CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

L. STRAMACCIA

Homotopy preserving functors

Cahiers de topologie et géométrie différentielle catégoriques, tome 29, n° 4 (1988), p. 287-296

<http://www.numdam.org/item?id=CTGDC_1988__29_4_287_0>

CAHIERS DE TOPOLOGIE VOL. XXIX-4 (1988)
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

HOMOTOPY PRESERVING FUNCTORS

by L. STRAMACCIA

RÉSUMÉ. On étudie dans cet article le comportement des épiréflecteurs et des pro-épiréflecteurs topologiques par rapport à l'homotopie. Des applications à la Théorie de la Forme sont données.

INTRODUCTION.

Let r : **TOP** → **TYCH** be the usual reflector from the category of topological spaces to its full subcategory of Tychonoff spaces. Morita has shown in ([9], Theorem 5.1) that every topological space X has the same shape as its reflection $r(X)$ in **TYCH**. It is worth noting that the same is not true with shape replaced by homotopy type (e.g., consider any countable set with cofinite topology; it cannot have the homotopy type of any Hausdorff space). Morita's Theorem depends essentially on the fact that r preserves products with the unit invterval I [12].

In this paper we extend Morita's result to every epireflector r: TOP → **R** such that

(i) I ∈ **R** and (ii) $r(X\times I) = r(X)\times I$, for every topological space X.

We show that, in case **R** is quotient reflective in **TOP**, then condition (ii) is automatically satisfied whenever (i) holds. We show furthermore that, in such a situation, the given epireflector induces a functor at the homotopical level, which is still a reflector.

In the second part of the paper we extend the results obtained to the case of a pro-reflector [6] p: **TOP** → **Pro-R** giving conditions in order that the category **Pro-Ho(R)** be reflective in **Pro-Ho(TOP)**, thus providing connections between non-homotopical shape theories and homotopical ones. Moreover we prove the analogous result concerning the categories π(**Pro-R**) and π(**Pro-TOP**) which are obtained by passing to homotopy classes of morphisms in **Pro-TOP** with respect to the cylinder functor (-)×I defined by extension on **Pro-TOP**.

Finally, we point out that all results above are still valid when **TOP** is replaced by any epireflective subcategory S of **TOP** itself.

----

*) This work was partially supported by funds (40%) of M.P.I., Italy.

1. In what follows **R** will denote a full epireflective subcategory of **TOP** with reflector r : **TOP** $\to$ **R**. Then, for every space X, there is an onto reflection map r_X: X $\to$ r(X) such that, for every continuous map f: X $\to$ R, R $\in$ **R**, there is a unique continuous map

$$f' : r(X) \to R \text{ with } f = f'.r_X.$$

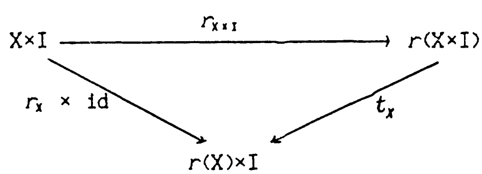
Let us assume that **R** contains the unit interval I. This is equivalent to say that **TYCH** $\subset$ **R**.

We refer to [7] for all that concerns the theory of reflections.

Let X be any topological space and consider the objects r(X×I) and r(X)×I of **R**. By the universal property of the reflection there exists a unique map

$$t_X: r(X\times I) \to r(X)\times I$$

which renders the following diagram commutative



We shall say that r preserves products with I, and write

$$r(X\times I) = r(X)\times I$$

to mean that the map t_X is a homeomorphism, for every space X.

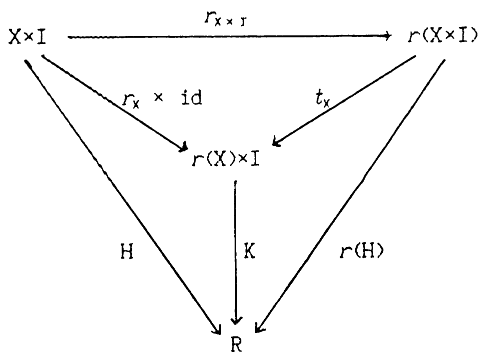
1.1. **DEFINITION.** We say that r preserves (resp. reflects) homotopies with respect to **R** if, given maps f,g: X $\to$ R, R $\in$ **R**. $f \simeq g$ implies $r(f) \simeq r(g)$ (resp. $r(f) \simeq r(g)$ implies $f \simeq g$).

1.2. **THEOREM.** *The following statements are equivalent:*

(i) r(X×I) = r(X)×I, *for every space* X.

(ii) r *preserves and reflects homotopies with respect to* **R**.

