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NON-ABELIAN COHOMOLOGY OF ASSOCIATIVE ALGEBRAS. THE 9-TERM EXACT SEQUENCE

by Antonio M. CEGARRA and Antonio R. GARZON

RÉSUMÉ. Cet article étudie une notion d'ensemble de cohomologie non-abélienne $\mathbf{H}^3(X,\Phi)$ de dimension 3 d'une algèbre associative X à coefficients dans un module croisé d'algèbres Φ. On obtient une suite exacte naturelle à 9 termes associée à une suite exacte courte de modules croisés, qui étend la suite exacte à 6 termes de Dedecker-Lue, et se ramène à la suite usuelle abélienne si les coefficients sont abéliens. Les méthodes utilisées étant purement catégoriques, des résultats analogues vaudraient pour les groupes, les algèbres de Lie, etc.

INTRODUCTION.

Given associative algebras X and B over a commutative and unitary ring R, the set Hom(X,B) of homomorphisms of algebras from X to B is pointed with distinguished element the zero morphism $0: X \to B$ and the functor

$$\mathrm{Hom}(X,-) : \text{Associative algebras} \longrightarrow \text{Pointed sets}$$

is left exact; i.e., for any exact sequence of associative algebras

$$(1) \qquad 0 \longrightarrow B' \xrightarrow{i} B \xrightarrow{p} B'' \longrightarrow 0$$

the associated sequence of pointed sets

$$(2) \quad 0 \longrightarrow \mathrm{Hom}(X,B') \xrightarrow{i_*} \mathrm{Hom}(X,B) \xrightarrow{p_*} \mathrm{Hom}(X,B'') \longrightarrow 0$$

is exact. When B', B and B" are zero-algebras, i.e., the multiplicative structure is trivial, the usual cohomology groups of X with coefficients in the trivial X-bimodules B', B, B" allow one to obtain a well known long exact sequence whose first 3 terms reduce to (2), so that these cohomology groups give an appropriate solution to the problem of measuring the deviation from exactness of the functor Hom(X,-) on short exact sequences of trivial algebras.

The main object of the non-abelian 2-cohomology for associative algebras is to measure the deviation from exactness of Hom(X,-) on general exact sequences of algebras such as (1). In 1966, by using "crossed modules" of algebras to define non-abelian 2-dimensional cohomology for algebras. Dedecker-Lue in [16] gave a solution to this problem: associated to the sequence (1) there always exists a short exact sequence of crossed modules

$$\text{(3)}\qquad \begin{array}{ccccc} \Phi': & B' & \longrightarrow & I_B \\ \downarrow & \downarrow & & \| \\ \Phi\ : & B' & \longrightarrow & I_B \\ \downarrow & \downarrow & & \downarrow \\ \Phi'': & B'' & \longrightarrow & I_{B''} \end{array}$$

where I_B is the algebra of inner bimultiplications of B and this short exact sequence induces a 6-term exact sequence of sets with distinguished elements

$$0 \longrightarrow \mathrm{Hom}(X,B') \longrightarrow \mathrm{Hom}(X,B) \longrightarrow \mathrm{Hom}(X,B'')$$
$$\longrightarrow \mathbf{H}^2(X,\Phi') \longrightarrow \mathbf{H}^2(X,\Phi) \longrightarrow \mathbf{H}^2(X,\Phi'')$$

where, for an arbitrary crossed module $\Phi = (\delta: B \to A, \mu: A \to M_B)$, $\mathbf{H}^2(X,\Phi)$ was defined in terms of non-abelian 2-cocycles and it was interpreted in terms of isomorphism classes of extensions of X by the crossed module Φ, i.e., commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\pi} & & & & \\ & & B & \xrightarrow{\ \delta\ } & A & & & & \end{array}$$

where the top row is a short exact sequence of algebras such that actions of E on B by translations coincide with those induced by π.

After Dedecker-Lue's 6-term exact sequence in non-abelian cohomology associated to a short exact sequence of crossed modules the main problem is that of giving a measure of the deviation from right exactness of $\mathbf{H}^2(X,-)$ adding new terms (functorial in X and the short exact sequence of crossed modules) by using an adequate notion of $\mathbf{H}^3(X,-)$. To give a solution for this problem is the main object of this paper.

Our solution uses the monadic non-abelian cohomology sets with coefficients in "hypergroupoids" studied in [8,9] and it was suggested by the results given by J. Duskin in [18], about monadic non-abelian cohomology with coefficients in groupoids. Duskin observes that, in the more usual algebraic categories, "crossed modules" [26] are equivalent to internal groupoids and arbitrary extensions of an object by a crossed module correspond to torsors under the associated groupoid, which are classified by the set of homotopy classes of simplicial morphisms from the cotriple standard resolution to the nerve of the groupoid. The key to define $\mathbf{H}^3(X,\Phi)$ is the observation that just as a crossed module Φ has associated a groupoid (1-hypergroupoid) $\underset{\sim}{G}(\Phi)$ in $\mathbf{V}$, in such a way that

$$\mathbf{H}^2(X,\Phi) \approx \mathbf{H}^1_{\mathbf{G}}(X,\underset{\sim}{G}(\Phi))$$

(see Corollary 9), it also has canonically associated a 2-hypergroupoid $\underset{\sim}{G}^2(\Phi)$, via an equivalence of categories between the category of crossed modules and a certain full subcategory of the category of 2-hypergroupoids in $\mathbf{V}$, and we take

$$\mathbf{H}^3(X,\Phi) = \mathbf{H}^2_{\mathbf{G}}(X,\underset{\sim}{G}^2(\Phi)).$$

These sets $\mathbf{H}^3(X,\Phi)$ have some distinguished elements (neutral and null classes) which allow us to describe the exactness of a sequence

$$\mathbf{H}^2(X,\Phi) \longrightarrow \mathbf{H}^2(X,\Phi'') \longrightarrow \mathbf{H}^3_{\Phi}(X,\Phi') \longrightarrow \mathbf{H}^3(X,\Phi) \longrightarrow \mathbf{H}^3(X,\Phi'')$$

associated to a short exact sequence of crossed modules

$$\Phi' = (\delta' : B' \to A, \mu') \longrightarrow \Phi = (\delta : B \to A, \mu) \xrightarrow{p=(p_1,p_0)} \Phi'' = (\delta'' : B'' \to A, \mu'')$$

satisfying in addition that $\mathrm{Ker}(p_0) = \mathrm{Im}(\delta')$. This condition is equivalent to the property of the morphism of groupoids $\underset{\sim}{G}(p) : \underset{\sim}{G}(\Phi) \to \underset{\sim}{G}(\Phi'')$ being a "quotient map" in the sense of Higgins [25], and we need that to establish the connecting map $\mathbf{H}^2(X,\Phi) \to \mathbf{H}^3_{\Phi}(X,\Phi')$ since every 1-cocyle under $\underset{\sim}{G}(\Phi'')$ must have a lifting to a 1-cochain under $\underset{\sim}{G}(\Phi)$. Moreover in such conditions the morphism of 2-hypergroupoids $\underset{\sim}{G}^2(p) : \underset{\sim}{G}^2(\Phi) \to \underset{\sim}{G}^2(\Phi'')$ is a surjective Kan fibration whose "2-hypergroupoid kernel" $\underset{\sim}{G}^2_{\Phi}(X,\Phi')$, is used to define the "relative" 3-rd cohomology set

$$\mathbf{H}^3_{\Phi}(X,\Phi') = \mathbf{H}^2_{\mathbf{G}}(X,\underset{\sim}{G}^2_{\Phi}(\Phi')).$$

Note that any short exact sequence of algebras such as (1) has always associated a short exact sequence of crossed modules with $\mathrm{Ker}(p_0) = \mathrm{Im}(\delta')$ taking $\delta : B \to A = I_B$ the canonical morphism into the algebra of inner bimultiplications and $A'' = I_B/\delta(B')$. Therefore for any algebra X there exists a 9-term exact sequence extending the sequence

$$0 \longrightarrow \mathrm{Hom}(X,B') \longrightarrow \mathrm{Hom}(X,B) \longrightarrow \mathrm{Hom}(X,B'')$$

Moreover, in this case, the resulting 9-term sequence is exact of pointed sets in the 3 last terms (i.e., the corresponding sets $\mathbf{H}^3$ have only one distinguished element).

The structure of this paper is as follows. In Section 1, §1.1 is devoted to give a quick review of Dedecker-Lue's non-abelian cohomology theory. In 1.2 we recall Duskin's low dimensional non-abelian monadic cohomology with coefficients in groupoids, showing explicitly its relationship with Dedecker-Lue's theory, and 1.3 is dedicated to establishing a general 6-term exact sequence in non-abelian monadic cohomology carrying as examples those classically shown in the more usual algebraic contexts (Groups, Associative algebras,...). The simplicial way in which this sequence is obtained will allow us to prove the existence of the 9-term exact sequence in non-abelian cohomology of algebras. In Section 2, §2.1 is devoted to set up the basic machinery of the 2-dimensional non-abelian monadic cohomology with coefficients in 2-hypergroupoids; in 2.2, using that general theory we define the 3-rd non-abelian cohomology set of an algebra X with coefficients in a crossed module Φ, $\mathbf{H}^3(X,\Phi)$, and in 2.3 we show the announced 9-term exact sequence in non-abelian cohomology of algebras, which is a generalization of the usual abelian one when the sequence of crossed modules is that associated to a short exact sequence of zero-algebras.

Although we have choosen the category of associative algebras as the context in which we develop this paper, we want to point out that the methods, constructions and concepts used here are essentially categorical and so they are applicable to many different algebraic contexts as Groups, Lie algebras, etc.

NOTATIONS AND PRELIMINARIES.

Throughout the paper $\mathbf{V}$ will denote the category of associative algebras over a commutative and unitary ring R.

For an algebra B, M_B will denote the "multiplication algebra" of B. For each element $b \in B$ a bimultiplication $\mu(b)$ is defined by

$$\mu(b)b_1 = bb_1,\ b_1\mu(b) = b_1 b,\ b_1 \in B;$$

$\mu(b)$ is the "inner bimultiplication" defined by b and the map $\mu: B \to M_B$ is a homomorphism of algebras. The image $\mu_*(B) = I_B$ of this homomorphism is a two-sided ideal in M_B and the quotient $P_B = M_B/I_B$ is called the "algebra of outer bimultiplications" of B. Two bimultiplications $\sigma_1, \sigma_2 \in M_B$ are permutable on the subset S of B if

$$(\sigma_1 b)\sigma_2 = \sigma_1(b\sigma_2), \ (\sigma_2 b)\sigma_1 = \sigma_2(b\sigma_1)$$

for every $b \in S$. A subset M of M_B is permutable on S if every pair of bimultiplications from M is permutable on S. For more details about these notions see [27,28].

Given $X, B \in \mathbf{V}$ an action of X on B is a homomorphism $\varphi: X \to M_B$ such that $\mathrm{Im}(\varphi)$ is permutable on B; as usual ${}^x b$ and b^x denote $\varphi(x)b$ and $b\varphi(x)$ respectively. An action of X on B determines an extension of X by B,

$$0 \longrightarrow B \longrightarrow B \rtimes X \longrightarrow X \longrightarrow 0$$

where $B \rtimes X$ is the semidirect product of B with X, i.e., $B \rtimes X$ is the direct sum as modules and the product of two elements is given by the rule

$$(b_1, x_1)(b_2, x_2) = (b_1 b_2 + b^{x_2} + {}^{x_1} b_2, x_1 x_2).$$

In this paper we will use standard simplicial terminology. The category of simplicial objects in a category $\mathbf{C}$ is denoted Simpl($\mathbf{C}$); (E., E'.) will denote the set of simplicial morphisms of E. into E'. and [E., E'.] the quotient set of (E., E'.) under the equivalence relation generated by homotopy.

Given a simplicial object E. and $n > 1$, the *n-th simplicial kernel* of E. is an object denoted $\Delta^n(E.)$ together with morphisms $d_i: \Delta^n(E.) \to E_{n-1}$, $0 \le i \le n$, universal with respect to satisfying $d_i d_j = d_{j-1} d_i$ for all $i < j$; and for $0 \le i \le n$ we denote $\Lambda_i^n(E.)$ the object universal with respect to having morphisms $d_j: \Lambda_i^n(E.) \to E_{n-1}$, $0 \le j \le n$, $j \neq i$ satisfying

$$d_j d_k = d_{k-1} d_j \text{ for } j < k, \ k \neq i,$$

which is called the *object of open i-horns* at dimension n. For these objects one has a commutative diagram of canonical morphisms

$$\begin{array}{ccccc} & & E_n & & \\ & \swarrow^{F_n} & & \searrow^{K_i^n} & \\ \Delta^n(E.) & & \xrightarrow[H_i^n]{} & & \Lambda_i^n(E.) \end{array}$$

$$F_n = (d_0, \dots, d_n), \quad K_i^n = (d_0, \dots, d_{i-1}, -, d_{i+1}, \dots, d_n)$$
$$H_i^n = (d_0, \dots, d_{i-1}, -, d_{i+1}, \dots, d_n).$$

An n-truncated simplicial object $F._{tr}$ consists only of $F_0, \dots, F_n$ and the usual face and degeneracy morphisms between them. The process of n-truncating is a functor (usually denoted by Tr^n) which has a right adjoint Cosk^n and built by iterating simplicial kernels to truncated simplicial objects. The universal adjunction property gives a natural bijection

$$(E., \mathrm{Cosk}^n(F._{tr})) \approx (\mathrm{Tr}^n(E.), F._{tr}).$$

We will denote by $\mathbf{G} = (\mathbf{G}, \rho, \varepsilon)$ the cotriple associated to the monadic forgetful functor $\mathbf{V} \to \mathrm{Sets}$. For $X \in \mathbf{V}$, the cotriple determines an augmented simplicial algebra

$$\mathbf{G}.(X) = \ldots\ \mathbf{G}^2(X) \rightrightarrows \mathbf{G}(X) \xrightarrow{\varepsilon_X} X$$

called the *standard cotriple resolution* of X. Let us note that this resolution is aspherical, i.e., the canonical morphisms F_n from $\mathbf{G}^{n+1}(X)$ to $\Delta^n(\mathbf{G}.(X))$ and $\mathbf{G}^2(X) \to \mathbf{G}(X) \times_X \mathbf{G}(X)$ are surjective epimorphisms. For more details about simplicial objects see [23, 29].

1. SIMPLICIAL VERSION OF DEDECKER-LUE'S NON-ABELIAN COHOMOLOGY FOR ASSOCIATIVE ALGEBRAS.

In 1967 J. Beck gave an interpretation theorem for the Barr-Beck monadic cohomology groups $\mathbf{H}^1_G$ in terms of isomorphism classes of torsors (i.e., principal homogeneous spaces) under internal abelian groups which, in the case of algebras, are identificable with singular extensions in the usual sense (i.e., extensions

$$0 \longrightarrow B \longrightarrow E \longrightarrow X \longrightarrow 0 \quad \text{with } B^2 = 0).$$

This fact has been generalized by J. Duskin in [18] where a low dimensional monadic cohomology with coefficients in internal groupoids is developed; here, the cotriple resolution gives rise to a cocomplex of groupoids where 1-cohomology turns out to be homotopy classes of simplicial cocycles, and a classification theorem, analogous to Beck's Theorem, is proved by the notion of torsors under groupoids. Now, the categories of crossed modules in algebras and groupoids in algebras are equivalent, and therefore extensions of an algebra by a crossed module correspond to torsors under the associated groupoid. Therefore, Duskin's non-abelian cohomology applied to the category of algebras gives an alternative simplicial description of Dedecker-Lue's $\mathbf{H}^2$ defined in [16].

Since this general cohomology with coefficients in groupoids is the first stage for establishing our non-abelian $\mathbf{H}^3$, we will give in this section some background on it, showing explicitly its relationship with Dedecker-Lue's non-abelian cohomology for associative algebras. Moreover, to obtain the 9-term exact sequence we will describe the Dedecker-Lue's 6-term exact sequence by using simplicial cocycles.

1.1. Quick review of Dedecker-Lue's non-abelian cohomology.

Let us remember that a crossed module in **V** is a system

$$\Phi = (\delta: B \to A, \mu: A \to M_B)$$

where $A, B \in \mathbf{V}$, M_B is the multiplication algebra of B and δ and μ are homomorphisms satisfying

i) $\mathrm{Im}(\mu)$ is permutable in B (i.e., ${}^{a}(b^{a'}) = ({}^{a}b)^{a'}$, $a, a' \in A$, $b \in B$),

ii) $\delta({}^{a}b) = a\delta(b)$, $\delta(b^{a}) = \delta(b)a$, $a \in A$, $b \in B$,

iii) The composite

$$B \xrightarrow{\ \delta\ } A \xrightarrow{\ \mu\ } M_B$$

maps each element of B onto the inner bimultiplication which it defines (i.e., $bb' = {}^{\delta(b)}b' = b^{\delta(b')}$),

where $\qquad {}^{a}b = \mu(a)(b)$ and $b^{a} = (b)\mu(a)$.

A *morphism of crossed modules in* **V** is a commutative diagram

$$\begin{array}{ccc} B' & \xrightarrow{\ \delta\ } & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B' & \xrightarrow[\ \delta'\ ]{} & A' \end{array}$$

where f and g are morphisms in **V** such that

$$f({}^{a}b) = {}^{g(a)}f(b),\ f(b^{a}) = f(b)^{g(a)} \text{ for all } a \in A \text{ and } b \in B.$$

We will denote by XM(**V**) the category of crossed modules in **V**.

In the following $\Phi = (\delta: B \to A, \mu: A \to M_B)$ will denote a crossed module in **V**. For $X \in \mathbf{V}$, F(X) will denote the free R-module on the generators $(\bar{x} \mid x \in X)$ and N(X) the kernel of the canonical epimorphism $F(X) \to X$.

A *1-cocycle* from X to Φ is a pair (f, φ) where $f: X \to B$ is a homomorphism of R-modules, $\varphi: X \to A$ is a homomorphism of R-algebras and the condition of *φ-crossed homomorphism*

$$f(x_1x_2) = f(x_1)f(x_2) + f(x_1)^{\varphi(x_2)} + {}^{\varphi(x_1)}f(x_2)$$

is verified for any $x_1, x_2 \in X$.

