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IGOR KŘÍŽ
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PECULIAR BEHAVIOR OF CONNECTED LOCALES

by Igor KŘÍŽ and Aleš PULTR

RÉSUMÉ. Si X et Y sont des espaces topologiques et si X est connexe, alors pour des raisons triviales X×Y se décompose le long d'une décomposition de X. Cependant pour des "locales", ceci n'est pas vrai en général. On donne un contr'exemple où X est localement compact (donc spatial). Par contre on obtient des énoncés positifs si X est compact ou héréditairement Lindelöf.

Although locales have found a broader area of application, the original aim to study them as a useful generalization of the classical topological spaces (to name just a few pioneering works, let us mention here e.g. [2,4,5]; the reader will find further references in [8]) is still a most important aspect of the theory.

Some facts about topological spaces generalize to locales in a straightforward way. Some need a more involved translation or modification of the original procedures. Then there are topological facts with no reasonable generalization at all. Of a particular interest, however, are the cases in between: those where a statement can be generalized (at least to some extent), but where the proofs have to be led along quite different lines than before. Here one profits in two ways (at least). First, a deeper insight into the fact in question is gained, and more fundamental reasons for the fact are discovered. Second, counterexamples for the validity of such a statement over some extent can indicate flaws in natural notions and stress the importance of some additional properties (as, e.g., the facts presented in this paper indicate that the bare notion of connectedness may not be quite so natural as it seems to be).

In this paper we are going to study some aspects of the behavior of connectedness in products, which we will show to belong in among the intermediate cases mentioned. Consider a connected topological space X and an arbitrary Y. There is a trivial fact that whenever $X\times Y$ decomposes into $U_1 \cup U_2$ with disjoint open U_1, U_2, there is a decomposition $V_1 \cup V_2$ of Y such that $U_i = X\times V_i$. The statement is easily proved using points of Y (one considers the connected subspaces $X\times\{y\}$ of $X\times Y$). This, of course, cannot be mimicked when trying to prove an analogous statement on locales. It does not immediately follow that a more involved modification may not do. But indeed, it may not, which follows from a counterexample we present. On the other hand, the fact does hold in some important special cases, and the mechanism of these positive cases is perhaps not without interest.

To simplify the following survey, let us agree to say a locale A to be *p*-connected if (in the frame notation) for any locale B, whenever the unit $1(A\oplus B)$ of $A\oplus B$ can be written as $u_1 \wedge u_2$ with $u_1 \wedge u_2 = 0$, there are

$$v_1, v_2 \in B \text{ such that } u_i = 1(A)\oplus v_i.$$

(0) Let us recall that any connected locally connected locale is *p*-connected (this is proved elsewhere - in [11] - but we mention it here to complete the overall picture).

(1) We will show that there is a locally compact (and hence spatial (!)) locale which is connected but not *p*-connected.

(2) For compact locales, however, the notions of connected and *p*-connected coincide, and,

(3) Furthermore, the same holds for hereditarily Lindelöf locales.

By [11], a locale with a dense *p*-connected part is *p*-connected, and if a locale is covered by a chained system of *p*-connected parts it is *p*-connected again. This yields an immediate extension of the positive statements above.

1. PRELIMINARIES.

Wishing to make the reading easier for those who are not (yet) locale enthusiasts we have made some of the points of this section more explicit than absolutely necessary. We hope the reader acquainted with the notions will agree that trying to save space here would not bring much gain.

1.1. The class of ordinals will be denoted by Ord and considered ordered by the natural ordering. The set of natural numbers will be denoted by $\mathbb{N}$. If X is a set, card X is its cardinality.

1.2. A *frame (locale)* is a complete lattice A satisfying the distributive law

$$a \wedge \bigvee_J b_i = \bigvee_J (a \wedge b_i)$$

(see, e.g., [7,8]). The bottom resp. top of A will be denoted by 0(A) resp. 1(A) (or simply 0 resp. 1). *Frame morphisms* are mappings preserving unions and finite meets. The category obtained will be denoted by Frm, its opposite, the *category of locales*, by Loc. It is well-known that the functor $\mathcal{D}$: Sob → Loc, where Sob is the category of sober topological spaces,

$$\mathcal{D}(T) = (\{U \mid U \text{ open in } T\}, \subset) \text{ and } (\mathcal{D}f)(U) = f^{-1}(U),$$

constitutes a full embedding (see e.g. [7]). We will consider, therefore, Loc as a natural extension of the category Sob.

Although it is the locale point of view we have in mind, the notation will be kept for obvious technical reasons as in Frm. Thus, we may speak about a product of locales and count with the corresponding coproducts of frames (see 1.5): a sublocale B of A will be represented as a surjective frame morphism A → B. In particular, we stress that the symbol A×B stands for the product of frames, not that of locales; it will play a technical role.

A *cover* of a frame (locale) A is a subset $U \subset A$ such that $\bigvee U = 1$.

The *complement* of an $x \in A$, denoted by $\bar{x}$, is the largest y such that $y \wedge x = 0$. An element x is said to be *complemented* if $x \vee \bar{x} = 1$.

A *basis* of a frame is a subset $U \subset A$ such that for each x in A,

$$x = \bigvee\{u \mid u \leq x \;\&\; u \in U\}.$$

If $a \in A$ we denote by $[a]$ the locale

$$(\{x \mid x \leq a\}, \wedge, \bigvee, 0([a]) = 0(A), 1([a]) = a).$$

It will be considered a sublocale of A in view of the surjection $x \mapsto x \wedge a$.

1.3.

1.3.1. Let A be a poset. A subset $X \subset A$ is said to be decreasing if

$$(x \in X \;\&\; y \leq x) \Rightarrow y \in X.$$

The set of all decreasing subsets of A , endowed by the relation of inclusion, will be denoted by D(A). Obviously, D(A) is a frame. For $a \in A$, $X \subset A$, we will use the notation

$$\downarrow a = \{x \in A \mid x \leq a\}, \; \downarrow X = \cup\{\downarrow a \mid a \in X\}.$$

It is easy to see that if A is a lower semilattice, δ_A: A → D(A) defined by $\delta(a) = \downarrow a$ preserves meets.

