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AN EXTERIOR PRODUCT FOR THE HOMOLOGY OF GROUPS WITH INTEGRAL COEFFICIENTS MODULO p

by G.J. ELLIS[1] *and C. RODRIGUEZ-FERNANDEZ*

RÉSUMÉ. Les auteurs généralisent la suite exacte à 8 termes de groupes d'homologie entière obtenue par Brown et Loday en une suite exacte de groupes d'homologie à coefficients dans $\mathbb{Z}_p$, où p est un entier non-négatif.

In this article we generalize Brown and Loday's eight term exact sequence in integral group homology [2] to an exact sequence in group homology with coefficients in $\mathbb{Z}_p$, where p is any nonnegative integer.

Let G be a group with a normal subgroup N, and consider $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ as a trivial G-module. We prove

THEOREM 1. *There is a natural exact sequence*

$$\begin{aligned} &H_3(G,\mathbb{Z}_p) \longrightarrow H_3(G/N,\mathbb{Z}_p) \longrightarrow \mathrm{Ker}(\partial : N\Delta^p G \to G) \longrightarrow H_2(G,\mathbb{Z}_p) \\ &\longrightarrow H_2(G/N,\mathbb{Z}_p) \longrightarrow N/N\#_p G \longrightarrow H_1(G,\mathbb{Z}_p) \longrightarrow H_1(G/N,\mathbb{Z}_p) \longrightarrow 0. \end{aligned}$$

Here $N\#_p G$ denotes the subgroup of N generated by the elements $[n,g]$ and n^p, for $g \in G$, $n \in N$. (When x,y are elements of a group, we write $[x,y] = xyx^{-1}y^{-1}$ and ${}^x y = xyx^{-1}$.)

The group $N\Delta^p G$ is a new construction. It is generated by the symbols $n \wedge g$ and $\{n\}$ for $n \in N$, $g \in G$, subject to the relations

(1) $$n \wedge gh = (n \wedge g)({}^g n \wedge {}^g h),$$

(2) $$nm \wedge g = ({}^n m \wedge {}^n g)(n \wedge g),$$

[1] This author would like to thank the University of Santiago de Compostela for its hospitality during the preparation of this article.

(3) $$n \wedge n = 1,$$

(4) $$\{n\}(m \wedge g)\{n\}^{-1} = {}^{n^p}m \wedge {}^{n^p}g,$$

(5) $$\{nm\} = \{n\}\Big(\prod_{i=1}^{p-1} (n^{-1} \wedge ({}^{n^{1-p+i}}m)^i)\Big)\{m\},$$

(6) $$[\{n\},\{m\}] = n^p \wedge m^p,$$

(7) $$\{[n,g]\} = (n \wedge m)^p$$

for $g, h \in G$, $m, n \in N$. Note that if $N = G$, then (2) and (7) are redundant.

Clearly $N\Delta^p G$ is functorial in N and G.

The homomorphism $\partial: N\Delta^p G \to G$ is defined by

$$\partial(n \wedge g) = [n,g] \text{ and } \partial\{n\} = n^p.$$

It is routine to check that ∂ is a well-defined homomorphism, and that its image is $N \#_p G$.

As an immediate consequence of Theorem 1 we have

COROLLARY 2. *There is an isomorphism*

$$H_2(G,\mathbb{Z}_p) \approx \mathrm{Ker}\,(\partial: G\Delta^p G \to G)$$

Also, for any presentation $R \rightarrowtail F \twoheadrightarrow G$ *of* G, *there is an isomorphism*

$$H_3(G,\mathbb{Z}_p) \approx \mathrm{Ker}\,(\partial: R\Delta^p F \to F)\,.$$

In order to prove Theorem 1 we need the following

LEMMA 3. *If* F *is a free group, then* ∂ *induces an isomorphism* $F\Delta^p F \approx F\#_p F$.