$\mathbf{Z}^1(\mathbf{X}, \Phi)$ denotes the set of 1-cocyles from X to Φ and taking $\varphi: X \to A$ to be a fixed homomorphism, $\mathbf{Z}^1_{\varphi}(X, \Phi)$ denotes the subset of $\mathbf{Z}^1(X, \Phi)$ of those elements of the form (f, φ). $\mathbf{Z}^1_{\varphi}(X, \Phi)$ is then endowed with a base point $(0, \varphi)$.

A *2-cocycle* from X to Φ is a system $(\Gamma_1,\Gamma_2,\varphi)$ where

$$\Gamma_1: X\times X \to B,\ \Gamma_2: N(X)\to B \text{ and } \varphi: X\to A$$

are maps satisfying:

1. ${}^{\varphi(x_1)}\Gamma_1(x_2,x_3)-\Gamma_1(x_1x_2,x_3)+\Gamma_1(x_1,x_2x_3)-\Gamma_1(x_1,x_2)^{\varphi(x_3)} = 0$,
2. $\Sigma r_i\Gamma_1(x,x_i) = \Gamma_2(\Sigma r_i\overline{xx_i}) - {}^{\varphi(x)}\Gamma_2(\Sigma r_i\overline{x_i})$,
3. $\Sigma r_i\Gamma_1(x,x_i) = \Gamma_2(\Sigma r_i\overline{x_ix}) - \Gamma_2(\Sigma r_i\overline{x_i})^{\varphi(x)}$,
4. Γ_2 is a homomorphism of R-modules,
5. $\delta\Gamma_2(\Sigma r_i\overline{x}_i) = \Sigma r_i\varphi(x_i)$,
6. $\delta\Gamma_1(x_1,x_2) = \varphi(x_1x_2)-\varphi(x_1)\varphi(x_2)$

for $r_i\in R$, $x,x_1,x_2,x_3,x_i\in X$ and $\Sigma r_i\overline{x_i}\in N(X)$.

We denote $\mathbf{Z}^2(X,\Phi)$ the set of 2-cocycles from X to Φ.

Two 2-cocycles $(\Gamma_1,\Gamma_2,\varphi)$ and $(\Gamma'_1,\Gamma'_2,\varphi')$ are equivalent if there exists a map $p: X\to B$ such that

i) $\varphi'(x) = \delta p(x)+\varphi(x)$,

ii) $\Gamma'_1(x_1,x_2) = p(x_1x_2)-p(x_1)p(x_2)-p(x_1)^{\varphi(x_2)}-{}^{\varphi(x_1)}p(x_2)+\Gamma_1(x_1,x_2)$,

iii) $\Gamma'_2(\Sigma r_i\overline{x}_i) = \Sigma r_i p(x_i)+\Gamma_2(\Sigma r_i\overline{x}_i)$

for $x,x_1,x_2,x_i\in X$, $\Sigma r_i\overline{x}_i\in N(X)$.

This establishes an equivalence relation in $\mathbf{Z}^2(X,\Phi)$ and the quotient set $\mathbf{H}^2(X,\Phi)$ is by definition the second cohomology set of X with coefficients in Φ. Note that $(0,0,\varphi)\in\mathbf{Z}^2(X,\Phi)$ iff φ: $X\to A$ is a homomorphism. These special 2-cocycles form a privileged subset of $\mathbf{Z}^2$, the elements of which are called *neutral*. Thus, $\mathbf{H}^2(X,\Phi)$ is a set with prefered subset $\mathbf{O}^2(X,\Phi)$ of *neutral classes* (i.e., containing a neutral cocycle); the neutral class containing the neutral cocycle $(0,0,\varphi)$ will be called the φ-neutral class.

The set $\mathbf{H}^2(X,\Phi)$ has an interpretation in terms of equivalence classes of *Φ-extensions of* X, i.e., commutative diagrams

$$\mathbf{E} = \begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\beta} & E & \xrightarrow{\alpha} & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\pi} & & & & \\ & & B & \xrightarrow{\delta} & A & & & & \end{array}$$

where

$$0 \longrightarrow B \longrightarrow E \longrightarrow X \longrightarrow 0$$

is a short exact sequence of algebras such that the actions of E on B by translation coincide with those induced by π, i.e.,

$$eb = {}^{\pi(e)}b,\ be = b^{\pi(e)},\ b\in B,\ e\in E.$$

A Φ-extension $\mathbf{E}$ of X has associated a 2-cocycle from X to Φ, $(\Gamma_1,\Gamma_2,\varphi)$ as follows: To each $x \in X$, choose a representative $u(x)$ in E, that is, an element $u(x)$ with $\alpha(u(x)) = x$; in particular let us choose $u(0) = 0$. We define

$$\varphi(x) = \pi(u(x)),\ \Gamma_1(x_1,x_2) = u(x_1x_2) - u(x_1)u(x_2),\ \Gamma_2(\Sigma r_i\overline{x}_i) = \Sigma r_i u(x_i)$$

for $x, x_1, x_2, x_i \in X$, $r_i \in R$, $\Sigma r_i\overline{x}_i \in N(X)$, and it is straightforward to see that $(\Gamma_1,\Gamma_2,\varphi)$ is really a 2-cocycle. Note that this cocycle is normalized in the sense that

$$\varphi(0) = 0,\ \Gamma_2(\overline{0}) = 0 \text{ and } \Gamma_1(x,0) = 0 = \Gamma_1(0,x),\ x \in X.$$

The 2-cocycle $(\Gamma_1,\Gamma_2,\varphi)$ depends on a choice of representatives, but if $u'(x)$ is a second set of representatives, $u'(x)$ and $u(x)$ lie in the same coset, so there is a map $p\colon X \to B$ given by $p(x) = u'(x) - u(x)$ and the new 2-cocyle is

$$\begin{aligned}
&\varphi'(x) = \delta p(x) + \varphi(x),\\
&\Gamma'_1(x_1,x_2) = p(x_1x_2) - p(x_1)p(x_2) - p(x_1)^{\varphi(x_2)} - {}^{\varphi(x_1)}p(x_2) + \Gamma_1(x_1,x_2),\\
&\Gamma'_2(\Sigma r_i\overline{x}_i) = \Sigma r_i p(x_i) + \Gamma_2(\Sigma r_i\overline{x}_i).
\end{aligned}$$

So both cocycles represent the same class in $\mathbf{H}^2(X,\Phi)$.

Therefore we have a canonical map $\Omega\colon \mathbf{Ext}(X,\Phi) \to \mathbf{H}^2(X,\Phi)$ where $\mathbf{Ext}(X,\Phi)$ denotes the set of isomorphism classes of Φ-extensions of X.

Now, in order to show that Ω is a bijection, note that any 2-cocycle $(\Gamma'_1,\Gamma'_2,\varphi')$ is equivalent to a normalized one $(\Gamma_1,\Gamma_2,\varphi)$ through the map $p\colon X \to B$ given by $p(x) = -\Gamma_2(\overline{0})$, $x \in X$. Then, given a normalized $(\Gamma_1,\Gamma_2,\varphi) \in \mathbf{Z}^2(X,\Phi)$ consider in $\mathbf{Ext}(X,\Phi)$ the class of the Φ-extension

$$\mathbf{E} = \quad \begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{\beta} & E & \xrightarrow{\alpha} & X & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle\gamma} & & & & \\
 & & B & \xrightarrow{\delta} & A & & & &
\end{array}$$

where $E = B \times X$ with the operations

$$\begin{aligned}
(b_1,x_1) + (b_2,x_2) &= (b_1 + b_2 + \Gamma_2(\overline{x}_1 + \overline{x}_2 - \overline{x_1 + x_2}),\ x_1 + x_2),\\
r(b,x) &= (rb + \Gamma_2(r\overline{x} - \overline{rx}),\ rx),\\
(b_1,x_1)(b_2,x_2) &= (b_1b_2 + b_1^{\varphi(x_2)} + {}^{\varphi(x_1)}b_2 - \Gamma_1(x_1,x_2),\ x_1x_2),\\
\beta(b) = (b,0),\ \alpha(b,x) &= x \text{ and } \gamma(b,x) = \delta(b) + \varphi(x),
\end{aligned}$$

and it is straightforward to see that $\overline{\Omega}[(\Gamma_1,\Gamma_2,\varphi)] = [\mathbf{E}]$ is well defined and it is an inverse of Ω. Therefore we have a natural bijection $\mathbf{H}^2(X,\Phi) \approx \mathbf{Ext}(X,\Phi)$ through which the elements of $\mathbf{O}^2(X,\Phi)$ correspond to isomorphism classes of split Φ-extensions, that is, extensions with E the semidirect product of B by

X where the action of X on B is given via a homomorphism φ: $X \to A$.

Suppose now that $\Phi' = (\delta': B \to A', \mu')$, $\Phi = (\delta: B \to A, \mu)$ and $\Phi'' = (\delta'': B'' \to A'', \mu'')$ are crossed modules in **V** and recall that a sequence of morphisms of crossed modules

$$\Phi' \xrightarrow{(in,1)} \Phi \xrightarrow{p=(p_1,p_0)} \Phi''$$

is called a short exact sequence [16] if the sequence

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

is an exact sequence of algebras and $p_0: A \to A''$ an epimorphism of algebras. Then, if $\Phi' \to \Phi \to \Phi''$ is a short exact sequence of crossed modules, $X \in \mathbf{V}$ $\varphi: X \to A$ is a homomorphism and $\vartheta = p_0\varphi$: $X \to A''$ one has

THEOREM 1 (Dedecker-Lue). *There exists a sequence*

$$* \longrightarrow \mathbf{Z}^1_\varphi(X,\Phi') \longrightarrow \mathbf{Z}^1_\varphi(X,\Phi) \longrightarrow \mathbf{Z}^1_\vartheta(X,\Phi'') \longrightarrow \mathbf{H}^2(X,\Phi') \longrightarrow \mathbf{H}^2(X,\Phi) \longrightarrow \mathbf{H}^2(X,\Phi'')$$

which is exact in the following sense: The sequence

$$* \longrightarrow \mathbf{Z}^1_\varphi(X,\Phi') \longrightarrow \mathbf{Z}^1_\varphi(X,\Phi) \longrightarrow \mathbf{Z}^1_\vartheta(X,\Phi'')$$

is an exact sequence of pointed sets. An element of $\mathbf{Z}^1_\vartheta(X,\Phi'')$ *is in the image of the preceding map iff its image under the following map is neutral. An element of* $\mathbf{H}^2(X,\Phi')$ *(resp.* $\mathbf{H}^2(X,\Phi)$*) lies in the image of the preceding map iff its image under the following map is the* φ*-neutral class (resp. neutral).*

1.2. Low dimensional monadic non-abelian cohomology.

In this paragraph we will recall Duskin's non-abelian monadic 0- and 1-cohomology theory with coefficients in internal groupoids in an algebraic category, and we will show explicitly its relationship with Dedecker-Lue's theory for associative algebras.

Throughout this paragraph **C** will denote an algebraic category, i.e., monadic over Sets, $\mathbf{G} = (\mathbf{G}, \eta, \varepsilon)$ the cotriple associated to the forgetful functor $\mathbf{C} \to$ Sets and for each $S \in \mathbf{C}$, $\mathbf{G}.(S)$ the cotriple standard resolution of S and $\eta_S: S \to \mathbf{G}(S)$ will denote the natural inclusion map given by the unit of the adjunction. We will suppose that $\mathbf{G}.(S)$ is aspherical and let us note that this condition is always verified in all the more usual algebraic categories (see [23]).

Let us remember that an internal groupoid $\underset{\sim}{G}$ in **C** is a simplicial truncated diagram in **C**:

$$G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0 \quad (s_0 : G_0 \to G_1)$$

together with a morphism in **C**, $m: G_1\ {}_{d_1}\!\times_{d_0} G_1 \to G_1$ (the multiplication of $\underset{\sim}{G}$) satisfying

i) $m(m(x,y),z) = m(x,m(y,z))$,

ii) $m(x,s_0d_1x) = x = m(s_0d_0x,x)$,

iii) for all $x \in G_1$ there exists a unique $x^{-1} \in G_1$ such that

$$m(x,x^{-1}) = s_0d_0x,\ m(x^{-1},x) = s_0d_1x.$$

Usually we will denote $xy = m(x,y)$.

A morphism of groupoids $\underset{\sim}{f}: \underset{\sim}{G} \to \underset{\sim}{G}'$ is a commutative diagram:

$$\begin{array}{ccccc}
G_1\ {}_{d_1}\!\times_{d_0} G_1 & \longrightarrow & G_1 & \rightrightarrows & G_0 \\
\downarrow{\scriptstyle f_1\times f_1} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
G'_1\ {}_{d_1}\!\times_{d_0} G'_1 & \xrightarrow{m'} & G'_1 & \rightrightarrows & G'_0
\end{array}$$

and the corresponding category of groupoids in **C** is denoted by GPD(**C**).

Now, in the case $\underset{\sim}{G}$ is a groupoid in **V**, the category of algebras, since $d_0s_0 = \mathrm{id}$, every element in G_1 can be expressed uniquely as a sum $b+s_0(x)$ with $b \in B = \mathrm{Ker}(d_0)$ and $x \in G_0$; so G_1 is the semidirect product algebra of B and G_0, and the multiplication morphism of the groupoid is necessarily given by

$$m(b+s_0(x),\ b'+s_0(d_1(b)+x)) = b+b'+s_0(x)$$

(because

$$m(b+s_0(x), b'+s_0d_1(b)+x)) = m(b+s_0(x), s_0(d_1(b)+x))+m(0,b')$$
$$= m(b+s_0(x),\ s_0(d_1(b+s_0(x)\ )+m(s_0d_0b',b') = b+s_0(x)+b'\ ).$$

Then m is a homomorphism iff

$$m((b,b'+s_0d_1b)(b_0,b'_0+s_0d_1b_0))$$
$$= m(b,b'+s_0d_1b)\,m(b_0,b'_0+s_0d_1b_0)$$

or equivalently iff

$${}^{d_1b}b' + b'\,{}^{d_1b_0} = bb'+b'b$$

which implies that m is a homomorphism iff

$$b\,b' = {}^{d_1 b}b' = b^{d_1 b'} \quad \text{for all } b, b' \in B.$$

Then, si $B = \mathrm{Ker}(d_0)$, translations in G_1 via s_0 define a homomorphism $\mu\colon G_0 \to M_B$, and $\Phi(\underset{\sim}{G}) = (\delta\colon B \to G_0, \mu)$ where $\delta = d_1/B$, is a crossed module in $\mathbf{V}$.

Moreover, a morphism of groupoids $\underset{\sim}{f}\colon \underset{\sim}{G} \to \underset{\sim}{G}'$ induces a morphism of crossed modules

$$\Phi(\underset{\sim}{f}) = (f_1/\mathrm{Ker}(d_0), f_0)\colon \Phi(\underset{\sim}{G}) \to \Phi(\underset{\sim}{G}')$$

and so we have a functor $\Phi(-)\colon \mathrm{GPD}(\mathbf{V}) \to \mathrm{XM}(\mathbf{V})$ which is an equivalence; a quasi-inverse for $\Phi(-)$ is defined by associating to any crossed module $\Phi = (\delta\colon B \to A, \mu)$ the groupoid

$$\underset{\sim}{G}(\Phi) = (B \rtimes A)_{d_1}\!\times_{d_0}(B \rtimes A) \xrightarrow{\;m\;} (B \rtimes A) \rightleftarrows A$$

where

$$d_0(b,a) = a,\quad d_1(b,a) = \delta(b) + a,\quad s_0(a) = (0,a)$$

and

$$m((b,a),(b',\delta(b)+a)) = (b+b',a),$$

and to a crossed module morphism $(f,g)\colon \Phi \to \Phi'$ the groupoid morphism $(g_1,g)\colon \underset{\sim}{G}(\Phi) \to \underset{\sim}{G}(\Phi')$ where $g_1(b,a) = (f(b), g(a))$.

Thus we have

PROPOSITION 2 [26,7]. *The category* XM($\mathbf{V}$) *of crossed modules in* $\mathbf{V}$ *is equivalent to the category* GDP($\mathbf{V}$) *of groupoids in* $\mathbf{V}$. ∎

The 1-cocyles from an algebra X to a crossed module Φ have a natural translation in terms of the groupoid $\underset{\sim}{G}(\Phi)$.

DEFINITION 3. Given a groupoid $\underset{\sim}{G}$ in $\mathbf{C}$ and a morphism $\varphi\colon S \to G_0$ in $\mathbf{C}$, $\Gamma_\varphi(S,\underset{\sim}{G})$ is the set of all morphisms $f\colon S \to G_1$ in $\mathbf{C}$ such that $d_0 f = \varphi$.

Now, considering a crossed module Φ and the associated groupoid $\underset{\sim}{G}(\Phi)$, on has

PROPOSITION 4. *There exists a natural bijection*

$$\mathbf{Z}^1_\varphi(X,\Phi) \approx \Gamma_\varphi(X,\underset{\sim}{G}(\Phi))$$

given by $(f,\varphi) \mapsto h$ *with* $h(x) = (f(x),\varphi(x))$, $x \in X$.

Let us note that $\Gamma_\varphi(X,\underset{\sim}{G}(\Phi))$ is pointed by $s_0\varphi$ and the base point $(0,\varphi)$ of $\mathbf{Z}^1_\varphi(X,\Phi)$ maps, by the above bijection, into $s_0\varphi$.

