\mathbf{E}$. From this monadicity is extracted in [1] the monadicity of the category *Grd* $\mathbf{E}$ of internal groupoids in $\mathbf{E}$ above the category *Pt* $\mathbf{E}$ of split epimorphisms in $\mathbf{E}$, when $\mathbf{E}$ is left exact. Now, via the nerve functor N, the category *Cat* $\mathbf{E}$ of internal categories in $\mathbf{E}$ has an intermediate position : *Grd* $\mathbf{E}$ < *Cat* $\mathbf{E}$ < *Simpl* $\mathbf{E}$. The aim of this note is to precise the place of *Cat* $\mathbf{E}$ with respect to this monadic complex.

If we denote by U the forgetful functor *Simpl* $\mathbf{E} \to$ *Sp Simpl* $\mathbf{E}$, then, given an internal category X_1 in $\mathbf{E}$, the split augmented simplicial object UNX_1 is the "nerve" of a category with a given choice of initial objects in each connected component. Let us denote by *In Cat* $\mathbf{E}$ the category whose objects are the internal categories in $\mathbf{E}$ equipped with such a choice and whose morphisms are the choice preserving functors. Let $\overline{U}$: *Cat* $\mathbf{E} \to$ *In Cat* $\mathbf{E}$ be the functor induced by U via the previous remark. This functor $\overline{U}$ has an adjoint, namely the restriction $\overline{F}$ of the adjoint F of U. The functor U is no more monadic, but the comparison functor K : *Cat* $\mathbf{E} \to Alg\,\overline{U}\cdot\overline{F}$ is fully faithful (let us say, then, that U is submonadic). Furthermore internal categories are exactly those algebras $z : \overline{U}\cdot\overline{F}Z \to Z$ in *In Cat* $\mathbf{E}$ which are cartesian with respect to a certain fibration *In Cat* $\mathbf{E} \to \mathbf{E}$.

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1 Initialized categories

An internal category X_1 in $\mathbf{E}$:

$$X_1 \; : \; X_0 \; \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{s_0} \\ \xleftarrow{d_1} \end{array} \; mX_1 \; \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{d_1} \\ \xleftarrow{d_2} \end{array} \; m_2X_1$$

will be said initialized when it is equipped with a split augmentation as a 2-truncated simplicial object :

$$X_{-1} \; \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow[s_0]{} \end{array} \; X_0 \; \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{s_0} \\ \xleftarrow{d_1} \\ \xrightarrow[s_1]{} \end{array} \; mX_1 \; \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{d_1} \\ \xleftarrow{d_2} \\ \xrightarrow[s_2]{} \end{array} \; m_2X_1$$

That means that there is a given choice of initial objects in each connected component and that X_{-1} represents the object of those distinguished elements.

Example : Given any category X_1, then the category $Dec\ X_1$ is canonically initialized :

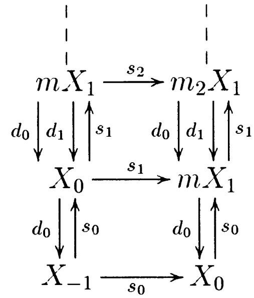
$$X_0 \; \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow[s_0]{} \end{array} \; mX_1 \; \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{d_1} \\ \xrightarrow[s_1]{} \end{array} \; m_2X_1 \; \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{d_1} \\ \xleftarrow{d_2} \\ \xrightarrow[s_2]{} \end{array} \; m_3X_1$$

We shall denote by $\underline{X}_1$ an initialized category and by *In Cat* $\mathbf{E}$ the category whose objects are the initialized categories, and whose morphisms are the functors preserving the split augmentation. This category is clearly left exact and the previous example induces a left exact functor $\overline{U}$: *Cat* $\mathbf{E} \to$ *In Cat* $\mathbf{E}$, where $\overline{U}(X_1)$ is *Dec* X_1 with its canonical initialization.