1.3.2. We will use the following fact from [12]:

PROPOSITION. *Let* A *be a lower semilattice,* B *a frame. Then for any meet-preserving* f: A → B *there is exactly one frame morphism* g: D(A)→B *such that* $g \circ \delta_A = f$.

1.3.3. Let X be a set. Denote by FLS(X) the free lower semilattice generated by X. It can be obtained e.g. as the set of all finite subsets of X with the union for meet; we will use, however, the more handy notation of formal meets $\bigwedge_{i=1}^{n} x_i$.

According to 1.3.2, FF(X) = D(FLS(X)) is the free frame generated by X.

1.3.4. Note that if T is the quasidiscrete space corresponding to the poset A, $\mathcal{D}$(T) = D(A).

1 4 Quotients of frames.

1.4.1. A relation R on a frame is called *pre-congruence* if, for each $(a,b) \in R$,

$$\{x \mid (a \wedge x, b \wedge x) \in R\}$$

is a basis of A. An element $u \in A$ is said to be R-*coherent* if

$$\text{for any } a, b \in A \quad aRb \Rightarrow (a \leq u \text{ iff } b \leq u).$$

For a pre-congruence R on a frame A and $a \in A$ put

$$j_R(a) = \bigwedge\{u \mid u \geq a,\ u \text{ R-coherent}\}.$$

Denote by A/R the set of all R-coherent elements of A. Obviously, j is a mapping of A onto A/R and $j(j(a)) = j(a)$. Following the lines of a slightly less generally formulated theorem from [9] one can easily prove (see [10]):

1.4.2. THEOREM. *The mapping j_R preserves meets. If we define the joins $\bigvee_R$ in A/R by putting $\bigvee_R X = j(\bigvee X)$, A/R becomes a frame and $j\colon A \to A/R$ a frame morphism. For any frame morphism $f\colon A \to B$ such that*

$$aRb \Rightarrow f(a) = f(b)$$

there is exactly one frame morphism $g\colon A/R \to B$ such that

$$g \circ j = f.$$

1.5. Products of locales (coproducts of frames).

In particular, one can construct the coproduct of frames A, B as (see, e.g., [7], cf. [3,6])

$$A \oplus B = D(A \times B)/S$$

where the relation S consists of all the couples

$$(\downarrow(\bigvee_J a_i, b), \bigcup_J \downarrow(a_i, b)),\ (\downarrow(a, \bigvee_J b_i), \bigcup_J \downarrow(a, b_i)),$$
$$(\downarrow(a,0), \emptyset),\ (\downarrow(0,b), \emptyset).$$

We immediately see that an $X \in D(A \times B)$ is S-coherent iff

$(x,0)$, $(0,x) \in X$ for all $x,y \in X$, and
$(\bigvee x_i, y)$, $(x, \bigvee y_i) \in X$ whenever (x_i, y) resp. $(x, y_i) \in X$ for all $i \in J$.

Let us write $a \oplus b$ for $j_S(\downarrow(a, b))$. We will use the following easy facts:

1.5.1. The elements of the form $a \oplus b$ constitute a basis of $A \oplus B$.

1.5.2. If X is an S-coherent subset of $A \times B$ and

$$a \oplus b \leq \bigvee\{x \oplus y \mid (x,y) \in X\}$$

then $(a, b) \in X$.

1.5.3. We have $a \oplus 0 = 0 \oplus b = 0$ for all $a \in A$, $b \in B$, but if $x \neq 0$, then $a \oplus x \le a' \oplus x$ implies $a \le a'$ and, similarly, $x \oplus b \le x \oplus b'$ implies $b \le b'$.

1.6. A non-trivial locale A (i.e., such that $0(A) \ne 1(A)$) is said to be *connected* if there is no complemented element but 0 and 1. An element a is connected if $[a]$ is (which amounts to non-existence of a non-trivial decomposition

$$a = x \vee y \text{ with } x \wedge y = 0 \text{).}$$

A locale is said to be *locally connected* if it has a basis consisting of connected elements.

1.6.1. Note that if A is a poset, D(A) is always locally connected.

1.7. We say that a couple of non-trivial locales (A,B) has *regular cuts* if each complemented element in A⊕B is of the form $1(A) \oplus b$.

Obviously, if (A,B) has regular cuts then A is connected.

We say that A is *p-connected* if (A,B) has regular cuts for each non-trivial B.

1.7.1. By a result from [11] we have

THEOREM. *Each connected locally connected locale is p-connected.*

1.8. Let A be a frame and let X be a subset of A. We say that x,y are joined by a chain in X if there are $x_1, \dots, x_n \in X$ such that

$$x = x_1,\ y = x_n \text{ and } x_i \wedge x_{i+1} \ne \emptyset \text{ for all } i = 1, \dots, n-1.$$

A subset $X \subset A$ is said to be *chained* if any two $x,y \in X$ are joined by a chain in X.

1.8.1. OBSERVATION. Let U be a cover of a connected A, and let $0 \notin U$. Then U is chained. (Realize that

$$\bigvee \{x \mid x \text{ is joined by chain with } u \text{ in } U\}$$

is complemented for any u.)

2. A COUNTEREXAMPLE.

2.1. Let B be a boolean algebra. Denote by $\mathcal{C} = \mathcal{C}(B)$ the set of all disjoint covers of B by non-zero elements and by FinB the set of all finite subsets of B. Put

$$X_B = \{(\alpha u \varepsilon) \mid \alpha \in \mathcal{C}(B),\ u \in \alpha,\ \varepsilon = 1,2\},$$
$$M = M_B = FLS(X_B) \text{ (recall 1.3.3) .}$$

On D(M) (= $FF(X_B)$) define a pre-congruence R as that consisting of all the pairs

$$((\downarrow(\alpha u1) \vee \downarrow(\alpha u2)) \wedge \downarrow x,\ (\downarrow g(\alpha v1) \vee \downarrow(\alpha v2)) \wedge \downarrow x) \quad \text{with } \alpha \in \mathcal{C},\quad u,v \in \alpha \quad \text{and} \quad x \in M,$$

and
$$(\downarrow((a u1) \wedge (\beta v2) \wedge x), \emptyset)$$
with $\alpha,\beta \in \mathcal{C}$, $u \in \alpha$, $v \in \beta$ such that $u \wedge v \neq 0$ and $x \in M$.