Now, if $S \in \mathbf{C}$ and $\underset{\sim}{G}$ is a groupoid in $\mathbf{C}$, following Duskin

[18] we define the 1-cohomology set S with coefficients in $\underset{\sim}{G}$ as the set of homotopy classes of simplicial morphisms from $\mathbf{G.}$(S) to the simplicial object Ner($\underset{\sim}{G}$), the nerve of the groupoid in the sense of Grothendieck, which is defined as follows:

$$(\mathrm{Ner}(\underset{\sim}{G}))_i = G_i,\ i = 0,1, \text{ and in general}$$
$$(\mathrm{Ner}(\underset{\sim}{G}))_n = \{(x_1,\dots,x_n) \in G_1^n \text{ such that } d_0 x_{j+1} = d_1 x_j,\ 0 \le j \le n-1\};$$

The face operators $d_i\colon \mathrm{Ner}(\underset{\sim}{G})_n \to \mathrm{Ner}(\underset{\sim}{G})_{n-1}$ are

$$d_0(x_1,\dots,x_n) = (x_1,\dots,x_{n-1}),$$
$$d_i(x_1,\dots,x_n) = (x_1,\dots,x_{n-i-1}, x_{n-i}x_{n-i+1}, x_{n-i+2},\dots,x_n),\ 1 \le i \le n-1$$
$$d_n(x_1,\dots,x_n) = (x_2,\dots,x_n),$$

and the degeneracies $s_i\colon \mathrm{Ner}(\underset{\sim}{G})_{n-1} \to \mathrm{Ner}(\underset{\sim}{G})_n$ are

$$s_0(x_1,\dots,x_{n-1}) = (x_1,\dots,x_{n-1}, s_0 d_1 x_{n-1}),$$
$$s_i(x_1,\dots,x_{n-1}) = (x_1,\dots,x_{n-i-1}, s_0 d_1 x_{n-i+1}, x_{n-i},\dots,x_n),\ 1 \le i \le n-2$$
$$s_{n-1}(x_1,\dots,x_{n-1}) = (s_0 d_0 x_1, x_1,\dots,x_{n-1}).$$

A 1-cocycle of S with coefficients in $\underset{\sim}{G}$ is by definition a simplicial morphism from $\mathbf{G.}$(S) to Ner($\underset{\sim}{G}$). We denote $\mathbf{Z}^1_G(S,\underset{\sim}{G})$ the set of such cocyles, i.e., $\mathbf{Z}^1_G(S,\underset{\sim}{G}) = (\mathbf{G.}(S),\mathrm{Ner}(\underset{\sim}{G}))$.

The homotopy relation defines an equivalence relation in $\mathbf{Z}^1_G(S,\underset{\sim}{G})$ and the corresponding quotient set

$$\mathbf{H}^1_G(S,\underset{\sim}{G}) = [\mathbf{G.}(S),\mathrm{Ner}(\underset{\sim}{G})]$$

is the set of cotriple cohomology of S with coefficients in $\underset{\sim}{G}$ [18]. There is in $\mathbf{H}^1_G(S,\underset{\sim}{G})$ a subset $\mathbf{O}^1_G(S,\underset{\sim}{G})$ of distinguished elements ("neutral elements"), those classes of 1-cocycles $(s_0\varphi d_0^n\varepsilon_S)_{n\ge 0}$ ("neutral 1-cocycles") with $\varphi\colon S \to G_0$ a homomorphism; the class containing the neutral cocycle $(s_0\varphi d_0^n\varepsilon_S)$ will be called the φ-neutral class.

These $\mathbf{H}^1_G$ are functorial in both variables (contravariant in the first one).

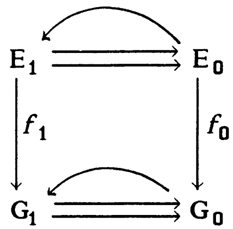
Let us note that a simplicial morphism $f. \in (E., \mathrm{Ner}(\underset{\sim}{G}))$ is completely determined by its 1-truncation (f_1, f_0), since for any $n \ge 2$ and $v \in E_n$

$$f_n(v) = (f_1 d_0^{n-1}(v), f_1 d_0^{n-2} d_n(v),\dots, f_1 d_0 d_3 \dots d_n(v), f_1 d_2 \dots d_n(v))$$

and conversely, an 1-truncated simplicial morphism as below extends to a simplicial morphism iff the "cocycle condition"

(CC1) $\qquad f_1 d_0(v) f_1 d_2(v) = f_1 d_1(v)$ for all $v \in E_2$

is verified.



A 1-truncated simplicial morphism

$$(f_1, f_0): \mathrm{Tr}^1(\mathbf{G.}(X)) \to \mathrm{Tr}^1(\mathrm{Ner}(\underset{\sim}{G}))$$

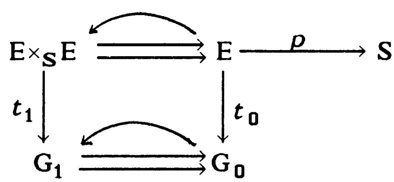
not necessarily satisfying the cocycle condition (CC1) will be called a 1-*cochain* and $\mathbf{C}^1_G(X,\underset{\sim}{G})$ will denote the set of such 1-cochains.

Also, any homotopy $h.: f. \to g.$ for $f., g. \in (E., \mathrm{Ner}(\underset{\sim}{G}))$ is completely determined by its truncation $h^0_0: E_0 \to G_1$; and a homomorphism h^0_0 defines a homotopy from $f.$ to $g.$ iff the conditions

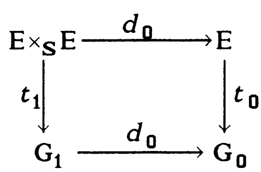
(HC1) | i) $d_0 h^0_0 = f_0$; ii) $d_1 h^0_0 = g_0$;
iii) $f_1(z) h^0_0 d_1(z) = h^0_0 d_0(z) g_1(z), \quad z \in E_1$

are satisfied.

The set $\mathbf{H}^1_G(S,\underset{\sim}{G})$ has an interpretation in terms of equivalence classes of $\underset{\sim}{G}$-torsors over S (principal homogeneous spaces over S under $\underset{\sim}{G}$). Let us recall that a $\underset{\sim}{G}$-torsor over S is a truncated simplicial morphism

$\mathbf{E} =$ 

such that p is a surjective epimorphism, the square

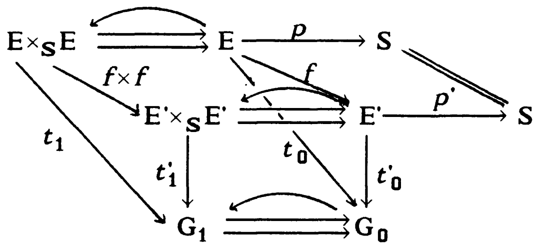


is a pullback, and for any $(z_0, z_1, z_2) \in E \times_S E \times_S E$ the cocycle condition

$$t_1(z_0, z_1)\, t_1(z_1, z_2) = t_1(z_0, z_2)$$

is satisfied.

A morphism of $\underset{\sim}{G}$-torsors over S is a commutative diagram



i.e., a homomorphism $f\colon E \to E'$ such that

$$p'f = p,\ t'_0 f = t_0 \text{ and } t'(f\times f) = t_1.$$

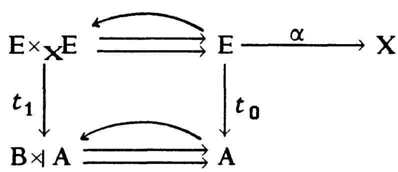
It is easy to see that every $\underset{\sim}{G}$-torsor morphism is an isomorphism, and we will denote the set of isomorphism classes of $\underset{\sim}{G}$-torsors over S by $\mathbf{Tors}^1[S,\underset{\sim}{G}]$.

PROPOSITION 5 (Duskin [18]). *There exists a natural bijection*

$$\mathbf{H}^1_G(S,\underset{\sim}{G}) = \mathbf{Tors}^1[S,\underset{\sim}{G}]. \blacksquare$$

As we said in the introduction it was observed by Duskin that torsors under groupoids correspond categorically to non-singular extensions in the usual algebraic categories. We can now make this fact explicit in the category **V** of associative algebras.

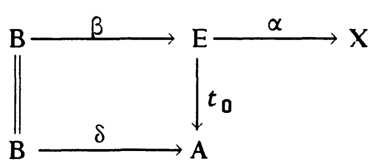
Let us suppose $\Phi = (B \to A, \mu)$ is a crossed mocule in **V** and $\underset{\sim}{G}(\Phi)$ is the associated groupoid as in Proposition 2. If



is a torsor over X under $\underset{\sim}{G}(\Phi)$, then

$$\mathrm{Ker}(\alpha) \approx \mathrm{Ker}(d_0\colon E\times_X E \to E) \approx \mathrm{Ker}\ (d_0\colon B \rtimes A \to A) = B$$

and so we have a commutative diagram



where $\beta(b)$ is the unique element of E such that

$$t_1(0,\beta(b)) = (b,0) \in B \rtimes A$$

$$(t_0\beta(b) = t_0 d_1(0,\beta(b)) = d_1 t_1(0,\beta(b)) = d_1(b,0) = \delta(b)).$$

Moreover, the actions of E on B by translation coincide with those induced by t_0. In fact:

$$t_1(0,e\beta(b)) = t_1[(e,e)(0,\beta(b))] = (0,t_0(e))(b,0) = ({}^{t_0(e)}b,0)$$

which implies $\beta({}^{t_0(e)}b) = e\beta(b)$ and

$$t_1(0,\beta(b)e) = t_1[(0,\beta(b)),(e,e)] = (b,0)(0,t_0(e)) = (b^{t_0(e)},0),$$

whence $\beta(b^{t_0(e)}) = \beta(b)e$. Conversely, if

$$\begin{array}{ccccc} B & \xrightarrow{\ \beta\ } & E & \xrightarrow{\ \alpha\ } & X \\ \| & & \downarrow{\scriptstyle t_0} & & \\ B & \xrightarrow{\ \delta\ } & A & & \end{array}$$

is a Φ-extension of X, it determines canonically, up to isomorphism, a torsor over X under $\underset{\sim}{G}(\Phi)$ which is given by

$$\begin{array}{ccccc} E\times_X E & \rightrightarrows & E & \xrightarrow{\ \alpha\ } & X \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle t_0} & & \\ B\rtimes A & \rightrightarrows & A & & \end{array}$$

where $t_1(e,\beta(b)+e) = (b,t_0(e))$ which is a morphism since

$$\begin{aligned} t_1[(e,\beta(b)+e)(e',\beta(b')+e')] &= t_1(ee',\beta(bb')+\beta(b)e'+e\beta(b')+ee') \\ &= t_1(ee',\beta(bb')+ee') + t_1(0,\beta(b)e') + t_1(0,e\beta(b')) \\ &= (bb',t_0(ee')) + t_1(0,\beta(b^{t_0(e')})) + t_1(0,\beta({}^{t_0(e)}b)) \\ &= (bb'+b^{t_0(e')}+{}^{t_0(e)}b',\ t_0(e)t_0(e')) \\ &= (b,t_0(e))(b',t_0(e')) = t_1(e,\beta(b)+e)\,t'_1(e',\beta(b')+e'). \end{aligned}$$

Finally the condition $E\times_X E \approx (B\rtimes A)\times_A E$ is clear, the isomorphism being $(e,\beta(b)+e) \mapsto ((b,t_0(e)),e)$. ∎

Since the Φ-extensions are classified by Dedecker-Lue's $\mathbf{H}^2$ and the $\underset{\sim}{G}(\Phi)$-torsors are classified by Duskin's $\mathbf{H}^1$ it is clear that there must be a natural isomorphism between them and we will give now the explicit relationship. For this, we will use the description of $\mathbf{H}^1$ in terms of the simplicial "covering" of S,

$\mathrm{Cosk}^0(\mathbf{G}(S) \longrightarrow S) =$

$$\cdots\ \mathbf{G}(S)\times_S\mathbf{G}(S)\times_S\mathbf{G}(S) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathbf{G}(S)\times_S\mathbf{G}(S) \;\rightrightarrows\; \mathbf{G}(S) \xrightarrow{\ \varepsilon_S\ } S$$

The asphericity of $\mathbf{G}.(S)$ implies:

i) The canonical simplicial morphism $q.:\mathbf{G}.(S)\to \mathrm{Cosk}^0(\mathbf{G}(S)\to S)$ given by

$$q_n(z) = (d_0^n z, d_0^{n-1}\, d'_n z, \dots, d_0 d_2 \dots d_n z, d_1 d_2 \dots d_n z),\ z \in \mathbf{G}^{n+1}(S),$$

is a surjective epimorphism in any dimension.

ii) Any simplicial morphism $f. \in (\mathbf{G}.(S), \mathrm{Ner}(\underset{\sim}{G}))$ factors uniquely through the simplicial morphism $q.$.

$$\begin{array}{ccc} \mathbf{G}.(S) & \xrightarrow{\ q.\ } & \mathrm{Cosk}^0(\mathbf{G}(S) \longrightarrow S) \\ {\scriptstyle f.}\downarrow & \swarrow{\scriptstyle -\,-\,-} & \\ \mathrm{Ner}(\underset{\sim}{G}) & & \end{array}$$

iii) Any homotopy between simplicial morphisms in $(\mathbf{G}.(S), \mathrm{Ner}(\underset{\sim}{G}))$ induces another between the corresponding factorizations through $q.$. Consequently, $q.$ induces natural bijections

$$\mathbf{Z}^1_G(S,\underset{\sim}{G}) \approx (\mathrm{Cosk}^0(\mathbf{G}(S)\to S), \mathrm{Ner}(\underset{\sim}{G})).$$
$$\mathbf{H}^1_G(S,\underset{\sim}{G}) \approx [\mathrm{Cosk}^0(\mathbf{G}(S)\to S), \mathrm{Ner}(\underset{\sim}{G})].$$

Let us recall that for any algebra X, $\mathbf{G}(X)$ is the tensor algebra over the free R-module F(X) on the generators $(\bar{x} \mid x \in X)$, i.e.,

$$\mathbf{G}(X) = F(X)\oplus(F(X)\otimes_R F(X))\oplus\dots\oplus(F(X)\otimes_R\dots\otimes_R F(X))\dots$$

Then, an element $v \in \mathbf{G}(X)$ has a unique expression such as

$$v = \Sigma r_i \bar{x}_i + \Sigma r_{ij} \bar{x}_i \otimes \bar{x}_j + \dots + \Sigma r_{ij\dots l}\, \bar{x}_i \otimes \bar{x}_j \otimes \dots \otimes \bar{x}_l$$

and therefore, $v \in \mathrm{Ker}(\varepsilon_X: \mathbf{G}(X) \to X)$ iff

$$\Sigma r_i x_i + \Sigma r_{ij} x_i x_j + \dots + \Sigma r_{ij\dots l}\, x_i x_j \dots x_l = 0$$

and one can write

$$v = \Sigma r_i \bar{x}_i + \Sigma r_{ij} \overline{x_i x_j} + \dots + \Sigma r_{ij\dots l} \overline{x_i x_j \dots x_l} + (\Sigma r_{ij}(\bar{x}_i \otimes \bar{x}_j - \overline{x_i x_j}) +$$
$$+ \dots + \Sigma r_{ij\dots l} \bar{x}_i \otimes \bar{x}_j \otimes \dots \otimes \bar{x}_l - \overline{x_i x_j \dots x_l})).$$

Using this, it is plain to see that, if T(X) is the R-submodule of $\mathrm{Ker}(\varepsilon_X)$ generated by all elements of the form

$$v_2 = \overline{x_1 x_2} - \bar{x}_1 \otimes \bar{x}_2,$$
$$v_3 = \overline{x_1 x_2 x_3} - \bar{x}_1 \otimes \bar{x}_2 \otimes \bar{x}_3,$$
$$\dots\dots\dots\dots$$
$$v_n = \overline{x_1 x_2 \dots x_n} - \bar{x}_1 \otimes \bar{x}_2 \otimes \dots \otimes \bar{x}_n,$$

one has

PROPOSITION 6. T(X) *is free, as a* R*-module, on the generators*

$v_2, v_3, \ldots, v_n, \ldots$, *and* $\mathrm{Ker}(\varepsilon_X)$ *is, as an* R*-module, the direct sum of* T(X) *and* $N(X) = \mathrm{Ker}(F(X) \to X)$. ∎

PROPOSITION 7. *Given an algebra* X *and a crossed module* $\Phi = (\delta: B \to A, \mu)$ *there exists a natural bijection*

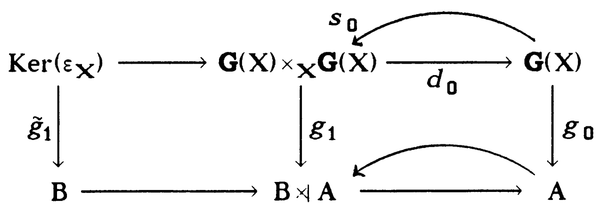
$$\vartheta: \mathbf{Z}^2(X,\Phi) \approx \mathbf{Z}^1_{\mathbf{G}}(X, \underset{\sim}{G}(\Phi))$$

which maps one to one neutral 2*-cocycles into neutral* 1*-cocycles.*

PROOF. Let us give $(\Gamma_1, \Gamma_2, \varphi) \in \mathbf{Z}^2(X,\Phi)$. The map φ induces a homomorphism $g_0: \mathbf{G}(X) \to A$ such that $g_0(\bar{x}) = \varphi(x)$, $x \in X$. Since every element in $\mathbf{G}(X) \times_X \mathbf{G}(X)$ has a unique expression such as $(z, v+z)$, $v \in \mathrm{Ker}(\varepsilon_X: \mathbf{G}(X) \to X)$, if we map

$$v_2 \longmapsto \Gamma_1(x_1, x_2),$$
$$v_3 \longmapsto \Gamma_1(x_1 x_2, x_3) + \Gamma_1(x_1, x_2)^{\varphi(x_3)},$$
$$v_4 \longmapsto \Gamma_1(x_1 x_2 x_3, x_4) + \Gamma_1(x_1 x_2, x_3)^{\varphi(x_4)} + \Gamma_1(x_1, x_2)^{\varphi(x_3)\varphi(x_4)},$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$v_n \longmapsto \Gamma_1(x_1 \ldots x_{n-1}, x_n) + \ldots + \Gamma_1(x_1, x_2)^{\varphi(x_3)\ldots\varphi(x_n)},$$

we have a well defined map (really a homomorphism of R-modules) $g_1: \mathbf{G}(X) \times_X \mathbf{G}(X) \to B \rtimes A$ given by $g_1(z, v+z) = (\tilde{g}_1(v), g_0(z))$, where $\tilde{g}_1: \mathrm{Ker}(\varepsilon_X) \to B$ is given by Γ_2 and the unique homomorphisms of R-modules induced by the above mappings, using that $\mathrm{Ker}(\varepsilon_X) = T(X) \oplus N(X)$. In other words, $\tilde{g}_1$ is the unique map making commutative the diagram



This map g_1 is a homomorphism iff the conditions

$$\text{i) } \tilde{g}_1(vz) = \tilde{g}_1(v)^{g_0(z)}, \qquad \text{ii) } \tilde{g}_1(zv) = {}^{g_0(z)}\tilde{g}_1(v)$$

are verified for all $z \in \mathbf{G}(X)$ and $v \in \mathrm{Ker}(\varepsilon_X)$; but if $Y = \{z \in \mathbf{G}(X) \mid$ i) and ii) are verified for all $v \in \mathrm{Ker}(\varepsilon_X)\}$, Y is a subalgebra of $\mathbf{G}(X)$ and it is straightforward to see that $\{\bar{x} \mid x \in X\} \subset Y$. So $Y = G(X)$ and g_1 is a homomorphism.