There is also a functor $\overline{F}$: *In Cat* $\mathbf{E} \to$ *Cat* $\mathbf{E}$ which associates to $\underline{X}_1$ its underlying category X_1. Furthermore $\overline{F} \cdot \overline{U} = Dec$ and there is a natural transformation : $\epsilon_1 X_1 : Dec\ X_1 \to X_1$:

$$\begin{array}{ccc} m_3X_1 & \xrightarrow{d_3} & m_2X_1 \\ \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\ m_2X_1 & \xrightarrow{d_2} & mX_1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ mX_1 & \xrightarrow[d_1]{} & X_0 \end{array}$$

where $\epsilon_1 X_1 : Dec\, X_1 \to X_1$ is a discrete cofibration. On the other hand there is a natural transformation $\eta_1 \underline{X}_1 : \underline{X}_1 \to \overline{U} \cdot \overline{F} \underline{X}_1$:



These natural transformations clearly satisfy the equations which make $\overline{F}$ a left adjoint of $\overline{U}$. We shall denote by (T, η, μ) and by (Dec, ϵ, ν) the monad and the comonad induced respectively on *In Cat* **E** and on *Cat* **E** by this adjunction.

2 *In Cat* E as a fibered category

Let us denote by $h_0 : In\ Cat\ \mathbf{E} \to \mathbf{E}$ the functor associating X_{-1} to $\underline{X}_1$. It is left exact and has a right inverse right adjoint Γ_1, where, for every object X in **E**, $\Gamma_1 X$ is the discrete category *dis* X with its unique possible initialization. Now, *In Cat* **E** being left exact, the functor h_0 is a fibration. A morphism $\underline{f}_1 : \underline{X}_1 :\to \underline{Y}_1$ is cartesian if and only if the following square is a pullback :

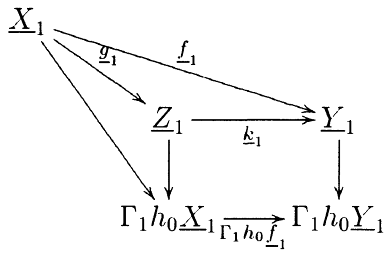
$$\begin{array}{ccc} \underline{X}_1 & \xrightarrow{\underline{f}_1} & \underline{Y}_1 \\ \downarrow & & \downarrow \\ \Gamma_1 h_0 \underline{X}_1 & \xrightarrow[\Gamma_1 h_0 \underline{f}_1]{} & \Gamma_1 h_0 \underline{Y}_1 \end{array}$$

Proposition 1 *The morphism* $\underline{f}_1$ *is cartesian if and only if its underlying functor* $f_1 = \overline{F}(\underline{f}_1)$ *is a discrete cofibration in Cat* **E**.

Demonstration :

The category $\Gamma_1\ h_0 \underline{X}_1$ being discrete, the functor $\Gamma_1\ h_0\ \underline{f_1}$ is a discrete cofibration. Now if $\underline{f_1}$ is cartesian, the previous square is a pullback and f_1 is a discrete cofibration.

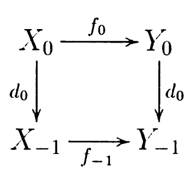
Conversely let us suppose that f_1 is a discrete cofibration and let us consider the following diagram where the lower right square is a pullback :



Then k_1 is a discrete cofibration and also the factorization g_1. Let us show that $\underline{g}_1$ is an isomorphism. Thanks to the Yoneda imbedding, it is sufficient to do this with **E** the category of sets. Let Z be an object of $\underline{Z}_1$ and $s_1 Z : s_0 Z \to Z$ be the associated initial map in its connected component. The object $s_0 Z$ is then a uniquely determined object in a connected component of $\underline{X}_1$. The functor g_1 being a discrete cofibration, it determines a unique map $s_0 Z \to X$ above $s_1 Z$. The object X is the unique object above Z. The functor f_1 is then bijective on the objects and a discrete cofibration. Consequently, it is an isomorphism. —— **QED** (Proposition 1)

Remarks

(1) That $\underline{f}_1$ is cartesian implies that the following square is a pullback :



(2) That $\underline{f}_1$ is cartesian implies that f_1 is also a discrete fibration.

(3) Clearly the functor $\eta_1 \underline{X}_1 : \underline{X}_1 \to \overline{U} \cdot \overline{F} \underline{X}_1$ is cartesian.

(4) The functor $f_1 : X_1 \to Y_1$ in *Cat* **E** is a discrete cofibration if and only if $\overline{U}(f_1)$ is cartesian.