Put
$$A = A_B = D(M)/R .$$

We will write briefly $\langle \alpha u \varepsilon \rangle$ for $j(\downarrow(\alpha u \varepsilon))$ and, more generally, $\langle x \rangle$ for $j(\downarrow x)$ ($x \in M$).

2.2. Some more notation. For $x = \bigwedge_{i=1}^{n} (\alpha_i u_i \varepsilon_i)$ put

$$\tau(x) = \{\alpha_1,\dots,\alpha_n\},\ \mathcal{T}(x) = \bigcup \tau(x).$$

Further put

$$N = \{x \in M \mid x = \bigwedge_{i=1}^{n} (\alpha_i u_i \varepsilon_i) \text{ such that if } u_i \text{ and } u_j \text{ are joined by chain in } \mathcal{T}(x) \text{ then } \varepsilon_i = \varepsilon_j\}.$$

For $\alpha \in \mathcal{C}$ and $\emptyset \neq J \subset \alpha$ finite put

$$\zeta(\alpha,J) = \{ \bigwedge_{u \in J} (\alpha u \varepsilon_u) \mid \varepsilon_u \text{ arbitrary}\}.$$

2.3. Counting in A_B.

2.3.1. The condition of R-coherence for a decreasing $X \subset M$ is easily rewritten to the pair of conditions

(C1) $\forall \alpha, \beta \in \mathcal{C}$, $u \in \alpha$ and $v \in \beta$,

$$u \wedge v \neq 0 \Rightarrow (\alpha u1) \wedge (\beta v2) \in X,$$

(C2) $\forall\, x \in M$, $\alpha \in \mathcal{C}$ and $u, v \in \alpha$

$$(\alpha u \varepsilon) \wedge x \in X \text{ for both } \varepsilon = 1,2 \Rightarrow (\alpha v \varepsilon) \wedge x \in X \text{ for both } \varepsilon = 1,2.$$

2.3.2. We immediately see that

$$\langle\alpha u 1\rangle \wedge \langle\beta v 2\rangle = 0 \text{ whenever } u \wedge v \neq 0$$

and $\langle\alpha u 1\rangle \wedge \langle\alpha u 2\rangle = a(\alpha)$ does not depend on the choice of $u \in \alpha$. Further, obviously,

$$\bigvee\{a(\alpha) \mid \alpha \in \mathcal{C}\} = 1.$$

By induction on the size of J we easily obtain that for any α and J,

$$\bigvee\{\langle x\rangle \mid x \in \zeta(\alpha, J)\} = a(\alpha).$$

Consequently, for any $x \in M$ and any collection of finite $J(\alpha) \neq 0$, $J(\alpha) \subset \alpha$,

$$\langle x\rangle \leq \bigwedge_{\alpha\in\tau(x)} \bigvee\{\langle y\rangle \mid y \in \zeta(\alpha, J(\alpha))\}.$$

2.3.3. LEMMA. *For each* $x \in N$, $\langle x\rangle \neq 0$.

PROOF. Put $X = M \setminus N$. Since N is obviously increasing, X is decreasing. Obviously the condition (C1) is satisfied. Now, let

$$y = (\alpha v \varepsilon) \wedge z \notin X \text{ (i.e., } y \in N).$$

Take $u \in \alpha$. If it is joined by a chain in $\alpha \cup \mathcal{T}(z)$ with some u_i where

$$z = \bigwedge_{i=1}^{n}(\alpha_i u_i \varepsilon_i),$$

put $\eta = \varepsilon_i$ (since $y \in N$, η is uniquely determined), otherwise choose η arbitrarily. Then we have $(\alpha u \eta) \wedge z \in N$ (and hence $\notin X$). Thus, also the condition (C2) is satisfied, and hence X is coherent. Since $x \notin X$, $\langle x\rangle \not\leq X$. ∎

2.4. THEOREM. *The frame* $A_B \oplus B$ *has a decomposition*

$$1(A_B \oplus B) = a_1 \vee a_2, \quad a_1 \wedge a_2 = 0$$

such that $a_i \leq 1 \oplus v$ *implies* $v = 1$ *for both* $i = 1,2$.

PROOF. Put

$$a_i = \bigvee\{\langle\alpha u i\rangle \oplus u \mid \alpha \in \mathcal{C},\ u \in \alpha\}.$$

We have

$$a_1 \wedge a_2 = \bigvee\{(\langle\alpha u i\rangle \wedge \langle\beta v 2\rangle) \oplus (u \wedge v) \mid \alpha,\beta \in \mathcal{C},\ u \in \alpha,\ v \in \beta\} = 0$$

since all the summands are zero (if $u \wedge v \neq 0$, $\langle\alpha u 1\rangle \wedge \langle\beta v 2\rangle = 0$ by 2.3.2) and

$$a_1 \vee a_2 = \bigvee\{(\langle\alpha u 1\rangle \vee \langle\alpha u 2\rangle) \oplus u \mid \alpha \in \mathcal{C},\ u \in \alpha\} =$$

$$\bigvee_{\alpha\in C}(a(\alpha)\oplus\bigvee_{u\in\alpha} u) = \bigvee(a(\alpha)\oplus 1) = (\bigvee a(\alpha))\oplus 1 = 1$$

by 2.3.2. Let $a_i \le 1\oplus v$. Then $\langle\alpha u i\rangle \oplus u \le 1\oplus v$ for all $\alpha\in C$, and $u\in\alpha$, and, since $\langle\alpha u i\rangle \ne 0$ by 2.3.3., we obtain $u\le v$ by 1.5.3. Hence $v=1$. ■

2.5. LEMMA. *Let* $j(\emptyset)\subset m\in{\downarrow}M$. *Put*

$$X = \{x\in M \mid \text{for } \alpha\in\tau(x) \text{ there exist finite } J_\alpha(x)\subset\alpha \text{ such that for all } y_\alpha\in\zeta(\alpha,J_\alpha(x)), \bigwedge_{\alpha\in\tau(x)} y_\alpha\wedge x\in m\}.$$

Then X *is* R*-coherent and, since obviously* $X\supset m$, *we have* $X\supset j(m)$.