Moreover, by construction $d_0 g_1 = g_0 d_1$ and $g_1 s_0 = s_0 g_0$; then the pair (g_1, g_0) is a truncated simplicial morphism

$$\begin{array}{ccc} \mathbf{G}(X)\times_X\mathbf{G}(X) & \rightrightarrows & \mathbf{G}(X) \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} \\ B\rtimes A & \rightrightarrows & A \end{array}$$

iff $d_1g_1 = g_0d_1$. Now, since $d_1g_1(z, v+z) = \delta\tilde{g}_1(v) + g_0(z)$, $d_1g_1 = g_0d_1$ iff $\delta\tilde{g}_1(v) = g_0(v)$ and, by using the conditions defining a Dedecker-Lue's 2-cocycle, it is straightforward to see that this relation is true for any $v \in \mathrm{Ker}(\varepsilon_X)$. So, (g_1,g_0) is a truncated simplicial morphism which satisfies the cocycle condition since for any $(z_0,z_1,z_2) \in \mathbf{G}(X)\times_X\mathbf{G}(X)\times_X\mathbf{G}(X)$,

$$\begin{aligned} g_1(z_0,z_1)g_1(z_1,z_2) &= (\tilde{g}_1(z_1-z_0),g_0(z_0))(\tilde{g}_1(z_2-z_1),g_0(z_1)) \\ &= (\tilde{g}_1(z_2-z_0),g_0(z_0)) = g_1(z_0,z_2). \end{aligned}$$

Conversely, a truncated simplicial morphism (g_1,g_0) has uniquely associated a Dedecker-Lue's 2-cocycle $(\Gamma_1,\Gamma_2,\varphi)$ where

$\Gamma_1: X\times X \longrightarrow B$ is given by $(\Gamma_1(x_1,x_2),0) = g_1(0,\overline{x_1x_2}-\bar{x}_1\otimes\bar{x}_2)$,
$\Gamma_2: N(X) \longrightarrow B$ is given by $(\Gamma_2(\Sigma r_i\bar{x}_i),0) = g_1(0,\Sigma r_i\bar{x}_i)$ and
$\varphi: X\rightarrow A$ is given by $\varphi(x) = g_0(\bar{x})$.

So we have just a bijection $\vartheta: \mathbf{Z}^2(X,\Phi) \approx \mathbf{Z}^1_{\mathbf{G}}(X,\underset{\sim}{G}(\Phi))$ which clearly maps neutral 2-cocycles into neutral 1-cocycles. ∎

The following proposition shows that two Dedecker-Lue's cocycles are equivalent iff their images by ϑ are homotopic.

PROPOSITION 8. *Given two 2-cocycles* $(\Gamma_1,\Gamma_2,\varphi)$, $(\Gamma'_1,\Gamma'_2,\varphi')$ *in* $\mathbf{Z}^2(X,\Phi)$ *there is a canonical bijection between the set of maps* $p: X\rightarrow B$ *which define Dedecker-Lue's equivalences from* $(\Gamma_1,\Gamma_2,\varphi)$ *to* $(\Gamma'_1,\Gamma'_2,\varphi')$ *and the set of homotopies from* $\vartheta(\Gamma_1,\Gamma_2,\varphi)$ *to* $\vartheta(\Gamma'_1,\Gamma'_2,\varphi')$.

PROOF. Let $\vartheta(\Gamma_1,\Gamma_2,\varphi) = (g_1,g_0)$ and $\vartheta(\Gamma'_1,\Gamma'_2,\varphi') = (g'_1,g'_0)$.

Since $\mathbf{G}(X)$ is the free algebra on X, to give a homomorphism $h^0_0: \mathbf{G}(X)\rightarrow B\rtimes A$ such that $d_0h^0_0 = g_0$, $d_1h^0_0 = g'_0$ is equivalent to give a map $p: X\rightarrow B$ such that $\varphi'(x) = \delta p(x) + \varphi(x)$ and $h^0_0(\bar{x}) = (p(x),\varphi(x))$, $x \in X$. Now, the homotopy condition

$$g_1(z,z')h^0_0(z') = h^0_0(z)g'_1(z,z'),\ (z,z') \in \mathbf{G}(X)\times_X\mathbf{G}(X)$$

is equivalent to the conditions

1. $\Gamma'_1(x_1,x_2) = p(x_1x_2) - p(x_1)p(x_2) - p(x_1)^{\varphi(x_2)} - {}^{\varphi(x_1)}p(x_2) + \Gamma_1(x_1,x_2)$
2. $\Gamma'_2(\Sigma r_i\bar{x}_i) = \Sigma r_ip(x_i) + \Gamma_2(\Sigma r_i\bar{x}_i)$, $x_1,x_2,x_i \in X$, $\Sigma r_i\bar{x}_i \in N(X)$,

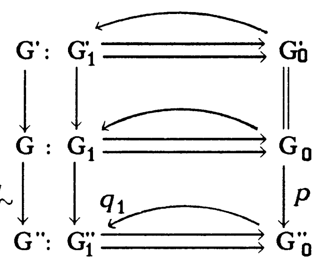
for Dedecker-Lue's equivalent cocycles. In fact, since any element $(z,z')\in \mathbf{G}(X)\times_X\mathbf{G}(X)$ can be expressed as $(0,v)+(z,z)$, $v\in \mathrm{Ker}(\varepsilon_X)$, and for elements as (z,z) the homotopy condition is easily verified, we reduce the homotopy condition to $g_1(0,v)h_0^0(v) = g'_1(0,v)$, which is equivalent to 1 if $v\in T(X)$ and to 2 if $v\in N(X)$. ∎ ●

COROLLARY 9. *For any algebra* X *and crossed module* Φ, ϑ *induces a canonical bijection* $\mathbf{H}^2(X,\Phi)\approx \mathbf{H}^1_{\mathbf{G}}(X,\underset{\sim}{G}(\Phi))$ *mapping bijectively* $\mathbf{O}^2(X,\Phi)$ *onto* $\mathbf{O}^1_{\mathbf{G}}(X,\underset{\sim}{G}(\Phi))$. ∎

1.3. The 6-term exact sequence in non-abelian monadic cohomology.

In the more usual algebraic categories the 6-term exact sequences in non-abelian cohomology [12,16,26,2,...] are associated to "short exact sequences" of crossed modules; this concept corresponds categorically to that of surjective precofibration (in the sense of Grothendieck [24]) of groupoids. We note this fact in the case of algebras.

PROPOSITION 10. *The equivalence of categories* $XM(\mathbf{V})\approx GPD(\mathbf{V})$ *carries short exact sequences of crossed modules into sequences of groupoids*

(3) 

such that

1. $\underset{\sim}{G}'$ *is the subgroupoid kernel of* $\underset{\sim}{q}$ *(i.e.,*

$$G'_1 = \{x\in G_1 \mid q_1(x) = s_0p_0d_0(x)\}).$$

2. p_0 *is surjective.*

3. $\underset{\sim}{q}$ *is a precofibration, i.e., the canonical morphism* (q_1,d_0) *from* G_1 *to* $G''_1{}_{d_0}\!\times_{p_0}G_0$ *is surjective.* ∎

Now we will establish a general 6-term exact sequence associated to a sequence of groupoids in an algebraic category **C** satisfying the above conditions 1, 2 and 3. That sequence restricted to the more algebraic contexts (algebras, groups,...) is equivalent to those classically established in them. Moreover, the simplicial way in which it is obtained will be used to prove the existence of the 9-term exact sequence for algebras.

PROPOSITION 11. *Let $(q_1, p_0): \underset{\sim}{G} \to \underset{\sim}{G}''$ be a precofibration of groupoids surjective on objects and let $\underset{\sim}{G}'$ be the subgroupoid kernel. For any morphism $\varphi: S \to \underset{\sim}{G}_0$ there exists a sequence*

$$* \longrightarrow \Gamma_\varphi(S,\underset{\sim}{G}') \xrightarrow{i_*} \Gamma_\varphi(S,\underset{\sim}{G}) \xrightarrow{q_*} \Gamma_\vartheta(S,\underset{\sim}{G}'') \xrightarrow{\chi^1} \mathbf{H}^1_G(S,\underset{\sim}{G}') \xrightarrow{i_*} \mathbf{H}^1_G(S,\underset{\sim}{G}) \xrightarrow{q_*} \mathbf{H}^1_G(S,\underset{\sim}{G}'')$$

where $\vartheta = p_0\varphi$, which is exact in the following sense:

The sequence

$$* \longrightarrow \Gamma_\varphi(S,\underset{\sim}{G}') \xrightarrow{i_*} \Gamma_\varphi(S,\underset{\sim}{G}) \xrightarrow{q_*} \Gamma_\vartheta(S,\underset{\sim}{G}'')$$

is an exact sequence of pointed sets. An element of $\Gamma_\vartheta(S,\underset{\sim}{G}'')$ is in the image of the preceding map iff its image under the following map is neutral: an element of $\mathbf{H}^1_G(S,\underset{\sim}{G}')$ (resp. $\mathbf{H}^1_G(S,\underset{\sim}{G})$) lies in the image of the preceding map iff its image under the following map is the φ-neutral class (resp. neutral).

PROOF. The exactness in the 2 first terms is clear.

We define the connecting map $\chi^1: \Gamma_\vartheta(S,\underset{\sim}{G}'') \to \mathbf{H}^1_G(S,\underset{\sim}{G}')$. Given $f \in \Gamma_\vartheta(S,\underset{\sim}{G}'')$, there exists a morphism $\mu: \mathbf{G}(S) \to G_1$ such that $(q_1, d_0)\mu = (f,\varphi)\varepsilon_S$ and we have a pair of morphisms (f_1, f_0) where $f_0: \mathbf{G}(S) \to G_0$ is given by $f_0 = d_1\mu$ and $f_1: \mathbf{G}(S) \times_S \mathbf{G}(S) \to G_1$ is given by $f_1(z_0,z_1) = \mu(z_0)^{-1}\mu(z_1)$ for any $(z_0,z_1) \in \mathbf{G}(S)\times_S\mathbf{G}(S)$.

Note that $f_1(z_0,z_1) \in G'_1$ since

$$\begin{aligned} q_1 f_1(z_0,z_1) &= q_1\mu(z_0)^{-1}q_1\mu(z_1) \\ &= (f\varepsilon_S(z_0))^{-1}(f\varepsilon_S(z_1)) = s_0 p_0 \varphi \varepsilon_S(z_0) \end{aligned}$$

and (f_1, f_0) is a truncated simplicial morphism which satisfies the cocycle condition since

$$\begin{aligned} d_0 f_1(z_0,z_1) &= d_0\mu(z_0)^{-1} = d_1\mu(z_0) = f_0(z_0) = f_0 d_0(z_0,z_1), \\ d_1 f_1(z_0,z_1) &= d_1\mu(z_1) = f_0(z_1) = f_0 d_1(z_0,z_1), \\ f_1\, s_0(z) &= f_1(z,z) = s_0 d_1\mu(z) = s_0 f_0(z), \\ f_1(z_0,z_1) f_1(z_1,z_2) &= \mu(z_0)^{-1}\mu(z_1)\mu(z_1)^{-1}\mu(z_2) \\ &= \mu(z_0)^{-1}\mu(z_2) = f_1(z_0,z_2). \end{aligned}$$

Then (f_1, f_0) defines a 1-cocycle whose class in $\mathbf{H}^1_G(S,\underset{\sim}{G}')$ does not depend of the election of μ since if μ' is another morphism with $(q_1,d_0)\mu' = (f,\varphi)\varepsilon_S$ and (f'_1, f'_0) is the corresponding cocycle as above, the morphism $h^0_0: \mathbf{G}(S) \to G'_1$ given by

$h_0^0(z) = \mu(z)^{-1}\mu'(z)$ defines a homotopy from (f_1, f_0) to (f_1', f_0').

Thus, we define $\xi^1: \Gamma_\vartheta(S, \underset{\sim}{G}'') \to \mathbf{H}^1_G(S, \underset{\sim}{G}')$ by $\xi^1(f) = [(f_1, f_0)]$.

The exactness in $\Gamma_\vartheta(S, \underset{\sim}{G}'')$: Let $g \in \Gamma_\varphi(S, \underset{\sim}{G})$ and $f = q_1 g: S \to G_1''$. The morphism $\mu: \mathbf{G}(S) \to G_1$ given by $\mu = g\varepsilon_S$ verifies $(q_1, d_0)\mu = (f, \varphi)\varepsilon_S$, so $\xi^1(f) = [(f_1, f_0)]$ where $f_0 = d_1\mu = d_1 g\varepsilon_S$ and

$$f_1(z_0, z_1) = \mu(z_0)^{-1}\mu(z_1) = s_0 d_1 g \varepsilon_S(z_0)$$

and therefore $\xi^1(f)$ is the φ'-neutral class where $\varphi' = d_1 g$.

Reciprocally, let $f \in \Gamma_\vartheta(S, \underset{\sim}{G}'')$ such that $\xi^1(f)$ is the φ'-neutral class. If $\mu: \mathbf{G}(S) \to G_1$ is a morphism such that $(q_1, d_0)\mu = (f, \varphi)\varepsilon_S$ then $\xi^1(f)$ is represented by the 1-cocycle defined by the pair (f_0, f_1) where $f_0 = d_1\mu$ and $f_1(z_0, z_1) = \mu(z_0)^{-1}\mu(z_1)$ and so it must exist a morphism $h_0^0: \mathbf{G}(S) \to G_1'$ defining a homotopy from the cocycle (f_0, f_1) to $(s_0\varphi'\varepsilon_S d_0, \varphi'\varepsilon_S)$. Then, the morphism $g': \mathbf{G}(S) \to G_1$ given by $g'(z) = \mu(z) h_0^0(z)$ factors through ε_S since for any $(z_0, z_1) \in \mathbf{G}(S) \times_S \mathbf{G}(S)$:

$$g'(z) = \mu(z_0) h_0^0(z_0) = \mu(z_0) f_1(z_0, z_1) h_0^0(z_1) = \mu(z_1) h_0^0(z_1) = g'(z_1)$$

and so there will exist $g: \mathbf{G}(S) \to G_1$ such that $g\varepsilon_S = g'$. It is plain to see that $q_1 g = f$ and $d_0 g = \varphi$, i.e., $g \in \Gamma_\varphi(S, \underset{\sim}{G})$ and $q_*(g) = f$.

The exactness in $\mathbf{H}^1_G(S, \underset{\sim}{G}')$: Let $f \in \Gamma_\vartheta(S, \underset{\sim}{G}'')$ and $\chi^1(f) = [(f_1, f_0)]$ with $f_0 = d_1\mu$ and $f_1(z_0, z_1) = \mu(z_0)^{-1}\mu(z_1)$ for a homomorphism $\mu: \mathbf{G}(S) \to G_1$ verifying $q_1\mu = f\varepsilon_S$ and $d_0\mu = \varphi\varepsilon_S$. Then $i_*\chi^1(f)$ is the φ-neutral class in $\mathbf{H}^1_G(S, \underset{\sim}{G})$ since $h_0^0 = \mu$ defines a homotopy from $(s_0\varphi\varepsilon_S d_0, \varphi\varepsilon_S)$ to $i_*[(f_1, f_0)]$.

Reciprocally, if $[(f_1, f_0)] \in \mathbf{H}^1_G(S, \underset{\sim}{G}')$ is such that $i_*[(f_1, f_0)]$ is the φ-neutral class, there will exist a homomorphism h_0^0: $\mathbf{G}(S) \to G_1$ such that $d_0 h_0^0 = \varphi\varepsilon_S$, $d_1 h_0^0 = f_0$ and for any (z_0, z_1) in $\mathbf{G}(S) \times_S \mathbf{G}(S)$, $h_0^0(z_0) f_1(z_0, z_1) = h_0^0(z_1)$. Then the morphism $q_1 h_0^0$: $\mathbf{G}(S) \to G_1''$ factors through ε_S so that there will exist $f: S \to G_1''$ such that $f\varepsilon_S = q_1 h_0^0$ and $f \in \Gamma_\vartheta(S, \underset{\sim}{G}'')$ since

$$d_0 f \varepsilon_S = d_0 q_1 h_0^0 = p_0 d_0 h_0^0 = p_0 \varphi \varepsilon_S$$

and consequently $d_0 f = p_0\varphi$. Now, recalling the definition of χ^1, it is clear that the cocycle associated to f via the choice of $\mu = h_0^0$ is just (f_1, f_0).

The exactness in $\mathbf{H}^1_G(S, \underset{\sim}{G})$: Let $[(f_1, f_0)] \in \mathbf{H}_G(S, \underset{\sim}{G}')$. Then

$$q_* i_*[(f_1, f_0)] = [(q_1 f_1, p_0 f_0)]$$

and since $p_0 f_0$ factors through ε_S, there will exist a morphism $\varphi'': S \to G_0''$ such that $p_0 f_0 = \varphi''\varepsilon_S$. Moreover, since f_1 has codomain G_1',

$$q_1 f_1 = s_0 p_0 f_0 d_0 = s_0 \varphi'' \varepsilon_S d_0$$

so that $q_* i_* [(f_1, f_0)]$ is the φ''-neutral class.