3 The comparison functor $K : Cat\,\mathbf{E} \to Alg\,T$ is fully faithful

The following diagram in *Cat* **E** determines a levelwise split fork in **E** and thus a coequalizer in *Cat* **E** :

$$Dec^2 X_1 \underset{\epsilon_1 Dec\, X_1}{\overset{Dec\, \epsilon_1 X_1}{\rightrightarrows}} Dec X_1 \xrightarrow{\epsilon_1 X_1} X_1$$

Proposition 2 *The comparison functor $K : Cat\,\mathbf{E} \to Alg\,T$ is fully faithful. This result is a consequence of the following proposition.*

Proposition 3 *Let $(U, F, \eta, \epsilon) : \mathbf{X} \to \mathbf{Y}$ be an adjunction, and let*

$$T = (U, F, \eta, U\epsilon F)$$

be the monad it defines on $\mathbf{Y}$. The comparison functor $K : \mathbf{X} \to Alg\,T$ is fully faithful (i.e. the functor U is submonadic) if and only if for every object X in $\mathbf{X}$, the map ϵ_X is the coequalizer of ϵFUX and $FU\epsilon X$.

The demonstration is straightforward.

4 The comparison functor K is not an equivalence

Let *Simpl* **E** and *Sp Simpl* **E** denote respectively the category of simplicial objects in **E** and the category of split angmented simplicial objects in **E**. Let $U : Simpl\,\mathbf{E} \to Sp\ Simpl\,\mathbf{E}$ denote the functor cancelling the upper indexed face maps. It has a left adjoint F.

Any internal category X_1 can be completed into a simplicial object NX_1 (its nerve) by means of simplicial kernels. In the same way, any initialized category $\underline{X}_1$ can be completed into a split augmented simplicial object $n\underline{X}_1$. Whence the following diagram:

$$\begin{array}{ccccc}
Grd\,\mathbf{E} & \xrightarrow{i} & Cat\,\mathbf{E} & \xrightarrow{N} & Simpl\,\mathbf{E} \\
\overline{\overline{U}}\downarrow\uparrow\overline{\overline{F}} & & \overline{U}\downarrow\uparrow\overline{F} & & U\downarrow\uparrow F \\
Pt\,\mathbf{E} & \xrightarrow{j} & In\ Cat\,\mathbf{E} & \xrightarrow{n} & Sp\ Simpl\,\mathbf{E}
\end{array}$$

The functors N and n are full embeddings. *Grd* $\mathbf{E}$ denotes the category of internal groupoids, i.e. internal categories such that the following square is a pullback :

$$\begin{array}{ccc} mX_1 & \xleftarrow{d_1} & m_2X_1 \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ X_0 & \xleftarrow[d_0]{} & mX_1 \end{array}$$

Pt $\mathbf{E}$ denotes the category whose objects are the split epimorphisms and whose morphisms are the coherent squares. The functor i is the inclusion, and the functor j associates to each split epimorphism the initialized groupoid obtained by the kernel groupoid of the given epimorphism. Clearly j is a full embedding.

Thus $(\overline{U}, \overline{F})$ and $(\overline{\overline{U}}, \overline{\overline{F}})$ appear to be successive restrictions of the adjunction (U, F). The functor (U, F) is always monadic (See [2]). When the idempotents split in $\mathbf{E}$, then furthermore F is comonadic. When $\mathbf{E}$ is left exact, $\overline{\overline{U}}$ is monadic (See [1]) and $\overline{\overline{F}}$ is comonadic.

It would be easy to show that $\overline{F}$ is comonadic (the dual of proposition 3, plus the existence of kernels). We are going to show that $\overline{U}$ is not monadic.

Let $\mathbf{2}$ be the category : $\mathbf{0} \xrightarrow{\alpha} \mathbf{1}$. It is clearly initialized in a unique possible way. The category $T\mathbf{2}$ has two connected components : $\mathbf{0} \xrightarrow{\overline{\alpha}} \alpha$ and $\mathbf{1}$. Let $\underline{h}_1$ be the unique possible initialized functor: $T\mathbf{2} \rightarrow \mathbf{2}$, which is a left inverse for $\eta_1\mathbf{2}$. It is easy to check that it determines an algebra on $\mathbf{2}$. Now, the simplicial set Z determined by $n\,\underline{h}_l$, as an algebra on $U \cdot F$, is not the nerve of a category. It is the smallest simplicial set associated to graph $\mathbf{1} : \mathbf{0} \rightarrow \mathbf{0}$.