PROOF. If $x\in X$ and $z\le x$, we have $\tau(z)\supset\tau(x)$; put $J_\alpha(z)=J_\alpha(x)$ for $\alpha\in\tau(x)$, otherwise choose $J_\alpha(z)$ arbitrarily. If $y_\alpha\in\zeta(\alpha,J_\alpha(z))$, we obtain

$$\bigwedge_{\alpha\in\tau(z)} y_\alpha\wedge z \le \bigwedge_{\alpha\in\tau(x)} y_\alpha\wedge x\in m.$$

Since m is decreasing, we conclude that X is, too. Since $X\supset m$ and $m\supset j(\emptyset)$, the condition (C1) is obviously satisfied.

The condition (C2): Let $(\alpha u\varepsilon)\wedge x\in X$ for $\varepsilon=1,2$. For

$$\beta\in\{\alpha\}\cup\tau(x) \;\; (= \tau((\alpha u\varepsilon)\wedge x) = \tau((\alpha v\varepsilon)\wedge x))$$

put

$$J_\beta((\alpha v\varepsilon)\wedge x) = J_\beta((\alpha u1)\wedge x)\cup J_\beta((\alpha u2)\wedge x) \text{ for } \beta\ne\alpha,$$
$$J_\alpha((\alpha v\varepsilon)\wedge x) = J_\alpha((\alpha u1)\wedge x)\cup J_\alpha((\alpha u2)\wedge x)\cup\{u\}.$$

Thus, a $y_\beta\in\zeta(\beta,J_\beta((\alpha v\varepsilon)\wedge x))$ can be written as

$$y_\beta = y_\beta^1\wedge y_\beta^2 \text{ with } y_\beta^i\in\zeta(\beta,J_\beta((\alpha u i)\wedge x)) \text{ for } \beta\ne\alpha,$$
$$y_\alpha = y_\alpha^1\wedge y_\alpha^2\wedge(\alpha u j) \quad\text{for some } j,$$

and

$$\bigwedge\{y_\beta^i\wedge(\alpha u i)\wedge x \mid \beta\in\{\alpha\}\cup\tau(x)\}\in m.$$

Since

$$\bigwedge\{y_\beta\wedge(\alpha v\varepsilon)\wedge x \mid \beta\in\{\alpha\}\cup\tau(x)\} \le \bigwedge y_\beta\wedge x \le \bigwedge y_\beta\wedge(\alpha u j)\wedge x,$$

we see that $(\alpha v\varepsilon)\wedge x\in K$ for $\varepsilon=1,2$. ■

2.6. LEMMA. *Let* $j(\emptyset)\subset m\in{\downarrow}M$ *and let* $x\in j(m)$. *Let* $\tau(x) = \{\alpha_1,\dots,\alpha_n\}$. *Then there are finite* $J_i\subset\alpha_i$ *such that, for all* $y_i\in\zeta(\alpha_i,J_i)$, $y_1\wedge\dots\wedge y_n\wedge x\in m$.

PROOF. Use 25: since $x \in j(m)$, we have $x \in X$. ■

2.7. In the following theorem we adopt the terminology and use a result from [1].

THEOREM. *Each* A_B *is a continuous lattice. Consequently, it is an* $\mathcal{D}(T)$ *for a sober locally compact space* T.

PROOF. Obviously it suffices to show that for each $\langle x \rangle$ with $x \in M$ we have $\langle x \rangle \ll \langle x \rangle$.

Let $\langle x \rangle \leq \bigvee_{\iota \in I} X_\iota$ with X_ι coherent. Thus, $x \in \langle x \rangle \subset j(\cup X_\iota)$. Put $m = \cup X_i$. Use 2.6 and put

$$Y_i = \zeta(\alpha_i, J_i), \quad Y = \bigtimes_{i=1}^{n} Y_i.$$

Thus, Y is finite. For $y = (y_1, \ldots, y_n) \in Y$ choose $\iota(y) \in I$ such that $y_1 \wedge \ldots \wedge y_n \wedge x \in X_{\iota(y)}$. We have, hence

$$\bigwedge_{i=1}^{n} \langle y_i \rangle \wedge \langle x \rangle \leq X_{\iota(y)} \text{ so that } \bigvee_{y \in Y} (\bigwedge_{i=1}^{n} \langle y_i \rangle \wedge \langle x \rangle \leq \bigvee_{y \in Y} X_{\iota(y)}.$$

On the other hand we have

$$\bigvee_{y \in Y} (\bigwedge \langle y_i \rangle) = \bigwedge_{i=1} (\bigvee_{y_i \in Y_i} \langle y_i \rangle) \geq \langle x \rangle$$

by 2.3.2 and hence

$$\langle x \rangle \leq \bigvee_{y \in Y} X_{\iota(y)}. \ ■$$

2.8. We say that a boolean algebra B has the property (*P*) if the following holds:

For each $\mathcal{C}' \subset \mathcal{C}(B)$ containing refinements of all $\alpha \in \mathcal{C}(B)$, and for each $\varphi: \mathcal{C}' \to \mathrm{Fin}\, B$ such that $\varphi(\alpha) \subset \alpha$ for all $\alpha \in \mathcal{C}'$, there exist $\alpha, \beta \in \mathcal{C}'$ such that no $u \in \varphi(\beta)$ is joined by chain in $\alpha \cup \beta$ with a

$$v \in \varphi(\alpha) \cup \varphi(\beta), \ v \neq u.$$

2.9. PROPOSITION. *Let* (X, ρ) *be a metric space such that for each non-void open* $U \subset X$ *we have* $\mathrm{card}\, U \geq 2^{2^\omega}$ *(such as, e.g., any* Y^ω *with a sufficiently large discrete* Y*). Then the boolean algebra* B *of the regular open subsets of* X *has property (P).*

PROOF. The closure of a subset Z in X will be denoted by cl(Z) (recall that the bar indicates the complement). We easily find $\sigma_n \in \mathcal{C}'$ such that

(1) for $m \geq n$, σ_m is a refinement of σ_n.

(2) for each $u \in \sigma_n$, diam $u \leq 1/n$.