Reciprocally, let $[(g_1, g_0)] \in \mathbf{H}^1_{\mathbf{G}}(S, \underset{\sim}{G})$ be such that

$$q_*[(g_1, g_0)] = [(q_1 g_1, p_0 g_0)]$$

is the φ''-neutral class. Then there will exist a morphism h^0_0: $\mathbf{G}(S) \to G''_1$ defining a homotopy from (g_1, g_0) to the neutral cocycle $(s_0 \varphi'' \varepsilon_S d_0, \varphi'' \varepsilon_S)$. Now, since q is a precofibration there exists a morphism α: $\mathbf{G}(S) \to G_1$ such that $(q_1, d_0)\alpha = (h^0_0, g_0)$. Then the pair (f_1, f_0) where

$$f_0 = d_1 \alpha \text{ and } f_1(z_0, z_1) = \alpha(z_0)^{-1} g_1(z_0, z_1) \alpha(z_1),$$

defines a 1-cocycle of $\mathbf{Z}^1_{\mathbf{G}}(S, \underset{\sim}{G}')$ since

$$q_1 f_1(z_0, z_1) = q_1 \alpha(z_0)^{-1} q_1 g_1(z_0, z_1) q_1 \alpha(z_1)$$
$$= h^0_0(z_0)^{-1} q_1 g_1(z_0, z_1) h^0_0(z_1) = s_0 \varphi'' \varepsilon_S(z_0),$$

and

$$f_1(z_0, z_1) f_1(z_1, z_2) = \alpha(z_0)^{-1} g_1(z_0, z_1) \alpha(z_1) \alpha(z_1)^{-1} g_1(z_1, z_2) \alpha(z_2) =$$
$$\alpha(z_0)^{-1} g_1(z_0, z_2)\ \alpha(z_2) = f_1(z_0, z_2).$$

It is clear that $h^0_0 = \alpha$ defines a homotopy from (g_1, g_0) to $i_*(f_1, f_0)$ and so the proof is finished. ■

2. NON-ABELIAN H^3 WITH COEFFICIENTS IN CROSSED MODULES. THE 9-TERM EXACT SEQUENCE.

As we said in the Introduction, the object of this section is to define for any algebra X and any crossed module Φ a 3-dimensional cohomology set $\mathbf{H}^3(X, \Phi)$, functorial in both variables, such that it allows to obtain a 9-term exact sequence associated to a short exact sequence of crossed modules, extending the Dedecker-Lue's 6-term exact sequence and reducing to the usual abelian one in the case the short exact sequence of crossed modules corresponds to a short exact sequence of zero-algebras.

Since the notion of $\mathbf{H}^3(X, \Phi)$ is given using the monadic $\mathbf{H}^*_{\mathbf{G}}$ non-abelian cohomology with coefficients in hypergroupoids we start in the next paragraph recalling, in an algebraic category **C**, some results about the monadic non-abelian cohomology $\mathbf{H}^2_{\mathbf{G}}$ which was studied in more generality in [9].

2.1. Non-abelian cohomology $\mathbf{H}^2_{\mathbf{G}}$.

DEFINITION 12 [23]. A *2-hypergroupoid* in **C** is a simplicial object G. in **C** such that for $0 \leq i \leq m$ and all $m > 2$ the morphism

$$K_i^m = (d_0, \dots, d_{i-1}, -, d_{i+1}, \dots, d_m) : G_m \longrightarrow \Lambda_i^m(G.)$$

is an isomorphism. The full subcategory of Simpl(**C**) whose objects are 2-hypergroupoids is denoted 2-HYPGD(**C**).

A canonical example of 2-hypergroupoid is given by the Eilenberg-MacLane complex K(M,2) for M an internal abelian group in **C**, which is defined as the 3-coskeleton of the truncated simplicial object

$$K(M,2)_m = \left| \begin{array}{ll} 0 & \text{for } m = 0,1 \\ M & \text{for } m = 2 \\ M^3 & \text{for } m = 3 \end{array} \right.$$

with $d_i : M^3 \to M$, $i = 0,1,2$ the projections and

$$d_3(x_0, x_1, x_2) = x_0 - x_1 + x_2.$$

In [17], Duskin proves that the usual monadic cohomology of an object S with coefficients in the internal abelian group M, $\mathbf{H}^2_{\mathbf{G}}(S,M)$, is isomorphic to [**G**.(S), K(M,2)], the group of homotopy classes of simplicial morphisms from the standard cotriple resolution of S to the 2-hypergroupoid K(M,2). This fact and the equivalence of Corollary 9 suggested the definition we give in [9] of the cohomology sets with coefficients in arbitrary hypergroupoids as:

DEFINITION 13. Let G. be a 2-hypergroupoid in **C** and $S \in \mathbf{C}$. The *second cotriple cohomology set* $\mathbf{H}^2_{\mathbf{G}}(S,G.)$ of S with coefficients in G. is defined as the set of homotopy classes of simplicial morphisms in **C** from the standard cotriple resolution of S into G., i.e., $\mathbf{H}^2_{\mathbf{G}}(S,G.) = [\mathbf{G}.(S), G.]$.

A *2-cocycle* of S with coefficients in G. is by definition a simplicial morphism from **G**.(S) to G. and the set of 2-cocyycles is denoted $\mathbf{Z}^2_{\mathbf{G}}(S,G.)$.

In the more usual algebraic categories (Groups, Associative algebras, Lie algebras,...) any 2-hypergroupoid G. has associated a sub-2-hypergroupoid defined as the simplicial subobject of G. generated by the degenerate 2-simplices. A 2-cocycle $f. \in \mathbf{Z}^2_{\mathbf{G}}(S,G.)$ which factors through that 2-hypergroupoid will be called *neutral* and so $\mathbf{H}^2_{\mathbf{G}}(S,G.)$ is a set with a subset of "neutral classes", i.e., containing a neutral 2-cocycle. Let us note that every morphism $\varphi : S \to G_0$ determines the neutral 2-cocycle $(s_0^n \varphi \varepsilon_S d_0^{n+1})_{n \geq 0}$.

Clearly $\mathbf{H}^2_{\mathbf{G}}$ is functorial in both variables.

Just as in the 1-dimensional monadic cohomology, 2-cocycles and homotopies between them are determined by their 2-truncations verifying appropriate "cocycle" and "homotopy" conditions. In fact, any simplicial morphism $f. \in (E.,G.)$, with G. a 2-hypergroupoid, is completely determined by its 2-truncation (f_2, f_1, f_0) since for $n \geq 3$ and $v \in E_n$:

$$f_n(v) = (K_n^n)^{-1}(f_{n-1}d_0(v), \dots, f_{n-1}d_{n-1}(v), -),$$

and conversely, a 2-truncated simplicial morphism (f_2, f_1, f_0) extends to a simplicial morphism iff the "cocycle condition"

(CC2) $$d_3(K_3^3)^{-1}(f_2d_0, f_2d_1, f_2d_2, -) = f_2d_3$$

is verified (see [9]).

Also, any homotopy $h.: f. \to g.$ for $f., g. \in (E.,G.)$ is completely determined by its truncation (h_0^1, h_1^1, h_0^0) since for $m \geq 2$

$$h_0^m = (K_1^{m+1})^{-1}(f_m, -, h_0^{m-1}d_1, \dots, h_0^{m-1}d_m),$$

....

$$h_i^m =$$
$$(K_{i+1}^m)^{-1}(h_{i-1}^m d_0, \dots, h_{i-1}^m d_{i-1}, d_i h_{i-1}^m, -, h_i^{m-1}d_{i+1}, \dots, h_i^{m-1}d_m),$$

....

$$h_m^m = (K_{m+1}^m)^{-1}(h_{m-1}^m d_0, \dots, h_{m-1}^m d_{m-1}, d_m h_{m-1}^m, -),$$

and conversely, a truncated homotopy (h_0^1, h_1^1, h_0^0) extends to a homotopy from $f.$ to $g.$ iff, with

$$h_0^2 = (K_1^3)^{-1}(f_2, -, h_0^1 d_1, h_0^1 d_2),\ h_1^2 = (K_3^3)^{-1}(h_0^1 d_0, d_1 h_0^2, -, h_1^1 d_2)$$

and $$h_2^2 = (K_2^3)^{-1}(h_1^1 d_0, h_1^1 d_1, d_2 h_1^2, -)$$

then the *homotopy condition*

(HC2) $$d_3 h_2^2 = g_2$$

is verified (see [9]).

The following proposition shows that in the more usual algebraic categories the homotopy relation between 2-cocycles is an equivalence relation.

PROPOSITION 14. *Let* G. *be a 2-hypergroupoid in* **C** *which is a Kan complex (i.e., the canonical morphisms* K_i^m *are surjective),* $X \in$ **C** *and 2-cocycles* $f., g., t. \in (\mathbf{G}.(S), G.)$.

(a) *If* $h.: f. \to g.$ *and* $h'.: g. \to t.$ *are homotopies, there exists a homotopy* $h''.: f. \to t.$.

(b) *If* $h.: f. \to g.$ *and* $h''.: f. \to t.$ *are homotopies, there exists a homotopy* $h'.: g. \to t.$.

PROOF. (a) Since the canonical morphism $K_1^2: G_2 \to \Lambda_1^2(G.)$ is surjective, there is a morphism $V: \mathbf{G}^2(S) \to G_2$ with $K_1^2 V =$

$(d_1h^1_0,-,h'^0_0d_1)$. Then we define

$$h''^0_0 = d_1Vs_0,\ h''^1_0 = d_1(K^3_1)^{-1}(h^1_0,-,V,Vs_0d_1)$$

and $$h''^1_1 = d_2(K^3_2)^{-1}(Vs_0d_0,Y,-,h'^1_1)$$

where $Y = d_1(K^3_1)^{-1}(h^1_1,-,V,h'^1_0)$. Thus

$$d_0h''^0_0 = d_0d_1Vs_0 = d_0d_0Vs_0 = d_0d_1h^1_0s_0 = f_0,$$
$$d_1h''^0_0 = d_1d_1Vs_0 = d_1d_2Vs_0 = d_1h'^0_0 = t_0,$$
$$d_0h''^1_0 = d_0h^1_0 = f_1,\ d_2h''^1_0 = d_1Vs_0d_1 = h''^0_0d_1,$$
$$d_0h''^1_1 = d_1Vs_0d_0 = h''^0_0d_0,$$
$$d_1h''^1_1 = d_1Y = d_1V = d_1h''^1_0,\ d_2h''^1_1 = d_2h'^1_1 = t_1,$$
$$h''^1_1s_0 = d_1s_2Vs_0 = s_1h''^0_0$$
(since $(K^3_1)^{-1}(h^1_0,-,V,Vs_0d_1)s_0 = s_2Vs_0$),
$$h''^1_1s_0 = d_2s_0Vs_0 = s_0h''^0_0$$
(since $(K^3_1)^{-1}(h^1_1,-,V,h'^1_0)s_0 = s_1Vs_0$ and so $(K^3_2)^{-1}(Vs_0d_0,Y,-,V,h'^1_1)s_0 = s_0Vs_0$).

Then $(h''^1_0,h''^1_1,h''^0_0)$ defines a truncated homotopy which extends to a homotopy from $f.$ to $t.$ since the homotopy condition is verified: In fact, let

$$h''^2_0 = (K^3_1)^{-1}(f_2,-,h''^1_0d_1,h''^1_0d_2),$$
$$h''^2_1 = (K^3_2)^{-1}(h''^1_0d_0,d_1h''^2_0,-,h''^1_1d_2)$$
and $$h''^2_2 = (K^3_3)^{-1}(h''^1_1d_0,h''^1_1d_1,d_2h''^2_1,-).$$

The homotopy condition is $d_3h''^2_2 = t_2$. To prove that consider the morphisms

$$X_1 = (K^3_1)^{-1}(h^1_0,-,V,Vs_0d_1),\ X_2 = (K^3_1)^{-1}(h^1_1,-,V,h'^1_0),$$
$$X_3 = (K^3_2)^{-1}(Vs_0d_0,Y,-,h'^1_1),\ L = (K^3_2)^{-1}(h^1_0d_0,d_1h''^2_0,-,Yd_2),$$
$$L_1 = (K^4_2)^{-1}(h^2_0,h''^2_0,-,X_1d_1,X_1d_2),\ L_2 = (K^4_3)^{-1}(h^2_1,L,d_2L_1,-,X_2d_2),$$
$$L_3 = (K^4_3)^{-1}(X^1d_0,L,h''^2_1,-,X_3d_2),$$
$$L_4 = (K^4_1)^{-1}(h^2_2,-,X_2d_1,d_3L_2,h'^2_0),$$
$$L_5 = (K^4_2)^{-1}(X_2d_0,d_1L_4,-,d_3L_3,h'^2_1),$$
and $$L_6 = (K^4_3)^{-1}(X_3d_0,X_3d_1,d_2L_5,-,h'^2_2).$$

It is straightforward to see that $d_id_3L_6 = d_ih''^2_2$ for $i = 0,1,2$, so $d_3L_6 = h''^2_2$ and therefore

$$d_3h''^2_2 = d_3d_3L_6 = d_3d_4L_6 = d_3h'^2_2 = t_2.$$

The proof of b is analogous. ∎

The well known Moore's Theorem [29], assuring that every group complex is a Kan complex, allows us to establish the following

COROLLARY 15. *For any 2- hypergroupoid* G. *in the category of algebras* **V** *and* $X \in \mathbf{V}$, *the homotopy relation in the set* $(\mathbf{G}.(X), G.)$ *is an equivalence relation.* ∎

As we saw in 1.2, $\mathbf{H}^1_G$ can be calculated using cocycles from $\mathrm{Cosk}^0(\mathrm{Tr}^0(\mathbf{G}.(S))$; the following is the corresponding 2-dimensional analogous result.

LEMMA 16. *Let be a 2-hypergroupoid in* **C** *and* $S \in \mathbf{C}$. *Then*

$$[\mathbf{G}.(S), G.] = [\mathrm{Cosk}^1(\mathrm{Tr}^1(\mathbf{G}.(S))), G.].$$

PROOF. We will show that every 2-cocycle $f. \in (\mathbf{G}.(S), G.)$ has a factorization, which is necessarily unique, by the canonical simplicial epimorphism $q.: \mathbf{G}.(S) \to \mathrm{Cosk}^1(\mathrm{Tr}^1(\mathbf{G}.(S)))$. Given $f.$, consider

$$f'._{tr} = f._{tr}: \mathrm{Tr}^1(\mathbf{G}.(S)) \to \mathrm{Tr}^1(G.) \text{ and } f'_2: \Delta^2(\mathbf{G}.(S)) \to G_2$$

as follows:

Since $\mathbf{G}.(S)$ is aspherical, the morphism $q_2: \mathbf{G}^3(S) \to \Delta^2(\mathbf{G}.(S))$ is a surjective epimorphism and for each $(x,y) \in \mathbf{G}^3(S) \times_{\Delta^2(\mathbf{G}.(S))} \mathbf{G}^3(S)$ we have that $f_2(x) = f_2(y)$. In effect, as

$$(s_1 d_0(x), s_1 d_1(x), x, y) \in \Delta^3(\mathbf{G}.(S)),$$

there is a $z \in \mathbf{G}^4$ such that $d_i(z) = s_1 d_i(x)$, $i = 0,1$, $d_2(z) = x$ and $d_3(z) = Y$. Then $d_i f_3(z) = d_i s_2 f_2(x)$, $0 \le i \le 2$, and as $G_3 \approx \Lambda^3_3(G.)$ we deduce that $d_3 f_3(z) = d_3 s_2 f_2(x)$, that is, $f_2(x) = f_2(y)$. Thus there is a unique morphism f'_2 from $\Delta^2(\mathbf{G}.(S))$ to G_2 such that $f'_2 q_2 = f_2$.

It is plain to see that $f'_{tr} = (f'_2, f_1, f_0)$ is really a 2-truncated simplicial morphism which satisfies the cocycle condition and therefore it has an extension $f'.: \mathrm{Cosk}^1(\mathrm{Tr}^1(\mathbf{G}.(S))) \to G.$ which satisfies $f'.\ q. = f.$ as required.

As $q.$ is an epimorphism, the correspondence $f. \mapsto f'.$ defines a bijection $(\mathbf{G}.(S), G.) = (\mathrm{Cosk}^1(\mathrm{Tr}^1(\mathbf{G}.(S))), G.)$.

Finally, if $h.: f. \to g.$ is a homotopy between 2-cocycles $f.$ and $g.$, the 2-truncated homotopy $h._{tr} = (h^1_0, h^1_1, h^0_0)$ from $f._{tr} = f'_{tr}$ to $g._{tr} = g'._{tr}$ also verifies the homotopy condition for $f'.$ and $g'.$ and so it extends to a homotopy $h'.: f'. \to g'.$. Clearly if $f'.$ and $g'.$ are homotopic, then $q.f'.$ and $q.g'.$ are homotopic too. ∎

The following lemma will allow us to give a general result which will be used to establish the 9-term exact sequence in non-abelian cohomology of algebras.

LEMMA 17. *Let* $S \in \mathbf{C}$ *and let* $q.: G. \to G''.$ *be a morphism between Kan 2-hypergroupoids in* $\mathbf{C}$ *which is a Kan-fibration. If* $g. \in \mathbf{Z}^2_G(S.G.)$ *is applied by the canonical map*

$$q_*: \mathbf{Z}^2_G(S,G.) \to \mathbf{Z}^2_G(S,G''.)$$

into a 2-cocycle homotopic to a 2-cocycle $f. \in \mathbf{Z}^2_G(S,G''.)$ *then there exists another 2-cocycle* $g'. \in \mathbf{Z}^2(S,G.)$ *homotopic to* $g.$ *such that* $q_* g'. = f.$.