PROOF. Recall from [2] that $N \wedge G$ is the group generated by the symbols $n \wedge g$ for $g \in G$, $n \in N$, subject to relations (1), (2), (3). There is thus a homomorphism $\iota: N \wedge G \to N\Delta^p G$, $n \wedge g \mapsto n \wedge g$. By (4) the image of ι is normal in $N\Delta^p G$. On taking $G = N = F$ we thus have a commutative diagram

$$\begin{tikzcd} F \wedge F \arrow[r] \arrow[d, "\partial'"'] & F\Delta^p F \arrow[r, two heads] \arrow[d, "\partial"] & F\Delta^p F/\mathrm{Im}(\iota) \arrow[d, "\partial''"] \\ {[F,F]} \arrow[r, tail] & F\#_p F \arrow[r, two heads] & pF^{ab} \end{tikzcd}$$

where ∂'' is induced by ∂, and where ∂' is the isomorphism proved in [2] (see [3] for an algebraic proof of this isomorphism).

The homomorphism ∂'' is clearly surjective, and hence has a splitting since pF^{ab} is free abelian and $F\Delta^pF/\mathrm{Im}(\iota)$ is abelian by (6). This splitting is surjective because of (5). Therefore ∂'' is an isomorphism. Since the rows of the diagram are both short exact, it follows that $\partial\colon F\Delta^pF \to F\#_pF$ is an isomorphism. ∎

In [1] the following natural exact sequence

$$H_3(G,\mathbb{Z}_p) \longrightarrow H_3(G/N,\mathbb{Z}_p) \longrightarrow \mathrm{Ker}(L_0V\mathcal{P}(\alpha)\to G) \longrightarrow H_2(G,\mathbb{Z}_p)$$
$$\longrightarrow H_2(G/N,\mathbb{Z}_p) \longrightarrow N/N\#_pG \longrightarrow H_1(G,\mathbb{Z}_p) \longrightarrow H_1(G/N,\mathbb{Z}_p) \longrightarrow 0.$$

is obtained. Thus to prove Theorem 1 it suffices to exhibit an isomorphism $\approx\colon N\Delta^pG \to L_0V\mathcal{P}(\alpha)$ such that

$$(*)\qquad \begin{array}{ccc} N\Delta^pG & \longrightarrow & G \\ \approx\big\downarrow & & \big\| \\ L_0V\mathcal{P}(\alpha) & \longrightarrow & G \end{array}$$

commutes.

For any surjection $\varepsilon\colon F\to G$ with F a free group let S be the kernel of the composite homomorphism

$$F \xrightarrow{\ \varepsilon\ } G \xrightarrow{\ \alpha\ } G/N.$$

Let $i\colon S'\to S$ be an isomorphism. Let T be the kernel of

$$\binom{\varepsilon i}{\varepsilon} : S'*F \longrightarrow G.$$

Then it is shown in ([1], Propositions 6.4 and 8.1) that

$$L_0V\mathcal{P}(\alpha) = \frac{(S'^{S'*F})\#_p(S'*F)}{((S'^{S'*F})\#_pT)((T\cap S'^{S'*F})\#_p(S'*F))}.$$

As in [1] let $\mu\colon G\to F$ be any set theoretic section of $\varepsilon\colon F\to G$. Then μ induces a section $N\to S \approx S'$; under this section we denote the image of $n\in N$ by $\mu(n)'\in S'$. Let

$$D = ((S'^{S'*F})\#_pT)((T\cap S'^{S'*F})\#_p(S'*F)).$$

With this notation, we have

LEMMA 4. *There is a homomorphism $h\colon N\Delta^pG\to L_0V\mathcal{P}(\alpha)$ defined by*

$$h(n\wedge g) = [\mu(n)',\mu(g)]D, \quad h(\{n\}) = (\mu(n)')^pD.$$

PROOF. We need to show that h preserves the relations (1)-(7). By [1] clearly h preserves (1)-(3). Relation (4) is preserved because

$$\mu({}^{n}m)' \in S'^{S'*F},\ ((\mu(n)')^{p}\mu(g))^{-1}\mu({}^{n^p}g) \in T$$
$$((\mu(n)')^{p}\mu(m)')^{-1}\,\mu({}^{n^p}m)' \in T\cap S'^{S'*F}$$

from which we see that

$$h({}^{n^p}m \wedge {}^{n^p}g) = (\mu(n)')^{p}[\mu(m)',\mu(g)](\mu(n)')^{p}D = h(\{n\}(g\wedge m)\{n\}^{-1}).$$