Choose finite $Z_n \subset X$ meeting all the elements of $\varphi(\sigma_n)$ and put $Z = \cup Z_n$. Thus, card $cl(Z) \leq 2^{\omega}$ and hence $cl(Z)$ is nowhere dense in X. If $\rho(x,Z) > 1/k$ we obviously have $x \in cl(\bigcup_{n\geq k} \cup\varphi(\sigma_n))$ and hence

$$cl(Z) \supset \bigcap_{k=1}^{\infty} cl(\bigcup_{n\geq k} \cup\varphi(\sigma_n)) \geq \bigcap_{k\in\omega} (\bigvee_{n\geq k} \vee \varphi(\sigma_n))$$

where $\vee$ indicates the join in B. Thus, if we put $a_k = \bigvee_{n\geq k} \vee\varphi(\sigma_n)$ we obtain ($\wedge$ is the meet in B)

$$(3) \qquad \bigwedge_{k=1}^{\infty} a_k = 0 \text{ and hence } \bigvee_{k=1}^{\infty} \bar{a}_k = 1.$$

Consider the system

$$\gamma = \{\bar{a}_1\} \cup \bigcup_{k=2}^{m} \{ a_{k-1} \wedge \bar{a}_k \wedge x \mid x \in \sigma_{k-1} \} .$$

For $k \geq m$ we have $a_k \leq a_m$ and hence $\bar{a}_k \geq \bar{a}_m$ so that

$$\vee\gamma \wedge \bar{a}_m = \bar{a}_1 \vee \bigvee_{k=2}^{m} (\vee\{a_{k-1} \wedge \bar{a}_k \wedge x \mid x \in \sigma_{k-1}\}) = \bar{a}_1 \vee \bigvee_{k=2}^{m} (a_{k-1} \wedge \bar{a}_k) = \bar{a}_m.$$

Thus, by (3), γ is a cover, obviously disjoint. Moreover,

(4) for each n, γ refines $\varphi(\sigma_n) \cup \{\overline{\vee\varphi(\sigma_n)}\}$.

(Indeed, let us take a $u = a_{k-1} \wedge \bar{a}_k \wedge x$ with $x \in \sigma_{k-1}$; if $k \leq n$, we have $a_k \geq \vee\varphi(\sigma_n)$ and hence $u \leq \overline{\vee\varphi(\sigma_n)}$, if $k-1 \geq n$ there is a $y \in \sigma_n$ such that $x \leq y$ and we have either $y \in \varphi(\sigma_n)$ or $y \leq \overline{\vee\varphi(\sigma_n)}$ again.)

Choose an $\alpha \in \mathcal{C}'$ which is a refinement of γ. Then we can find, for $u \in \alpha$, numbers $k(u)$ such that $u \leq \bar{a}_{k(u)}$. Put

$$n = \max\{k(u) \mid u \ \varphi(\alpha)\} \text{ and } \beta = \sigma_n.$$

Now, α refines $\varphi(\beta) \cup \{\overline{\vee\varphi(\beta)}]$ (recall (4)) and

$$\vee\varphi(\alpha) \leq \bar{a}_n = \overline{\bigvee_{j\geq n} \vee\varphi(\sigma_j)} \leq \overline{\vee\varphi(\beta)}$$

so that $\vee\varphi(\alpha) \wedge \vee\varphi(\beta) = 0$. Hence, no $u \in \varphi(\beta)$ is joined by chain in $\alpha \cup \beta$ with a $v \in \varphi(\alpha) \cup \varphi(\beta)$, $v \neq u$. ∎

2.10. LEMMA. *Let* $j(\emptyset) \subset m \in \downarrow M$ *and* $j(m) = 1$. *Then there is a* φ: $\mathcal{C}(B) \to \mathrm{Fin}\,B$ *such that* $\varphi(\alpha) \subset \alpha$ *and* $\zeta(\alpha, \varphi(\alpha)) \subset m$ *for each* α.

PROOF. Use Lemma 2.5. Now, $X = M$. For $\alpha \in \mathcal{C}$ choose a $u \in \alpha$ and put

$$\varphi(\alpha) = J_\alpha((\alpha u1)) \cup J_\alpha((\alpha u2)) \cup \{u\}.$$

Let $y \in \zeta(\alpha,\varphi(\alpha))$. Then

$$y = y_1 \wedge y_2 \wedge (\alpha u j) \text{ with } y_i \in \zeta(\alpha, J_\alpha((\alpha u i)));$$

Hence $y_i \wedge (\alpha u i) \in m$ and hence finally $y \in m$. ■

2.11. THEOREM. *Let* B *have the property (P). Then* A_B *is connected.*

PROOF. Let $m_1, m_2 \in A$ be such that

$$m_1 \vee m_2 = 0 \text{ and } m_1 \vee m_2 = j(m_1 \cup m_2) = 1.$$

By 2.10 there is a $\varphi : \mathcal{C}(B) \to \mathrm{Fin}\, B$ such that

$$\varphi(\alpha) \subset \alpha \text{ and } \zeta(\alpha,\varphi(\alpha)) \subset m_1 \cup m_2 \text{ for all } \alpha.$$

Put

$$\mathcal{C}_0 = \{\alpha \in \mathcal{C} \mid \zeta(\alpha,\varphi(\alpha)) \subset m_i \text{ for some } i\},\quad \mathcal{C}' = \mathcal{C} \setminus \mathcal{C}_0 .$$

I. Let $\mathcal{C}'$ contain a subdivision for any $\alpha \in \mathcal{C}$. Because of (P) we can choose $\alpha,\beta \in \mathcal{C}'$ such that a $u \in \varphi(\alpha)$ is never joined by chain in $\alpha \cup \beta$ with a $v \in \varphi(\alpha) \cup \varphi(\beta)$, $v \neq u$. Since

$$\zeta(\alpha,\varphi(\alpha)) \subset m_1 \cup m_2 ,$$

we have, say,

$$x = \bigwedge_{u \in \varphi(\alpha)} (a u 1) \in m_1;$$

since $\beta \notin \mathcal{C}_0$ there are ε_v such that

$$y = \bigwedge_{v \in \varphi(\beta)} (\beta v \varepsilon_v) \in m_2 .$$

Now, however,

$$\langle x \wedge y \rangle = \langle x \rangle \wedge \langle y \rangle \leq m_1 \wedge m_2 \text{ and } \langle x \wedge y \rangle \neq 0$$

by 2.3.3 in contradiction with $m_1 \wedge m_2 = 0$.