PROOF. Let $h.$ be a homotopy from $q_*g.$ to $f.$. We find the 2-cocycle $g'.$ with the required conditions as follows: Since the canonical morphism $G_1 \to G''_1 \;{}_{d_0}\times_{q_0} G_0$ is surjective there exists a morphism $H^0_0: \mathbf{G}(S) \to G_1$ such that $q_1 \mathbf{H}^0_0 = h^0_0$ and $d_0 H^0_0 = g_0$. We consider the morphism $g'_0 = d_1 H^0_0$. Now, for $z \in \mathbf{G}(S)$ such that z is not in $\eta_S(S)$ let $v_z \in G_2$ be such that

$q_2(v_z) = h^1_0(\eta_{G(S)}(z)),\; d_0(v_z) = g_1(\eta_{G(S)}(z)),\; d_2(v_z) = H^0_0\, d_1(\eta_{G(S)}(z))$

(this element exists since $q.$ is a Kan-fibration). Then we define the morphism $H^1_0: \mathbf{G}^2(S) \to G_2$ as the unique one verifying

$$H^1_0(\eta_{G(S)}(z)) = v_z \text{ if } z \notin \eta_S(S) \text{ and } H^1_0 \eta_{G(S)} \eta_S = s_1 H^0_0 \eta_S.$$

Now we define $H^1_1: \mathbf{G}^2(S) \to G''_2$ as the unique morphism verifying $H^1_1 \eta_{G(S)} \eta_S = s_0 H^0_0 \eta_S$ and for $z \in \mathbf{G}(S)$, $z \notin \eta_S(S)$, $H^1_1(\eta_{G(S)}(z) = w_z$ where $w_z \in G_2$ is such that

$$g_2(w_z) = h^1_1(\eta_{G(S)}(z)),\; d_0(w_z) = H^0_0 d_0(\eta_{G(S)}(z))$$

and

$$d_1(w_z) = d_1 H^1_0(\eta_{G(S)}(z))$$

(this element w_z exists since $q.$ is a Kan-fibration). Let $g'_1 = d_2 H^1_1$ and $g'_2: \Delta^2(\mathbf{G}.(S)) \to G_2$ given by

$$g'_2 = d_3(K^3_3)^{-1}(H^1_1 d_0, H^1_1 d_1,$$
$$d_2(K^3_2)^{-1}(H^1_0 d_0, d_1(K^3_1)^{-1}(g_2, -, H^1_0 d_1, H^1_0 d_2), -, H^1_1 d_2), -).$$

It is tedious but straightforward to see that (g'_2, g'_1, g'_0) is a 2-truncated simplicial morphism, which is really a 2-cocycle and (H^1_1, H^1_0, H^0_0) is a truncated homotopy defining a homotopy from $g.$ to $g'.$. Moreover $g'.$ is applied by q_* into $f.$ since

$$q_0 g'_0 = q_0 d_1 H^0_0 = d_1 q_1 H^0_0 = d_1 h^0_0 = f_0$$

and analogously $q_1 g'_1 = d_2 h^1_1 = f_1$ and $q_2 g'_2 = d_3 h^2_2 = f_2$ and so the proof is completed. ■

2.2. Non-abelian $H^3(X, \Phi)$.

Now, we will use the general monadic cohomology with coefficients in 2-hypergroupoids for our purpose of defining an adequate 3-dimensional cohomology of an algebra with coefficients in a crossed module. The key for that is the observation that just as a crossed module $\Phi = (\delta: B \to A, \mu)$ in $\mathbf{V}$ has canonically associated an internal groupoid in $\mathbf{V}$ (i.e., a 1-hypergroupoid)

$$\underset{\sim}{G}(\Phi) = \ldots\ B \rtimes A \rightrightarrows A$$

having in fact an equivalence of categories $XM(\mathbf{V}) \approx GPD(\mathbf{V})$ (see Prop. 2), it also has associated a 2-hypergroupoid in $\mathbf{V}$, denoted $\underset{\sim}{G}^2(\Phi)$, in such a way that one has a full and faithful functor

$$\underset{\sim}{G}^2(-): XM(\mathbf{V}) \longrightarrow 2\text{-}HYPGD(\mathbf{V})$$

and so there is an equivalence between $XM(\mathbf{V})$ and a certain full subcategory of $2\text{-}HYPGD(\mathbf{V})$. We will use this functor to define $\mathbf{H}^3(X,\Phi)$.

The 2-hypergroupoid $\underset{\sim}{G}^2(\Phi)$ associated to a crossed module Φ is defined as

$$\underset{\sim}{G}^2(\Phi) = \mathrm{Cosk}^2\Big((B\times_A B\times_A B)^3 \rtimes A \ \overrightarrow{\Rrightarrow}\ (B\times_A B)^2 \rtimes A \Rrightarrow B \rtimes A \rightrightarrows A\Big)$$

where $B \rtimes A$ is the semidirect product algebra, $(B\times_A B)^2 \rtimes A$ is the R-submodule of $B^4 \oplus A$ whose elements are those 5-th uples (b_0, b_1, b_2, b_3, a) such that $\delta(b_0) = \delta(b_1)$ and $\delta(b_2) = \delta(b_3)$ and whose structure of algebra is given by the product

$$(b_0, b_1, b_2, b_3, a)(b'_0, b'_1, b'_2, b'_3, a') = (b''_0, b''_1, b''_2, b''_3, aa')$$

where $\quad b''_i = b_i b'_i + b_i^{a'} + {}^a b'_i\ , \quad i = 0,1,2,3,$

$(B\times_A B\times_A B)^3 \rtimes A$ is the R-submodule of $B^9 \oplus A$ which consists of those 10-th uples $(b_0, \ldots, b_8, a)$ satisfying

$\delta(b_0) = \delta(b_1) = \delta(b_2),\ \delta(b_3) = \delta(b_4) = \delta(b_5),\ \delta(b_6) = \delta(b_7) = \delta(b_8)$

and whose product is given by

$$(b_0, \ldots, b_8, a)(b'_0, \ldots, b'_8, a') = (b''_0, \ldots, b''_8, aa')$$

where $\quad b''_i = b_i b'_i + b_i^{a'} + {}^a b'_i\ , \quad i = 0, \ldots, 8,$

the face and degeneracy operators are given by:

$$d_0(b,a) = a,\ d_1(b,a) = \delta(b) + a,\ s_0(a) = (0,a),$$
$$d_0(b_0,b_1,b_2,b_3,a) = (b_1,a),\ d_1(b_0,b_1,b_2,b_3,a) = (b_3,a),$$
$$d_2(b_0,b_1,b_2,b_3,a) = (b_2 - b_1, a),$$
$$s_0(b,a) = (b,b,b,b,a),\ s_1(b,a) = (0,0,b,b,a),$$
$$d_0(b_0,\ldots,b_8,a) = (b_0,b_1,b_2,b_3,a),$$
$$d_1(b_0,\ldots,b_8,a) = (b_4,b_1,b_5,b_6,a),$$
$$d_2(b_0,\ldots,b_8,a) = (b_7,b_3,b_8,b_6,a),$$

$$d_3(b_0,\dots,b_8,a) =$$
$$= (b_0 - b_4 + b_7 - b_3 + b_2 - b_1,\ b_2 - b_1,\ b_2 - b_1 + b_8 - b_3,\ b_5 - b_1, a),$$
$$s_0(b_0,b_1,b_2,b_3,a) = (b_0,b_1,b_2,b_3,b_0,b_2,b_3,b_3,b_3,a),$$
$$s_1(b_0,b_1,b_2,b_3,a) = (b_1,b_1,b_1,\ b_0,b_2,b_3,b_0,b_2,a),$$
$$s_2(b_0,b_1,b_2,b_3,a) = (0,0,b_1,b_1,0,\ b_3,\ b_3,b_0,b_2,a).$$

It is straightforward to see that $\underset{\sim}{G}^2(\Phi)$ really is a 2-hypergroupoid in $\mathbf{V}$ and besides, if $(p_1,p_0):\Phi\to\Phi''$ is a morphism of crossed modules one has an induced morphism of 2-hypergroupoids $q.:\underset{\sim}{G}^2(\Phi)\to\underset{\sim}{G}^2(\Phi'')$ determined by $q_0 = p_0$, $q_1(b,a) = (p_1(b),p_0(a))$ and

$$q_2(b_0,b_1,b_2,b_3,a) = (p_1(b_0),p_1(b_1),p_1(b_2),p_1(b_3),p_0(a)).$$

So that $\underset{\sim}{G}^2(\,):\mathrm{XM}(\mathbf{V})\to$2-HYPGD($\mathbf{V}$) is a functor. This clearly is a full and faithful functor and it is not difficult to observe that it gives an equivalence between XM($\mathbf{V}$) and the full subcategory of 2-HYPGD($\mathbf{V}$) consisting of those 2-hypergroupoids G. satisfying:

i) The canonical morphism $G.\to \mathrm{Cosk}^1(\mathrm{Tr}^1(G.))$ is a split epimorphism;

ii)

$$\mathrm{Tr}^1(G.) = G_1 \rightrightarrows G_0$$

is a groupoid (with the only one possible multiplication morphism, see 1.2):

iii) $\pi_1(G.) = 0$ and $\pi_2(G.) \approx \pi_1(\mathrm{Tr}^1(G.))$ as G_0-modules.

DEFINITION 18. The *third cohomology set* of an algebra X with coefficients in a crossed module Φ, denoted $\mathbf{H}^3(X,\Phi)$, is defined as $\mathbf{H}^3(X,\Phi) = \mathbf{H}^2_G(X,\underset{\sim}{G}^2(\Phi))$.

Clearly $\mathbf{H}^3(X,\Phi)$ is functorial in both variables.

REMARK 19. As we observed in general in 2.1, a 2-cocycle $g.\in \mathbf{Z}^2_G(X,\underset{\sim}{G}^2(\Phi))$ is equivalent to a truncated simplicial morphism (g_2,g_1,g_0) from $\mathrm{Cosk}^1(\mathbf{G}.(X))$ to $\underset{\sim}{G}^2(\Phi)$ satisfying the condition (CC2); now, if we express $g_1(z)$, $z\in \mathbf{G}^2(X)$, as an element of $B\rtimes A$ by $(\tilde{g}_1(z), g_0 d_0(z))$, to give g_1 is equivalent to give the $g_0 d_0$-derivation $\tilde{g}_1:\mathbf{G}(X)\to B$ and the simplicial identities reduce to $\tilde{g}_1$ must verify $\delta\tilde{g}_1(z) + g_0 d_0(z) = g_0 d_1(z)$ for $z\in\mathbf{G}^2(X)$ and $\tilde{g}_1 s_0(x) = 0$ for $x\in\mathbf{G}(X)$. Likewise, the simplicial identities imply that for any $(z_0,z_1,z_2)\in\Delta^2(\mathbf{G}.(X))$, $g_2(z_0,z_1,z_2)$, as an element of $(B\times_A B)^2\rtimes A$, has the form

$$(\tilde{g}_2(z_0,z_1,z_2),\tilde{g}_1(z_0),\tilde{g}_1(z_0+z_2),\tilde{g}_1(z_1),g_0 d_0(z_0))$$

and the fact of g_2 being a morphism of algebras is equivalent to that of the map $\tilde{g}_2\colon \Delta^2(\mathbf{G}.(X))\to B$ being a $g_0 d_0^2$-derivation, reducing the simplicial identities on $\tilde{g}_2$ to the equalities

$$\delta\tilde{g}_2(z_0,z_1,z_2)+g_0 d_0(z_0) = g_0 d_1(z_0),\ \tilde{g}_2(z,z,s_0 d_1(z)) = \tilde{g}_1(z)$$

and $$\tilde{g}_2(s_0 d_0(z),z,z)=0$$

and the condition (CC2) takes the form:

$$\tilde{g}_2(z_0,z_1,z_2)-\tilde{g}_2(z_0,y_1,y_2)+\tilde{g}_2(z_1,y_1,w_2)-\tilde{g}_2(z_2,y_2,w_2) = \tilde{g}_1(z_1)-\tilde{g}_1(z_2)$$

for all

$$(z_0,z_1,z_2),\ (z_0,y_1,y_2),\ (z_1,y_1,w_2),\ (z_2,y_2,w_2)\in\Delta^3(\mathrm{Cosk}^1(\mathbf{G}.(X))).$$

Consequently we will identify a 2-cocycle $g.$ with the system $(g_0\colon \mathbf{G}(X)\to A,\ \tilde{g}_1\colon \mathbf{G}^2(X)\to B;\ \tilde{g}_2\colon \Delta^2(\mathbf{G}.(X))\to B)$.

In the same way, if $g.\equiv(g_0,\tilde{g}_1,\tilde{g}_2)$ and $g'.\equiv(g'_0,\tilde{g}'_1,\tilde{g}'_2)$ are 2-cocycles, a homotopy $h.\colon g.\to g'.$ is determined by its truncation $h_0^0\colon \mathbf{G}(X)\to B\rtimes A$, $h_0^1. h_1^1\colon \mathbf{G}^2(X)\to(B\times_A B)^2\rtimes A$ satisfying the homotopy condition (HC2). Now, if we express $h_0^0(x)=(\tilde{h}_0^0(x),g_0 d_0(x))$, $x\in\mathbf{G}(X)$, as an element of $B\rtimes A$, to give h_0^0 is equivalent to give the g_0-derivation $\tilde{h}_0^0\colon \mathbf{G}(X)\to B$ verifying $\delta\tilde{h}_0^0(x)+g_0(x)=g'_0(x)$. Likewise, the simplicial identities imply that, for any $z\in\mathbf{G}^2(X)$, $h_0^1(z)$ and $h_1^1(z)$ take, as elements of $(B\times_A B)^2\rtimes A$, the form

$$h_0^1(z) = (\tilde{h}_0^1(z),\tilde{g}_1(z),\tilde{g}_1(z)+\tilde{h}_0^0 d_1(z),W(z),g_0 d_0(z)),$$
$$h_1^1(z) = (\tilde{h}_1^1(z),\tilde{h}_0^0 d_0(z),\tilde{g}'_1(z)+\tilde{h}_0^0 d_0(z),W(z),g'_0 d_0(z))$$

and the fact of h_0^1, h_1^1 being morphisms of algebras is equivalent to that of $\tilde{h}_0^1,\tilde{h}_1^1,W\colon \mathbf{G}^2(X)\to B$ being $g_0 d_0$-derivations, reducing the simplicial identities on $\tilde{h}_0^1,\tilde{h}_1^1,W$ to the equalities

$$\delta\tilde{h}_0^1(z)+g_0 d_0(z) = g_0 d_1(z),\ \delta\tilde{h}_1^1(z)+g_0 d_0(z) = g'_0 d_0(z),$$
$$\delta W(z)+g_0 d_0(z) = g'_0 d_1(z),\ \tilde{h}_0^1(s_0(x)) = 0$$

and $W(s_0(x)) = \tilde{h}_1^1(s_0(x)) = \tilde{h}_0^0(x)$ for $z\in\mathbf{G}^2(X)$, $x\in\mathbf{G}(X)$,

and the homotopy condition (HC2) takes the form

$$\tilde{g}'_2(z_0,z_1,z_2)-\tilde{g}_2(z_0,z_1,z_2) = \tilde{h}_1^1(z_0-z_1+z_2)-\tilde{h}_0^1(z_0-z_1+z_2)+\tilde{g}_1(z_2-z_1)-\tilde{h}_0^0 d_1(z_0)+\tilde{g}'_1(z_0)$$

for all $(z_0,z_1,z_2)\in\Delta^2(\mathbf{G}.(X))$.

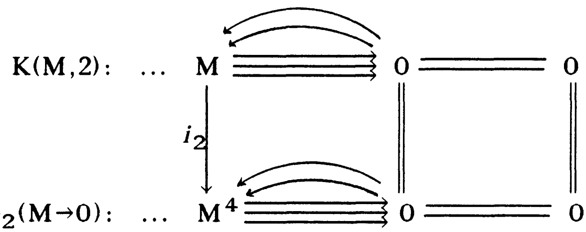
Consequently we will identify a homotopy $h.$ between 2-cocycles $g.$ and $g'.$ with the system

$$(\tilde{h}_0^0\colon \mathbf{G}(X)\to B,\ \tilde{h}_0^1,\tilde{h}_1^1,W\colon \mathbf{G}^2(X)\to B).\quad\blacksquare$$

The cohomology set $\mathbf{H}^3(X,\Phi)$ reduces to the usual monadic abelian cohomology when one considers the crossed module $(M\to 0)$ defined by a zero algebra M according to [17] (3.7) and the following lemma:

LEMMA 20. *Let* M *be a zero algebra, then the* 2-*hypergroupoid* $\underset{\sim}{G}^2(M\to 0)$ *is homotopically equivalent to* K(M,2).