5 The monad (T, η, μ) is transversely cartesian with respect to h_o

We saw that $\eta_1\underline{X}_1$ is cartesian. Now $\mu\underline{X}_1 = \overline{U}\epsilon X_1$. But ϵX_1 is a discrete cofibration and $\overline{U}$ sends discrete cofibrations on cartesian maps. So, $\mu\underline{X}_1$ is cartesian. Furthermore $T = \overline{U} \cdot \overline{F}$ preserves cartesian maps following proposition 1 and remark 4.

We shall say then that (T, η, μ) is transversely cartesian with respect to the fibration h_0.

6 Characterization of *Cat* E

Proposition 4 *Cat* $\mathbf{E}$ *is isomorphic to the full subcategory of* Alg T *whose objects are the algebras* $\underline{x}_1 : T\underline{X}_1 \to \underline{X}_1$ *in In Cat* $\mathbf{E}$ *such that* $\underline{x}_1$ *is cartesian.*

Demonstration :

Proof: Let X_1 be a category ; then the algebra $K(X_1)$ is : $\overline{U}\epsilon_1 X_1$: $\overline{U}Dec\, X_1 \to \overline{U}X_1$. But $\epsilon_1 X_1$ is a discrete cofibration and $\overline{U}\epsilon_1 X_1$ is cartesian.

Conversely if $\underline{x}_1 : T\underline{X}_1 \to \underline{X}_1$ is cartesian, then, following remark 1, the following diagram is a pullback :

$$\begin{array}{ccc} X_0 & \xleftarrow{x_0} & mX_1 \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ X_{-1} & \xleftarrow[x_{-1}]{} & X_0 \end{array}$$

and the 3-truncated simplicial object it determines is underlying to an intennal category.

7 The case of *Grd* E

Why is *Grd* $\mathbf{E}$ monadic and not *Cat* $\mathbf{E}$? If we denote again by (T, η, μ) the restriction to *Pt* $\mathbf{E}$ of the monad (T, η, μ) defined on *In Cat* $\mathbf{E}$, this monad is again transversely cartesian, but furthermore it has the particularity to be normal, i.e. the following diagram is a always pullback :

$$\begin{array}{ccc} T^2X & \xleftarrow{\mu TX} & T^3X \\ {\scriptstyle \mu X}\downarrow & & \downarrow{\scriptstyle T\mu X} \\ TX & \xleftarrow[\mu X]{} & T^2X \end{array}$$

Then any algebra $x : TX \to X$ in *Pt* $\mathbf{E}$ induces an internal groupoid in *Pt* $\mathbf{E}$ (see [1]) :

$$TX \underset{Tx}{\overset{\mu X}{\leftleftarrows}} T^2X \; \overset{\mu TX}{\underset{T^2x}{\overset{T\mu X}{\longleftarrow}}} \; T^3X$$

Now μX is cartesian. So Tx, being "equal to μX up to isomorphism" (thanks to the previous groupoid structure), is again cartesian. Then $Tx \cdot \lambda TX$ is cartesian since both Tx and λTX are cartesian. But $Tx \cdot \lambda TX = \lambda X \cdot x$ and, λX being cartesian, x is cartesian.

Consequently, every algebra $x : TX \to X$ is cartesian and the comparison functor $K : \mathit{Grd}\, \mathbf{E} \to \mathit{Alg}\, T$ is an equivalence.

A last remark : if again (T, η, μ) denotes the monad on *Sp Simpl* **E** induced by the adjunction (U, F), the objects of *In Cat* **E** are precisely the objects S of *Sp Simpl* **E** which have their map $\mu S : T^2 S \to TS$ cartesian in *Sp Simpl* **E** with respect to the fibration k_0 : *Sp Simpl* **E** $\to$ **E** defined by $k_0(S) = S_{-1}$

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