II. Thus, there is an $\alpha \in \mathcal{C}$ such that each of its subdivisions is in $\mathcal{C}_0$. We will show that $\mathcal{C}' = \emptyset$. Indeed, let $\beta \in \mathcal{C}'$. Then

$$\alpha \wedge \beta = \{u \wedge v \mid u \in \alpha,\ v \in \beta\} \in \mathcal{C}_0 .$$

Let, say, $\zeta(\alpha \wedge \beta, \varphi(\alpha \wedge \beta)) \subset m_1$. There are ε_v such that

$$y = \bigwedge_{v \in \varphi(\beta)} (\beta v \varepsilon_v) \in m_2 .$$

For $u \in \varphi(\alpha \wedge \beta)$ put $\varepsilon_u = \varepsilon_v$ whenever $u \subset v \in \varphi(\beta)$. Put

$$x = \bigwedge_{u \in \varphi(\alpha \wedge \beta)} (\alpha \wedge \beta, u, \varepsilon_u) .$$

Now we have $x \wedge y \in \mathrm{N}$, hence $\langle x \wedge y \rangle \neq 0$, and we have a contradiction $\langle x \wedge y \rangle \leq m_1 \wedge m_2$ again.

III. Consequently, $\mathcal{C} = \mathcal{C}_0$. If we had

$$\zeta(\alpha, \varphi(\alpha)) \subset m_1 \text{ and } \zeta(\beta, \varphi(\beta)) \subset m_2$$

we would again obtain a contradiction

$$0 \neq \langle \bigwedge_{u \in \varphi(\alpha)} (\alpha u 1) \wedge \bigwedge_{v \in \varphi(\beta)} (\beta v 1) \rangle \leq m_1 \wedge m_2 .$$

Thus, say, $\zeta(\alpha, \varphi(\alpha)) \subset m_1$ for all α and by 2.3.2

$$m_1 \geq \bigvee_{\alpha \in \mathcal{C}} \bigvee \{ \langle y \rangle \mid y \in \zeta(\alpha, \varphi(\alpha)) \} = \bigvee_{\alpha \in \mathcal{C}} a(\alpha) = 1 . \quad \blacksquare$$

2.12. From 2.4, 2.7, 2.9 and 2.11 we immediately obtain

CONCLUSION. *There exists a locally compact connected sober space* T *which is not p-connected* (more exactly, such that $\mathcal{D}$(T) is not *p*-connected).

3. POSITIVE RESULTS.

3.1. A *semitree* is a couple (T, R) where R is a binary relation on T such that there is no infinite sequence $t_1 \mathrm{R} t_2 \mathrm{R} t_3 \ldots$. A *semitree with ordinals* (T,R,φ) is, moreover, endowed with a mapping φ: T$\rightarrow$Ord such that

$$t_1 \mathrm{R} t_2 \Rightarrow \varphi(t_1) > \varphi(t_2).$$

3.2. Conventions and notation.

A subset U of a semitree is always viewed as the semitree $(\mathrm{U}, \mathrm{R} \cap (\mathrm{U} \times \mathrm{U}))$. We set

$$\tau(\mathrm{T},\mathrm{R}) = \{ t \in \mathrm{T} \mid t\mathrm{R} = \emptyset \}$$

and usually write just τ(T). Thus, in particular, if $\mathrm{U} \subset \mathrm{T}$,

$$\tau(\mathrm{U}) = \{ t \in \mathrm{U} \mid t\mathrm{R}s \Rightarrow s \notin \mathrm{U} \} .$$

Let $\mathrm{A}_1, \mathrm{A}_2$ be frames. For $x = (x_1, x_2) \in \mathrm{A}_1 \times \mathrm{A}_2$ and $i = 1,2$ we write

$$x =_i \bigvee \{ y \mid y \in \mathrm{Y} \} \text{ (or } x =_i \bigvee \mathrm{Y}, \text{ or } x =_i \bigvee_{\mathrm{Y}} y)$$

to indicate that $x_i = \bigvee \{ y_i \mid y \in \mathrm{Y} \}$ (where, of course, $y = (y_1, y_2)$) and all the coordinates y_{3-i} of $y \in \mathrm{Y}$ coincide.

3.3. PROPOSITION. *Let* A,B *be frames and let* $m \subset A \times B$ *be a decreasing set such that*

$$\bigvee\{a \oplus b \mid (a,b) \in m\} = 1 \text{ in } A \oplus B.$$

Then there is a semitree with ordinals (T,R,φ) *and subsets* $T_1, T_2 \subset T$ *such that*

(1) $T \subset A \times B$ *and* $\tau(T) \subset m$,

(2) $T_1 \cup T_2 = T \setminus \tau(T)$,

(3) $\forall x \in T_i \quad x = {}_i\bigvee\{y \mid xRy\}$.

PROOF. First, let us construct a transfinite sequence $T(\alpha)$ ($\alpha \in \mathrm{Ord}$) of subsets of $A \times B$ as follows: $T(0) = m$, and for $\alpha > 0$,

$$T(\alpha) = \{x \mid \exists i \in \{1,2\}\ \exists \beta < \alpha\ \exists Q \subset T(\beta) \text{ such that } x = {}_i\bigvee Q\}.$$

Obviously all the $T(\alpha)$ are decreasing sets and if $T(\alpha) = T(\alpha+1)$ then $T(\alpha)$ is coherent (recall 1.5). Also obviously

$$\alpha \le \beta \Rightarrow T(\alpha) \subset T(\beta)$$

and hence there has to be an α_0 with $T(\alpha_0) = T(\alpha_0+1)$. Since

$$1 \oplus 1 = \bigvee\{a \oplus b \mid (a,b) \in m\} \subset \bigvee\{a \oplus b \mid (a,b) \in T(\alpha_0)\}$$

we conclude (recall 1.5.2) that $1 \oplus 1 \in T(\alpha_0)$.