PROOF. Consider the natural inclusion $i.: K(M,2)\to \underset{\sim}{G}^2(M\to 0)$:

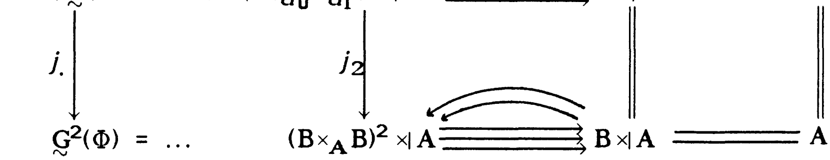


where $i_2(x)=(x,0,0,0)$, and the simplicial morphism $\alpha.$ from $\underset{\sim}{G}^2(M\to 0)$ to K(M,2) determined by

$$\alpha_2(x_0,x_1,x_2,x_3) = x_0 - x_1 + x_2 - x_3.$$

It is clear tha $\alpha.i. = 1_{K(M,2)}$. Moreover it is plain to see that the morphisms $h_0^0 = 0: 0\to M$ and $h_0^1, h_1^1: M\to M^4$ given by $h_0^1=0$ and $h_1^1(x)=(x,0\,x,0)$, determine a homotopy from $i.\alpha.$ to $1_{G2(M\to 0)}$; that K(M,2) is a deformation retract of $\underset{\sim}{G}^2(M\to 0)$. ∎

Let us recall now that, in general, a 2-cocycle $g.$ in $\mathbf{Z}^2_G(S,G.)$ is neutral if it factors through the simplicial subobject of G. generated by the degenerate 2-simplices. Now, for a crossed module Φ in $\mathbf{V}$, we are going to analyze in more detail the neutral 2-cocycles with coefficients in $\underset{\sim}{G}^2(\Phi)$ using for this the canonical morphism



determined by

$$j_2((b_0.a),(b_1,\delta(b_0)+a)) = (b_0,b_0,b_0+b_1,b_0+b_1,a).$$

In fact, it is plain to observe that the subalgebra of $\underset{\sim}{G}^2(\Phi)$ generated by the degenerate 2-simplices has as 1-truncation

$$B \ltimes A \;\overset{\longleftarrow}{\rightrightarrows}\; A$$

and it consists in those 2-simplices which are of the particular form (b, b, b', b', a); thus clearly one has that a 2-cocycle $g. \in \mathbf{Z}^2_G(S, \underset{\sim}{G}^2(\Phi))$ determined by the system $(g_0, \tilde{g}_1, \tilde{g}_2)$ is neutral iff for any $(z_0, z_1, z_2) \in \Delta^2(\mathbf{G}.(X))$,

$$\tilde{g}_2(z_0, z_1, z_2) = \tilde{g}_1(z_0) \text{ and } \tilde{g}_1(z_0 - z_1 + z_2) = 0$$

i.e., iff it factors through $\mathrm{Ner}(\underset{\sim}{G}(\Phi))$, but the simplicial morphisms under $\mathrm{Ner}(\underset{\sim}{G}(\Phi))$ are just the 1-cocycles under $\underset{\sim}{G}(\Phi)$ or equivalently the Dedecker-Lue's 2-cocycles with coefficients in Φ. So we have

PROPOSITION 21. *The canonical morphism* $j.: \mathrm{Ner}(\underset{\sim}{G}(\Phi)) \hookrightarrow \underset{\sim}{G}^2(\Phi)$ *induces a bijection between* $\mathbf{Z}^2(X, \Phi) \approx \mathbf{Z}^1_G(X, \underset{\sim}{G}(\Phi))$ *and the set of neutral 2-cocycles over* X *under* $\underset{\sim}{G}^2(\Phi)$. ▪

In order to establish the 9-term exact sequence it is necessary to point out two larger subsets than that consisting in the neutral cocycles which, recalling the terminology used by Dedecker in [13], will consists of "null cocycles". We introduce these cocycles as follows: The simplicial morphism $j.$: $\mathrm{Ner}(\underset{\sim}{G}(\Phi)) \hookrightarrow \underset{\sim}{G}^2(\Phi)$ has two factorizations through the simplicial algebra $\mathrm{Cosk}^1(\underset{\sim}{G}(\Phi))$

(I)
$$\begin{array}{ccc} \mathrm{Ner}(\underset{\sim}{G}(\Phi)) & \xhookrightarrow{\;j.\;} & \underset{\sim}{G}^2(\Phi) \\ & {\scriptstyle i.}\searrow \quad \nearrow{\scriptstyle r.}\;\nearrow{\scriptstyle t.} & \\ & \mathrm{Cosk}^1(\underset{\sim}{G}(\Phi)) & \end{array}$$

where the simplicial morphisms $i., r.$ and $t.$ are determined by:

$$i_0 = r_0 = t_0 = 1_A, \quad i_1 = r_1 = t_1 = 1_{B \ltimes A},$$
$$i_2((b_0, a), (b_2, \delta(b_0) + a)) = ((b_0, a), (b_0 + b_2, a), (b_2, \delta(b_0) + a)),$$
$$r_2((b_0, a), (b_1, a), (b_2, \delta(b_0) + a)) = (b_1 - b_2, b_0, b_0 + b_2, b_1, a)$$
and $t_2((b_0, a), (b_1, a), (b_2, \delta(b_0) + a)) = (b_0, b_0, b_0 + b_2, b_1, a)$.

Then we introduce the *r-null 2-cocycles* (resp. *t-null 2-cocycles*) under $\underset{\sim}{G}^2(\Phi)$ as those which factor through $r.$ (resp. $t.$). A class in $\mathbf{H}^3(X, \Phi)$ is called *r-null* (resp. *t-null*) if it contains an r-null (resp. t-null) 2-cocycle.

Let us observe that $\mathrm{Im}(r.)$ is just the simplicial subalgebra of $\underset{\sim}{G}^2(\Phi)$ generated by those 2-simplices (b_0,b_1,b_2,b_3,a) in $(B\times_A B)^2\rtimes A$ such that $b_0-b_1+b_2-b_3=0$ and $\mathrm{Im}(t.)$ is the simplicial subalgebra generated by those 2-simplices (b_0,b_1,b_2,b_3,a) such that $b_0=b_1$. So, a 2-cocycle $g.\in \mathbf{Z}^2_G(X,\underset{\sim}{G}^2(\Phi))$ represented by the system $(g_0,\tilde{g}_1,\tilde{g}_2)$ is r-null iff $\tilde{g}_2(z_0,z_1,z_2) = \tilde{g}_1(z_1)-\tilde{g}_1(z_2)$ and it is t-null iff $\tilde{g}_2(z_0,z_1,z_2)= \tilde{g}_1(z_0)$ for any $(z_0,z_1,z_2)\in \Delta^2(\mathbf{G}.(X))$.

Since to give a simplicial morphism from $\mathrm{Cosk}^1(\mathbf{G}.(X))$ to $\mathrm{Cosk}^1(\underset{\sim}{G}(\Phi))$ is equivalent to give its 1-truncation, we have

PROPOSITION 22. *There is a canonical bijection between the set of r-null 2-cocyles (t-null 2-cocycles) and* $\mathbf{C}^1_G(X,\underset{\sim}{G}(\Phi))$. ∎

REMARK 23. Because $\mathrm{Cosk}^1(\underset{\sim}{G}(\Phi))$ is aspherical, any simplicial morphisms $f.,g.$ from $\mathrm{Cosk}^1(\mathbf{G}.(X))$ to $\mathrm{Cosk}^1(\underset{\sim}{G}(\Phi))$ inducing the same morphism between the augmentations are always homotopic; likewise, any morphism $g: X\to A/\delta(B)$ has a "lifting" to a simplicial morphism $g.$ from $\mathrm{Cosk}^1(\mathbf{G}.(X))$ to $\mathrm{Cosk}^1(\underset{\sim}{G}(\Phi))$ and therefore the set of r-null elements (t-null elements) in $\mathbf{H}^3(X,\Phi)$ is bijective to the set $\mathrm{Hom}(X,A/\delta(B))$ by the map which associates to each r-null (t-null) class the induced morphism between the augmentations by any 2-cocycle representing it.

It is clear that every neutral 2-cocycle is an r-null 2-cocycle and also a t-null 2-cocycle, and the following proposition expresses a necessary and sufficient condition to the coincidence among neutral, r-null and t-null classes.

PROPOSITION 24. *Let* $\Phi=(\delta: B\to A,\mu)$ *be a crossed module in* $\mathbf{V}$. *For each* $X\in\mathbf{V}$ *let* $\pi_X:\mathbf{Z}^1_G(X,\underset{\sim}{G}(\Phi))\to \mathrm{Hom}(X,A/\delta(B))$ *be the map which associates to a cocycle the induced morphism between the augmentations*

$$\begin{array}{ccccc} \mathbf{G}^2(X) & \rightrightarrows & \mathbf{G}(X) & \longrightarrow & X \\ \downarrow g_1 & & \downarrow g_0 & & \downarrow g \\ B\rtimes A & \rightrightarrows & A & \longrightarrow & A/\delta(B) \end{array} \qquad \overset{\pi_X}{\longmapsto} \qquad g$$

(such a map corresponds to that called "crest" by Dedecker-Lue in [16]). *Then the following conditions are equivalent:*

i) *For each* $X \in \mathbf{V}$ *an element in* $\mathbf{H}^3(X,\Phi)$ *is neutral iff it is r-null.*

ii) *For each* $X \in \mathbf{V}$ *an element in* $\mathbf{H}^3(X,\Phi)$ *is neutral iff it is t-null.*

iii) *For each* $X \in \mathbf{V}$, π_X *is surjective.*

iv) *The morphism* $1_{A/\delta(B)}$ *is in the image of* $\pi_{A/\delta(B)}$.

v) *There exists a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A/\delta(B) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ & & B & \longrightarrow & A & \longrightarrow & A/\delta(B) & \longrightarrow & 0 \end{array}$$

representing a Φ*-extension of* $A/\delta(B)$.

Note that this last condition expresses that the class of 2-fold extensions

$$0 \longrightarrow M = \mathrm{Ker}(\delta) \longrightarrow B \xrightarrow{\delta} A \longrightarrow A/\delta(B) \longrightarrow 0$$

of $A/\delta(B)$ by the $A/\delta(B)$-bimodule M in $\mathbf{H}^2_{\mathbf{G}}(A/\delta(B),M)$ is zero (see [7] Prop. 8.15, [9]), i.e., the obstruction to the abstract kernel $(\delta: B \to A, 1_{A/\delta(B)})$ vanishes.

PROOF. The equivalences i)⇔iii) and ii)⇔iii) are immediate consequences of the above remark; iii)⇒iv) is trivial and iv)⇒iii) is proved as follows: since for any $f \in \mathrm{Hom}(X,A/\delta(B))$ the composition of $\mathbf{G}.(f): \mathbf{G}.(X) \to \mathbf{G}.(A/\delta(B))$ with a 1-cocycle

$$g. \in \mathbf{Z}^1_{\mathbf{G}}(A/\delta(B),\underset{\sim}{G}(\Phi)) \text{ such that } \pi_{A/\delta(B)}(g.) = 1_{A/\delta(B)}$$

gives a 1-cocycle over X under $\underset{\sim}{G}(\Phi)$ which is applied by π_X into f. Finally iv) and v) are clearly equivalent. ■

It is clear to observe that the diagram (I) is natural on the crossed module Φ and this fact allows us to use the notion of r-null 2-cocycle to define a "2-hypergroupoid kernel" of the 2-hypergroupoid morphism $\underset{\sim}{G}^2(p): \underset{\sim}{G}^2(\Phi) \to \underset{\sim}{G}^2(\Phi'')$ induced by a short exact sequence of crossed modules

$$\Phi' \longrightarrow \Phi \xrightarrow{\ p\ } \Phi''.$$

Let us note that the groupoid kernel of $\underset{\sim}{G}(p): \underset{\sim}{G}(\Phi) \to \underset{\sim}{G}(\Phi''), \underset{\sim}{G}(\Phi')$, is just the simplicial subalgebra of $\underset{\sim}{G}(\Phi)$ consisting of those 1-simplices $(b,a) \in B \rtimes A$ such that $p_1(b) = 0$.

DEFINITION 26. Let

$$\Phi' = (\delta' : B' \to A, \mu') \longrightarrow \Phi = (\delta : B \to A, \mu) \xrightarrow{p=(p_1,p_0)} \Phi'' = (\delta'' : B'' \to A'', \mu'')$$

be a short exact sequence of crossed modules. The *2-hypergroupoid kernel of* $\underset{\sim}{G}^2(p)$, denoted $\underset{\sim}{G}^2_\Phi(\Phi')$, is defined as the simplicial subalgebra of $\underset{\sim}{G}^2(\Phi)$ generated by those 2-simplices which are applied by $\underset{\sim}{G}^2(p)$ into the image of $r.: \mathrm{Cosk}^1(\underset{\sim}{G}(\Phi'')) \to \underset{\sim}{G}^2(\Phi'')$.

Explicitly, $\underset{\sim}{G}^2_\Phi(\Phi')$ has

$$B \rtimes A \rightrightarrows A$$

as 1-truncation and consists of those 2-simplices

$(b_0, b_1, b_2, b_3, a) \in \underset{\sim}{G}^2(\Phi)$ such that $p_1(b_0 - b_1 + b_2 - b_3) = 0$.

For an algebra X we will denote

$$\mathbf{H}^3_\Phi(X, \Phi') \approx \mathbf{H}^2_G(X, \underset{\sim}{G}^2_\Phi(\Phi')).$$

The definition of $\underset{\sim}{G}^2_\Phi(\Phi')$ is justified by all the following development; see especially the following remark, the definition of the connecting map and the proof of the exactness in the sequence.

REMARK 27. In the case of

$$\Phi' = (M' \to 0) \longrightarrow \Phi = (M \to 0) \longrightarrow \Phi'' = (M'' \to 0)$$

being the short exact sequence of crossed modules associated to a short exact sequence of zero-algebras $M' \to M \to M''$, it is easy to observe that the homotopical equivalence $\alpha.: \underset{\sim}{G}^2(\Phi) \to K(M,2)$ established in the proof of Lemma 20 restricts to an equivalence between $\underset{\sim}{G}^2_\Phi(\Phi')$ and $K(M',2)$ so that $\mathbf{H}^3_\Phi(X, \Phi')$ is the usual abelian cohomology with coefficients in M'.

2.3. The 9-term exact sequence.

In this paragraph we establish the 9-term exact sequence in non-abelian cohomology of algebras, using for that the description of Dedecker-Lue's 6-term exact sequence in the simplicial way shown in 1.3, i.e., that obtained as example of Proposition 11 when one considers the sequence of groupoids in **V**, $\underset{\sim}{G}(\Phi') \to \underset{\sim}{G}(\Phi) \to \underset{\sim}{G}(\Phi'')$ associated to a short exact sequence of crossed modules $\Phi' \to \Phi \to \Phi''$:

$$* \longrightarrow \Gamma_\varphi(X, \underset{\sim}{G}(\Phi')) \longrightarrow \Gamma_\varphi(X, \underset{\sim}{G}(\Phi)) \longrightarrow \Gamma_\vartheta(X, \underset{\sim}{G}(\Phi''))$$

$$\longrightarrow \mathbf{H}^1_G(X, \underset{\sim}{G}(\Phi')) \longrightarrow \mathbf{H}^1_G(X, \underset{\sim}{G}(\Phi)) \longrightarrow \mathbf{H}^1_G(X, \underset{\sim}{G}(\Phi'')).$$

Obtaining the elongation will necessitate a restrictive condition on the sequence of crossed modules since we will need that a cocycle under $\underset{\sim}{G}(\Phi'')$ has a lifting to a cochain under $\underset{\sim}{G}(\Phi)$. The following proposition shows a sufficient condition for that.

PROPOSITION 28. i) *If* $q_{\sim}: \underset{\sim}{G} \to \underset{\sim}{G}''$ *is a morphism of groupoids which is a quotient map in the sense of Higgins [25], i.e.,* q_0 *is surjective and the canonical morphism*

$$(q_1,(d_0,d_1)): G_1 \to G''_1\,{}_{(d_0,\,d_1)}\times_{(q_0\times q_0)} (G_0\times G_0)$$

is surjective, then any cocycle $f_{\cdot} \in \mathbf{Z}^1_G(X,\underset{\sim}{G}'')$ *has a lifting to a 1-cochain under* $\underset{\sim}{G}$*, i.e., there exists* $(g_1,g_0) \in \mathbf{C}^1_G(X,\underset{\sim}{G})$ *such that* $q_i g_i = f_i$, $i = 0,1$.

ii) *Let*

$$\Phi' = (\delta': B' \to A, \mu') \longrightarrow \Phi = (\delta: B \to A, \mu) \xrightarrow{p=(p_1,p_0)} \Phi'' = (\delta'': B'' \to A'', \mu'')$$

be a short exact sequence of crossed modules. Then $\underset{\sim}{G}(\Phi) \to \underset{\sim}{G}(\Phi'')$ *is a quotient map iff* $\mathrm{Ker}(p_0) = \mathrm{Im}(\delta')$. *Such a sequence verifying this condition will be called a "short strongly exact sequence of crossed modules".*

PROOF. Given $f_{\cdot} \in \mathbf{Z}^1_G(X,\underset{\sim}{G}'')$ let $g_0: \mathbf{G}(X) \to G_0$ be any morphism satisfying $q_0 g_0 = f_0$. For each $z \in \mathbf{G}(X)$ such that $z \notin \eta_X(X)$ let $v_z \in G_1$ be an element verifying

$$q_1(v_z) = f_1(\eta_{G(X)}(z)),\ d_0(v_z) = g_0 d_0(z) \text{ and } d_1(v_z) = g_0 d_1(z);$$

then we define g_1: $\mathbf{G}^2(X) \to G_1$ as the unique morphism satisfying $g_1 \eta_{G(X)} \eta_X(x) = s_0 g_0 \eta_X(x)$ and $g_1 \eta_{G(X)}(z) = v_z$ for each $z \in \mathbf{G}(X)$ and $z \notin \eta_X(X)$.

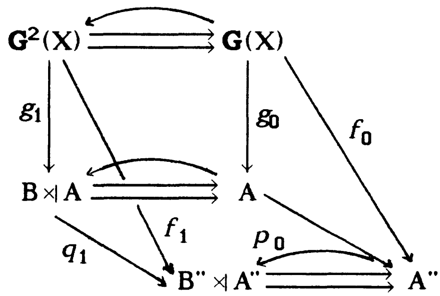
It is straightforward to see that $q_1 g_1 = f_1$ and (g_1,g_0) is a truncated simplicial morphism from $\mathbf{G}.(X)$ to $\mathrm{Ner}(\underset{\sim}{G})$ and so i) is proved.

ii) In these conditions, $\underset{\sim}{G}((p_1,p_0))$ is a quotient map iff for each $b'' \in B''$ and $a_0, a_1 \in A$ such that $\delta''(b'') + p_0(a_0) = p_0(a_1)$, there exists $b \in B$ satisfying $p_1(b) = b''$ and $\delta(b) + a_0 = a_1$; but, since p_1 is surjective, that condition is verified iff for each $a \in \mathrm{Ker}(p_0)$ there exists $b \in \mathrm{Ker}(p_1) = B'$ such that $\delta(b) = a$, i.e., iff $\mathrm{Ker}(p_0) = \mathrm{Im}(\delta')$. ■

If

$$\Phi' = (\delta': B' \to A, \mu') \longrightarrow \Phi = (\delta: B \to A, \mu) \xrightarrow{p=(p_1,p_0)} \Phi'' = (\delta'': B'' \to A'', \mu'')$$

is a short strongly exact sequence of crossed modules, there exists an adequate "connecting map" $\chi^2: \mathbf{H}^2(X,\Phi'') \to \mathbf{H}^3_{\Phi}(X,\Phi')$ defined as follows: Each 1-cocycle $f. \in \mathbf{Z}^1_{\mathrm{G}}(\mathrm{X},\underset{\sim}{\mathrm{G}}(\Phi''))$ has a lifting to a 1-cochain under $\underset{\sim}{\mathrm{G}}(\Phi)$, say $(g_1,g_0) \in \mathbf{C}^1_{\mathrm{G}}(\mathrm{X},\underset{\sim}{\mathrm{G}}(\Phi))$, i.e., a truncated simplicial morphism from $\mathrm{Tr}^1(\mathbf{G}.(\mathrm{X}))$ to $\mathrm{Tr}^1(\underset{\sim}{\mathrm{G}}(\Phi))$ such that $q_1 g_1 = f_1$ and $p_0 g_0 = f_0$:



which in general is not a 1-cocycle, but according to Proposition 22 it canonically defines through $t.: \mathrm{Cosk}^1(\mathrm{G}.(\mathrm{X})) \to \underset{\sim}{\mathrm{G}}^2(\Phi)$ a t-null 2-cocycle $g.$ under $\underset{\sim}{\mathrm{G}}^2(\Phi)$ (explicitly, $g.$ is the 2-cocycle corresponding to the system $(g_0, \tilde{g}_1, \tilde{g}_1 d_0)$. This 2-cocycle $g.$ is carried by composition with $\underset{\sim}{\mathrm{G}}^2(p)$ into an r-null 2-cocycle under $\underset{\sim}{\mathrm{G}}^2(\Phi'')$ (really into a neutral 2-cocycle) and so it factors through $\underset{\sim}{\mathrm{G}}^2_{\Phi}(\Phi')$.