Relation (5) is preserved because

$$\mu(n^{-1})'\mu(n)' \in T\cap S'^{S'*F}$$

and $\quad \mu(n)' \in S'^{S'*F},\ \mu({}^{n^{1-p+i}}m)^{i}((\mu(n)')^{1-p+i}\mu(m)')^{-1} \in T$

which implies

$$h(\{n\}\Big(\prod_{i=1}^{p-1}(n^{-1}\wedge({}^{n^{1-p+i}}m)^{i})\Big)\{m\}) = (\mu(n)'\mu(m)')^{p}D.$$

Since

$$(\mu(nm)')^{-p}(\mu(n)'\mu(m)')^{p} \in T\cap S'^{S'*F}$$

we have

$$h(\{n\}\Big(\prod_{i=1}^{p-1}(n^{-1}\wedge({}^{n^{1-p+i}}m)^{i})\Big)\{m\}) = (\mu(nm)')^{p}D = h(\{nm\}).$$

Relation (6) is preserved because we have

$(\mu(n)')^{p}(\mu(n^{p})')^{-1} \in T\cap S'^{S'*F}$ and $\mu(n^{p})' \in S'^{S'*F}$, $\mu(m^{p})^{-1}(\mu(m)')^{p} \in T$.

Clearly relation (7) is preserved. ∎

Consider the homomorphism

$$d = \binom{\varepsilon i}{\varepsilon} : S'*F \longrightarrow G.$$

By Lemma 3 we have a homomorphism $\varphi: S'\#_{p}S' \approx S'\Delta^{p}S' \to N\Delta^{p}G$ defined by

$$\varphi([s'_1,s'_2]) = d(s'_1)\wedge d(s'_2),\ \varphi((s')^{p}) = \{d(s')\}\ .$$

We therefore have a set theoretic map $g: S'^{S'*F}\#_{p}(S'*F) \to N\Delta^{p}G$ defined as follows (cf. [1], 8.12):

if $\prod_{i=1}^{n} s'_i f_i \in S'^{S'*F}\#_{p}(S'*F)$ then $\prod_{i=1}^{n} s'_i \in S'\#_{p}S'$ and we can define

$$g(\prod_{i=1}^{n} s'_i f_i) = \Big(\prod_{i=1}^{n-1}(d(\prod_{j=1}^{i} s'_j)\wedge d(\prod_{j=1}^{i} f_j))(d(\prod_{j=1}^{i} s'_j)\wedge d(\prod_{j=1}^{i+1} f_j))^{-1}\Big)\varphi(\prod_{i=1}^{n} s'_i).$$

LEMMA 5. *The composite function*

$$S'^{S'*F}\#_{p}(S'*F) \xrightarrow{\ g\ } N\Delta^{p}G \longrightarrow N\Delta^{p}G/\iota(N\wedge G)$$

is a homomorphism, and induces a homomorphism

$$\psi : L_0 V_1^P(\alpha) \longrightarrow N\Delta^P G/\iota(N\wedge G).$$

PROOF. The first homomorphism is clear, and certainly D is in the kernel of the first homomorphism. By Lemmas 4 and 5 we have a commutative diagram

$$\begin{tikzcd}
N\wedge G \arrow[r, "\iota"] \arrow[d, "h'"', "\approx"] & N\Delta^P G \arrow[r] \arrow[d, "h"] & N\Delta^P G/\iota(N\wedge G) \arrow[d, "\bar{h}"] \\
L_0 V_1^0(\alpha) \arrow[r, tail] & L_0 V_1^P(\alpha) \arrow[r, two heads] \arrow[ur, "\psi"] & L_0 V_1^P(\alpha)/L_0 V_1^0(\alpha)
\end{tikzcd}$$

where h' is the restriction of $h\iota$, and is an isomorphism by Section 8 of [1]. Clearly $L_0 V_1^0(\alpha)$ lies in the kernel of ψ, and so ψ induces a splitting of $\bar{h}$. But ψ is surjective and hence $\bar{h}$ is an isomorphism. It follows that h is an isomorphism. It is readily seen that the above diagram (*) commutes. So Theorem 1 is proved.

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