Consider the least α with $(1,1) \in T(\alpha)$ and put $T = T(\alpha)$. For $x \in T$ denote by $\varphi(x)$ the least ordinal β such that $x \in T(\beta)$. Now if $x \in T$, $\varphi(x) \neq 0$, choose arbitrarily (but fixedly) an $i = i(x)$ and a $Q_x \subset T(\gamma)$ with $\gamma < \varphi(x)$ such that $x = {}_i\bigvee Q$. Put

$$xR = Q_x,\ T_j = \{x \mid i(x) = j\}. \ \blacksquare$$

3.4. We will use the abbreviation Ch(A) to indicate that a frame A has the following property:

For each frame B and any $m \subset A \times B$ such that

$$1 \oplus 1 = \bigvee\{a \oplus b \mid (a,b) \in m\} \text{ in } A \oplus B,$$

whenever (a,b) and (c,d) are in m and $a,b,c \neq 0$ then $a \oplus b$ and $c \oplus d$ are joined by chain in $\{x \oplus y \mid (x,y) \in m\}$.

3.5. PROPOSITION. *If* Ch(A) *then* A *is p-connected.*

PROOF. Let $x \in A \oplus B$ be complemented. Put

$$m' = \{(a,b) \mid a \oplus b \le x,\ a,b \neq 0\},$$
$$m'' = \{(a,b) \mid a \oplus b \le \bar{x},\ a,b \neq 0\},$$

$$m = m' \cup m'' \ , \ m_2' = \{b \mid (a,b) \in m'\} \text{ and } m_2'' = \{b \mid (a,b) \in m''\}.$$

Let there exist $b' \in m'$ and $b'' \in m''$ with $b = b' \cap b'' \neq 0$. We have $(a,b') \in m'$, $(c,b'') \in m''$ for some $a,c \neq 0$, hence $(a,b) \in m' \subset m$, $(c,b) \in m'' \subset m$ so that $a \oplus b$ and $c \oplus b$ are joined by chain in $\{x \oplus y \mid (x,y) \in m\}$ which is contradicted by $a \oplus b \le x$ and $c \oplus b \le \overline{x}$. Thus, $u = \bigvee m_2'$ and $v = \bigvee m_2''$ are disjoint. Since

$$x = \bigvee\{a \oplus b \mid (a,b) \in m'\}$$

we have $x \le 1 \oplus u$ and, similarly, $\overline{x} \le 1 \oplus v$ and we immediately conclude

$$v = \bar{u} \text{ and } x = 1 \oplus u. \ \blacksquare$$

3.6. LEMMA. *Let* A *be a frame, let* $b \in$ A, X,Y $\subset$ A *and* z *in* X *be such that*

$$b \wedge \bigwedge X \neq 0 \text{ and } z = \bigvee Y.$$

Then there is a $y \in$ Y *such that*

$$0 \neq b \wedge \bigwedge X \wedge y = b \wedge \bigwedge(X \setminus \{z\}) \wedge y.$$

PROOF. We have

$$0 \neq b \wedge \bigwedge(X \setminus \{z\}) \wedge \bigvee Y = \bigvee\{b \wedge \bigwedge(X \setminus \{z\}) \wedge u \mid u \in Y\}. \ \blacksquare$$

3.7. Denote by $\mathcal{K}$ the class of all mappings $k\colon \mathrm{Ord} \to \mathbb{N}$ such that $k(\alpha) \neq 0$ for finitely many α only. Write $k > k'$ if

$$k(\alpha) \le k'(\alpha) \Rightarrow \exists \beta > \alpha \text{ such that } k(\beta) \neq 0.$$

OBSERVATION. There is no strictly decreasing sequence in $(\mathcal{K}, <)$.

3.8. Let (T,R,φ) be a semitree with ordinals. For a finite $X \subset T$ define $K(X) \in \mathcal{K}$ by putting

$$K(X)(\alpha) = \mathrm{card}\{x \in X \mid \varphi(x) = \alpha\}.$$

3.9. THEOREM. *All compact connected locales* A *are p-connected.*

PROOF. By 3.5 it suffices to prove Ch(A). Consider a frame B, a fixed non-zero $b \in$ B and an $m \subset$ A$\times$B such that

$$\bigvee\{x_1 \oplus x_2 \mid (x_1, x_2) \in m\} = 1 \ .$$

We will construct a finite $M \subset m$ such that

$$b \wedge \bigwedge_{x \in M} x_2 \neq 0 \text{ and } \bigvee_{x \in M} x_1 = 1$$

so that Ch(A) will follow by applying 1.8.1 for the cover $\{x_1 \mid x \in M\}$ of A.

Consider the semitree from 3.3. Put $M_0 = \{(1,1)\}$. Let us have M_n constructed so that

$$b \wedge \bigwedge_{x \in M_n} x_2 \neq 0 \text{ and } \bigvee_{x \in M_n} x_1 = 1.$$

If $M_n \subset m$ put $M = M_n$. Otherwise choose one of the $x \in M_n$ with $\varphi(x)$ maximal. Obviously $\varphi(x) \neq 0$. If $x \in T_2$ choose by 3.6 a y such that

$$xRy \text{ and } b \wedge \bigwedge\{ z_2 \mid z \in M_n \backslash \{x\}\} \wedge y_2 \neq 0,$$

and set

$$M_{n+1} = (M_n \backslash \{x\}) \cup \{y\}.$$

If $x \in T_1$, we have

$$\bigvee\{z_1 \mid z \in M_n \backslash \{x\}\} \vee \bigvee\{y_1 \mid xRy\} = 1$$

and hence we can find a finite

$$M_{n+1} \subset (M_n \backslash \{x\}) \cup xR \text{ such that } \bigvee\{z_1 \mid z \in M_{n+1}\} = 1$$

using the compactness.

In any case $K(M_{n+1}) < K(M_n)$ and hence, by 3.7, there is an n with $M_n \subset m$. ∎

3.10. LEMMA. *Let* A *be a frame,* (T,R) *a semitree and* f: T → A *a mapping such that*

(1) $1 \in f(T)$, *and*

(2) $t \in \tau(T) \Rightarrow f(t) = \bigvee\{ f(u) \mid tRu\}$.