We claim that the class of $g.$ in $\mathbf{H}^3_{\Phi}\ (\mathrm{X},\Phi')$ only depends on the class of $f.$ in $\mathbf{H}^2(\mathrm{X},\Phi'')$. In effect, let $h.: f. \to f'.$ be a homotopy between 1-cocycles under $\underset{\sim}{\mathrm{G}}(\Phi'')$ and $g.$ and $g'.$ the associated 2-cocycles under $\underset{\sim}{\mathrm{G}}^2_{\Phi}(\Phi')$ as above. Then a homotopy $\mathrm{H}.: g. \to g'.$ is built as follows: Since $\underset{\sim}{\mathrm{G}}(p): \underset{\sim}{\mathrm{G}}(\Phi) \to \underset{\sim}{\mathrm{G}}(\Phi'')$ is a quotient map of groupoids there exists $\mathrm{H}^0_0: \mathbf{G}(\mathrm{X}) \to \mathrm{B} \rtimes \mathrm{A}$ such that

$$q_1 \mathrm{H}^0_0 = h^0_0,\ d_0 \mathrm{H}^0_0 = g_0 \text{ and } d_1 \mathrm{H}^0_0 = g'_0;$$

denoting

$$\mathrm{H}^0_0(x) = (\tilde{\mathrm{H}}^0_0(x), g_0(x)) \in \mathrm{B} \rtimes \mathrm{A}$$

we define $\mathrm{H}^1_0, \mathrm{H}^1_1: \mathbf{G}^2(\mathrm{X}) \to \underset{\sim}{\mathrm{G}}^2_{\Phi}(\Phi')_2$ by

$$\mathrm{H}^1_0(z) = (\tilde{g}_1(z), \tilde{g}_1(z), \tilde{g}_1(z) + \tilde{\mathrm{H}}^0_0 d_0(z), \tilde{g}'_1(z) + \tilde{\mathrm{H}}^0_0 d_0(z), g_0 d_0(z))$$

and $\mathrm{H}^1_1(z) =$

$$(\tilde{\mathrm{H}}^0_0 d_0(z), \tilde{\mathrm{H}}^0_0 d_0(z), \tilde{g}'_1(z) + \tilde{\mathrm{H}}^0_0 d_0(z), \tilde{g}'_1(z) + \tilde{\mathrm{H}}^0_0 d_0(z), g_0 d_0(z))$$

(note that

$$p_1(\tilde{g}'_1(z) + \tilde{\mathrm{H}}^0_0 d_0(z)) = \tilde{f}'_1(z) = \tilde{h}^0_0 d_0(z)$$
$$= \tilde{f}_1(z) + \tilde{h}^0_0 d_1(z) = p_1(\tilde{g}_1(z) + \tilde{\mathrm{H}}^0_0 d_1(z))$$

the last equality by the homotopy condition).

It is easy to see that $(\mathrm{H}^1_0, \mathrm{H}^1_1; \mathrm{H}^0_0)$ is a truncated homotopy from $g.$ to $g'.$ verifying in addition the homotopy condition (HC2) (see Remark 19) and so it extends to a homotopy H. as required. Therefore, the correspondence $f. \mapsto g.$ according to the previous construction induces a well defined map

$$\chi^2 : \mathbf{H}^2(X,\Phi'') \to \mathbf{H}^3_\Phi(X,\Phi').$$

The following proposition expresses that $\chi^2([f.])$ is precisely the obstruction to $[f.]$ being in the image of p_*: $H^2(X,\Phi) \to \mathbf{H}^2(X,\Phi'')$.

PROPOSITION 29. *Let*

$$\Phi' = (\delta' : B' \to A, \mu') \longrightarrow \Phi = (\delta : B \to A, \mu) \xrightarrow{p=(p_1,p_0)} \Phi'' = (\delta'' : B'' \to A'', \mu'')$$

be a short strongly exact sequence of crossed modules and X *an algebra. The sequence*

$$H^2(X,\Phi) \xrightarrow{p_*} \mathbf{H}^2(X,\Phi'') \xrightarrow{\chi^2} \mathbf{H}^3_\Phi(X,\Phi')$$

is exact in the sense that an element is in the image of p_* iff it is applied by χ^2 into a neutral class.

PROOF. Let $g. \in \mathbf{Z}^1_G(X,\underset{\sim}{G}(\Phi))$ be a 1-cocycle: then $\chi^2 p_*[g.]$ is represented by the 2-cocycle which belongs to $\mathbf{Z}^2_G(X,\underset{\sim}{G}^2_\Phi(\Phi'))$ determined by the system $(g_0,\tilde{g}_1,\tilde{g}_1 d_0)$ which is neutral since $g.$ is a 1-cocycle and so $\tilde{g}_1(z_0-z_1+z_2)=0$ for all $(z_0,z_1,z_2)\in\Delta^2(\mathbf{G}.(X))$.

Conversely, let $f. \in \mathbf{Z}^1_G(X,\underset{\sim}{G}(\Phi''))$ be such that $\chi^2[f.]$ is a neutral class in $\mathbf{H}^3_\Phi(X,\Phi')$. To prove that $[f.]\in\mathbf{H}^2(X,\Phi'')$ is in the image of p_* let us suppose $(g_1,g_0)\in\mathbf{C}^1_G(X,\underset{\sim}{G}\Phi))$ is a 1-cochain lifting of $f.$ and $g. \equiv (g_0,\tilde{g}_1,\tilde{g}_1 d_0)\in\mathbf{Z}^2_G(X,\underset{\sim}{G}^2_\Phi(\Phi'))$ the corresponding 2-cocycle such that $\chi^2[f.] = g.$; since $[g.]$ is neutral there will exist a neutral 2-cocycle $g'. \equiv (g'_0,\tilde{g}'_1,\tilde{g}'_1 d_0)$ (so verifying $\tilde{g}'_1(z_0-z_1+z_2)=0$, $(z_0,z_1,z_2)\in\Delta^2(\mathbf{G}.(X)))$ and a homotopy $h. \equiv (\tilde{h}^0_0,\tilde{h}^1_0,\tilde{h}^1_1,W)$ from $g.$ to $g'.$. Then, using this homotopy, we build the following 1-cocycle $f'.\in\mathbf{Z}^1_G(X,\underset{\sim}{G}(\Phi))$: take $f'_0 = g_0$ and $f'_1 : \mathbf{G}^2(X)\to B\rtimes A$ given by

$$f'_1(z) = (\tilde{h}^1_1(z)-\tilde{h}^1_0(z)+\tilde{g}_1(z)-\tilde{h}^0_0 d_1(z)+\tilde{g}'_1(z), g_0 d_0(z)),\ z\in\mathbf{G}^2(X).$$

One has

$$\delta(\tilde{h}^1_1(z)-\tilde{h}^1_0(z)+\tilde{g}_1(z)-\tilde{h}^0_0 d_1(z)+\tilde{g}'_1(z)) = g_0 d_1(z)-g_0 d_0(z),$$

$$\tilde{h}^1_1(s_0(x))-\tilde{h}^1_0(s_0(x))+\tilde{g}_1(s_0(x))-\tilde{h}^0_0 d_1(s_0(x))+\tilde{g}'_1(s_0(x))) = \tilde{h}^0_0(x)-\tilde{h}^0_0(x) = 0,$$

$$\tilde{h}^1_1(z_0-z_1+z_2)-\tilde{h}^1_0(z_0-z_1+z_2)+\tilde{g}_1(z_0-z_1+z_2)-\tilde{h}^0_0 d_1(z_0)+\tilde{h}^0_0 d_1(z_1) -\tilde{h}^0_0 d_1(z_2)+\tilde{g}'_1(z_0-z_1+z_2) = \tilde{g}'_1(z_0)-\tilde{g}_1(z_0)+\tilde{g}_1(z_0)-\tilde{g}'_1(z_0) = 0$$

(using the condition (HC2)) and

$$p_1(\tilde h_1^1(z)-\tilde h_0^1(z)+\tilde g_1(z)-\tilde h_0^0 d_1(z)+\tilde g_1'(z)) = p_1\tilde g_1(z) = \tilde f_1(z)$$

(since $h_0^1, h_1^1 \in (\underset{\sim}{G}^2_\Phi(\Phi'))_2$); from where (f_1', f_0') defines a 1-cocycle $f'. \in \mathbf{Z}^1_G(X,\underset{\sim}{G}(\Phi))$ such that $p_*(f'.) = f.$ so that the proof is completed. ∎

Now, recalling the definition of χ^2, it is clear that $i_*\chi^2$ maps any element of $\mathbf{H}^2(X,\Phi'')$ into a t-null element of $\mathbf{H}^3(X,\Phi)$. Really one has:

PROPOSITION 30. *In the same conditions as in the previous proposition, the sequence*

$$\mathbf{H}^2(X,\Phi'') \xrightarrow{\chi^2} \mathbf{H}^3_\Phi(X,\Phi') \xrightarrow{i_*} \mathbf{H}^3(X,\Phi)$$

is exact in the sense that an element is in the image of χ^2 iff its image by i_* is a t-null element.

PROOF. Let $g. \equiv (g_0,\tilde g_1,\tilde g_2) \in \mathbf{Z}^2_G(X,\underset{\sim}{G}^2_\Phi(\Phi'))$ be such that $i_*[g.]$ is represented by a t-null 2-cocycle $g'. \equiv (g_0',\tilde g_1',\tilde g_1' d_0)$ and let $h. \equiv (\tilde h_0^0, \tilde h_0^1, \tilde h_1^1, W)$ be a homotopy from $i_*(g.)$ to $g'.$. Using this homotopy we define the following t-null 2-cocycle $g''. \equiv (g_0'', \tilde g_1'', \tilde g_1'' d_0)$: Take

$$g_0'' = g_0' \text{ and } \tilde g_1''(z) = \tilde h_0^1(z)-\tilde h_1^1(z)+ \tilde h_0^0 d_1(z)$$

(note that $\delta(\tilde g_1''(z)) = g_0' d_1(z) - g_0' d_0(z)$ and $\tilde g_1''(s_0(x)) = 0$). Since

$$\begin{aligned} p_1(\tilde g_1''(z_0-z_1+z_2)) &= p_1(\tilde h_0^1(z_0-z_1+z_2)-\tilde h_1^1(z_0-z_1+z_2)+\tilde h_0^0 d_1(z_0)) \\ &= p_1(\tilde g_2(z_0,z_1,z_2)-\tilde g_1'(z_0)+\tilde g_1(z_2-z_1)+\tilde g_1'(z_0)) = \\ &= p_1(\tilde g_2(z_0,z_1,z_2)+\tilde g_1(z_2)-\tilde g_1(z_1)) = 0 \end{aligned}$$

(using successively (HC2), see Remark 19, and the fact that $g_2(z_0,z_1,z_2)$ is in $\underset{\sim}{G}^2_\Phi(\Phi')$), we have that $g''.$ factors through $\underset{\sim}{G}^2_\Phi(\Phi')$ and the pair $(q_1 g_1'', p_0 g_0'')$ defines a 1-cocycle over X under $\underset{\sim}{G}(\Phi'')$ whose class in $\mathbf{H}^2(X,\Phi'')$ is clearly sent by χ^2 to $[g''.]$; but $g''.$ represents in $\mathbf{H}^3_\Phi(X,\Phi')$ the same element that $g.$ since the system $(\tilde h_0^0, \tilde H_0^1, \tilde H_1^1, W')$ where

$$\tilde H_0^1(z) = \tilde h_0^1(z)-\tilde h_1^1(z)+\tilde h_0^0 d_0(z), \quad \tilde H_1^1(z) = \tilde h_0^0 d_0(z),$$
$$W'(z) = \tilde h_0^1(z)-\tilde h_1^1(z)+\tilde h_0^0 d_0(z)+ \tilde h_0^0 d_1(z),$$

determines a homotopy from $g.$ to $g''.$ so that the proof is completed. ∎

The following is a consequence of Lemma 17 and the definition of $\underset{\sim}{G}^2_\Phi(\Phi')$.

LEMMA 31. *If*

$$\Phi' = (\delta' : B' \to A, \mu') \longrightarrow \Phi = (\delta : B \to A, \mu) \xrightarrow{p=(p_1,p_0)} \Phi'' = (\delta'' : B'' \to A'', \mu'')$$

is a short exact sequence of crossed modules such that the induced morphism $p_1 : \mathrm{Ker}(\delta) \to \mathrm{Ker}(\delta'')$ *is surjective (or equivalently that the induced morphism* $\mathrm{Ker}(p_0)/\delta(B') \to A/\delta(B)$ *is injective) the sequence*

$$H^3_{\Phi}(X, \Phi') \xrightarrow{\ i_*\ } H^3(X, \Phi) \xrightarrow{\ p_*\ } H^3(X, \Phi'')$$

where i_* *is induced by the inclusion* $\underset{\sim}{G}^2_{\Phi}(\Phi') \to G^2(\Phi)$ *and* p_* *by* $\underset{\sim}{G}^2(p)$ *is exact in the sense that an element is in the image of* i_* *iff it is applied by* p_* *into an r-null class.*

PROOF. The condition of $p_1 : \mathrm{Ker}(\delta) \to \mathrm{Ker}(\delta'')$ being surjective is equivalent to that of $\underset{\sim}{G}^2(p) : \underset{\sim}{G}^2(\Phi) \to \underset{\sim}{G}^2(\Phi'')$ being a Kan-fibration; then if $g. \in \mathbf{Z}^2_G(X, \underset{\sim}{G}^2(\Phi))$ is such that $p_*(g.)$ is homotopic to an r-null 2-cocycle, Lemma 17 implies that there is another 2-cocycle $g'.$ homotopic to $g.$ such that $p_*(g'.)$ is an r-null 2-cocycle, but then $g'.$ factors through $\underset{\sim}{G}^2_{\Phi}(\Phi')$and so the class of $g.$ is in the image if i_*. That $p_* i_*$ carries all element into an r-null class is obvious. ■

COROLLARY 32. *In the same conditions as in Proposition* 29 *the sequence*

$$H^3_{\Phi}(X, \Phi') \xrightarrow{\ i_*\ } H^3(X, \Phi) \xrightarrow{\ p_*\ } H^3(X, \Phi'')$$

is exact in the sense that an element is in the image of i_* *iff it is applied by* p_* *into an r-null class.*

PROOF. The condition $\mathrm{Ker}(p_0) = \mathrm{Im}(\delta')$ implies that the map p_1: $\mathrm{Ker}(\delta) \to \mathrm{Ker}(\delta'')$ is surjective. ■

In summary, by Theorem 1, Propositions 29 and 30, Corollary 32 and Remarks 23 and 25, we have:

THEOREM 33. *Let*

$$\Phi' = (\delta' : B' \to A, \mu') \longrightarrow \Phi = (\delta : B \to A, \mu) \xrightarrow{p=(p_1,p_0)} \Phi'' = (\delta'' : B'' \to A'', \mu'')$$

be a short strongly exact sequence of crossed modules in **V** *(i. e., such that* $\mathrm{Ker}(p_0) = \mathrm{Im}(\delta')$ *). For any algebra* X *and any homomorphism* $\varphi : X \to A$ *there exists an exact sequence. extending Dedecker-Lue's 6-term exact sequence*

$$* \longrightarrow \mathbf{Z}^1_{\varphi}(X, \Phi') \xrightarrow{i_*} \mathbf{Z}^1_{\varphi}(X, \Phi) \xrightarrow{p_*} \mathbf{Z}^1_{p_0\varphi}(X, \Phi'') \xrightarrow{\chi} H^2(X, \Phi') \xrightarrow{i_*}$$

$$H^2(X, \Phi) \xrightarrow{p_*} H^2(X, \Phi'') \xrightarrow{\chi^2} H^3_{\Phi}(X, \Phi') \xrightarrow{i_*} H^3(X, \Phi) \xrightarrow{p_*} H^3(X, \Phi'')$$

whose exactness in the three last points is as follows: An element is in the image of p_* *iff its image by* χ^2 *is neutral; an element is in the image of* χ^2 *iff its image by* i_* *is t-null; an element is in the image of* i_* *iff its image by* p_* *is r-null.*

Moreover, if there exists a Φ*-extension of* $A/\delta(B)$ *with crest* $1_{A/\delta(B)}$ *then neutral = t-null = r-null and the exactness is established in the sense that an element is in the image of a map in the sequence iff its image by the following map is neutral.* ∎

In the particular case of Φ being the crossed module of inner bimultiplications in B and $\Phi'' = (B'' \to I_B/\delta(B'))$ the exactness in the 3 last points is established in terms of pointed sets.

Finally, by Lemma 20 and Remark 27, the sequence of Theorem 33 reduces to the usual abelian one when the sequence of crossed modules is that corresponding to a short exact sequence of zero-algebras.

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