Then $\bigvee\{f(t) \mid t \in \tau(T)\} = 1$.

PROOF. Let

$$a = \bigvee_{\tau(T)} f(t) < 1 .$$

Choose a $t_1 \in T$ such that $f(t_1) = 1$. Thus, $a \not\geq f(t_1)$. Let us have found $t_1, t_2, \ldots, t_n$ such that $t_{i-1} R t_i$ and $a \not\geq f(t_i)$ for $i \leq n$. Then, in particular, $t_n \notin \tau(T)$ and hence

$$a \not\geq f(t_n) = \bigvee\{f(t) \mid t_n R t\}.$$

Thus, there is a t_{n+1} such that $t_n R t_{n+1}$ and $a \not\geq f(t_{n+1})$. The sequence $(t_n)_{n=1,2,...}$ contradicts the definition of a semitree. ∎

3.11. THEOREM. *All hereditarily Lindelöf connected locales are p-connected.*

PROOF. Again, we will prove Ch(A). Let B be arbitrary, let

$$\bigvee\{a\oplus b \mid (a,b)\in m\} = 1(A\oplus B)$$

and let $a,b,c \neq 0$, $(a,b)\in m$, $(c,d)\in m$. We will prove that (a,b) and (c,b) are joined by a chain by finding a finite $M\subset m$ such that

$$b\wedge\bigwedge\{x_2 \mid x\in M\} \neq 0$$

and

$$\{x_1 \mid x\in M\}\cup\{a,c\} \text{ is chained.}$$

Let us take the semitree (T,R) from 3.3 again. Now, we will construct by induction a system of elements

$$x(i,j)\in A\oplus B,\ i\in\mathbb{N},\ j\in J_i\subset\mathbb{N}$$

such that, for $M_i=\{x(i,j) \mid j\in J_i\}$,

$$M_n\subset M_{n+1},\ b\wedge\bigwedge\{z_2 \mid z\in M_n\}\neq 0$$

and

$$\bigvee\{z_1 \mid z\in M_n\} = 1.$$

Put $J_0=0$ and $x(0,0)=1\oplus 1$. Let the $x(i,j)$ be determined for $i\leq n-1$. If $\tau(M_{n-1})\subset\tau(T)$ we put $J_n=J_{n-1}$ and $x(n,j)=x(n-1,j)$. Otherwise choose in $\tau(M_{n-1})\setminus\tau(T)$ one of the $x(i,j)$ with the least possible $i+j$. Denote this element simply by x.

If $x\in T_2$ we have $x={}_2\bigvee\{y \mid xRy\}$ and we can choose, by 3.6, a y such that

$$xRy \text{ and } 0\neq b\wedge\bigwedge\{z_2 \mid z\in M_{n-1}\}\wedge y_2.$$

Put $M_n=M_{n-1}\cup\{y\}$ and order this set into a sequence $x(n,0)$, $x(n,1),\ldots$.

If $x\in T_1$, $x={}_1\bigvee\{y \mid xRy\}$ and hence $x_1=\bigvee\{y_1 \mid xRy\}$ and, according to the Lindelöf property there is an at most countable $Q\subset xR$ such that $x_1=\bigvee\{y_1 \mid y\in Q\}$. Put $M_n=M_{n-1}\cup Q$ and order this set into a sequence $x(n,0)$, $x(n,1),\ldots$.

For

$$M' = \bigcup_{n=}^{\infty} M_n$$

we easily see that $\tau(M')\subset\tau(T)\subset m$. Since the $f\colon M'\to A$ defined by $f(x) = x_1$ satisfies the assumptions of 3.10, we must have

$$\bigvee\{x_1 \mid x \in \tau(M')\} = 1.$$

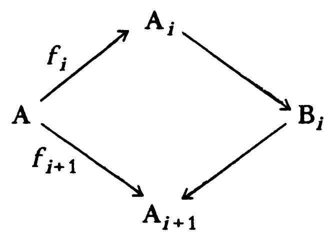
Hence, by 1.8.1 there is a (finite) $M \subset \tau(M')$ $(\subset m)$ such that

$$\{x_1 \mid x \in M\} \cup \{a, b\}$$

forms a chain. Since M is finite, it is contained in some of the M_n and hence

$$b \wedge \bigwedge\{x_2 \mid x \in M\} \neq 0. \ \blacksquare$$

3.12. REMARKS. Generalizing 1.8 we say that a system $\mathcal{F}$ of sublocales of A is *chained* if for any $f, g \in \mathcal{F}$, there are $f_1, f_2, \ldots, f_n$ in $\mathcal{F}$ $(f_i\colon A \to A_i)$ such that $f = f_1$, $g = f_i$ and for any $i = 1, \ldots, n-1$ there is a commutative diagram



with a non-trivial B_i.

A system $\mathcal{F}$ of sublocales of A is said to be *collectionwise dense* if

$$(\forall f \in \mathcal{F} \quad f(a) = 0) \Rightarrow a = 0.$$

By a result from [11], if $(f_i\colon A \to A_i)_{i \in J}$ is a collectionwise dense chained system and if all the A_i are p-connected then A is.

Now let $\mathcal{C}$ be a class of locales. Denote by $DC(\mathcal{C})$ the class of all the locales A for which there is a collectionwise dense chained system of connected sublocales $(f_i\colon A \to A_i)_{i \in J}$ with $A_i \in \mathcal{C}$.

In consequence of the above mentioned fact, if we already know that each connected locale from $\mathcal{C}$ is p-connected, we can conclude the same for all locales from $DC(\mathcal{C})$. Thus, e.g., if we denote by $\mathcal{C}_1$ the class of compact locales and by $\mathcal{C}_2$ that of hereditarily Lindelöf ones, we see that each locale from $DC(\mathcal{C}_1 \cup \mathcal{C}_2)$ is p-connected.

In particular, if a dense sublocale of A is covered by an expanding system of compact connected sublocales then A is p-connected.

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DEPARTMENT OF APPLIED MATHEMATICS
FACULTY OF MATHEMATICS AND PHYSICS
CHARLES UNIVERSITY
MALOSTRANSKÉ NÁM 25
11800 PRAHA 1. TCHÉCOSLOVAQUIE