PROOF. The implication (i) $\Rightarrow$ (ii) is obvious. Assume that (ii) holds. Given a homotopy H: X×I $\to$ R, R $\in$ **R**, there is a commutative diagram



In fact, since r preserves and reflects homotopies with respect to **R**, if f,g: X → R, R ∈ **R**, are homotopic maps by means of H, then there is a homotopy K: r(X)×I → R between $r(f)$ and $r(g)$. The composition $K.(r_X\times id)$ is then a homotopy H' between f and g. We can suppose, without any restriction, to have taken H = H' from the beginning. Now, since r_X is onto, it follows that $r_X\times$id, and hence t_X, are onto maps. Finally, taking R = r(X×I), it is easily seen that t_X has a left inverse, so that it is a homeomorphism.

As an immediate consequence of the theorem one obtains:

1.3. COROLLARY. *If r:* **TOP** → **R** *satisfies either of the conditions (i), (ii) of the theorem, then* Ho(**R**) *is reflective in* Ho(**TOP**).

By the prefix "Ho" we denote the passage to the homotopy categories.

1.4. PROPOSITION. *Let* **R** *be a quotient reflective subcategory of* TOP. *Then, if* **R** *satisfies condition (i) of the theorem, it follows that* **R** *also satisfies (ii) and, moreover,* Ho(**R**) *is reflective in* Ho(**TOP**).

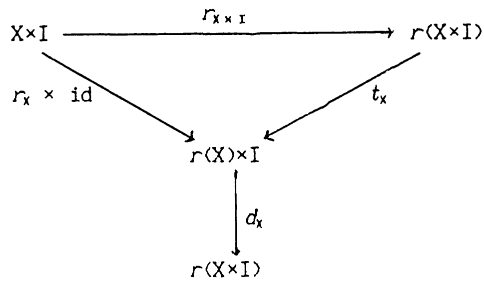
PROOF. For every space X the reflection map r_X: X → r(X) is a quotient map. By the Whitehead Theorem ([5], p. 200) it follows that $r_X\times$id: X×I → r(X)×I is also a quotient map. Let us define a function d_X: r(X)×I → r(X×I) by

$$d_X(\{x\},s) = r_{X\times I}(x,s) \quad \text{for every} \quad x \in X \text{ and } s \in I.$$

Since

$$d_X.(r_X\times id) = r_{X\times I},$$

it follows that d_X is continuous. From



one realizes that $d_X . t_X$ = id and, since t_X is already an onto map, it has to be a homeomorphism.

1.5. EXAMPLES.

(a) The following categories are all quotient reflective in **TOP**, hence they satisfy conditions (i) and (ii) of Theorem 1.2 and their homotopy categories are all reflective in Ho(**TOP**) :

TOP$_i$, i = 0,1,2,3.
URY, the category of Urysohn spaces.
FHAUS, the category of functionally Hausdorff spaces [2].
S(α), the category of S(α)-spaces, for every ordinal α [10].
HAUS(α), α an infinite cardinal, the category of spaces in which every subspace of cardinality α is Hausdorff [2].
HAUS(COMP), the category of spaces whose compact subsets are Hausdorff [2].
HAUS($\mathbf{N}_\infty$), $\mathbf{N}_\infty$ the Alexandroff compactification of **N**, the category of spaces in which every convergent sequence has a unique cluster point [2].

(b) As mentioned in the Introduction, it was proved in [12] that the category **TYCH** of Tychonoff spaces satisfies conditions (i) and (ii) of Theorem 1.2. Ho(**TYCH**) is reflective in Ho(**TOP**).

(c) Let **UNIF** and **CUNIF** be the categories of uniform, resp. complete uniform spaces. Let r : **UNIF** → **COUNIF** be the functor "completion with respect to the finest uniformity". r preserves and reflects products with I [12]. By techniques similar to those of Theorem 1.2, one shows that Ho(**COUNIF**) is reflective in Ho(**UNIF**).

1.6. THEOREM. *Let r be one of the epireflectors listed in the examples. Then, for every* X ∈ **TOP** (X ∈ **UNIF**), X *and* $r(X)$ *have the same shape.*

This theorem extends Theorem 5.1 of [9], which is concerned with the Tychonoff reflector. Morita's proof works as well since each of the categories considered in the examples contains that of ANR-spaces.

1.7. REMARK. After proving Proposition 1.4 we became acquainted with the paper of Schwarz [13], where he proved, in Theorem 3.5, a similar result. In fact, the unit interval I is a so called exponentiable object for the category **TOP**. However our proof is much more immediate and topological in nature, and it allows easily the generalization we have in mind (see the Remark at the end of the paper).

2. The Shape Theory of topological spaces is based on a property of the homotopy category Ho(**CW**) of spaces having the homotopy type of CW-complexes. Namely, Ho(**CW**) is pro-reflective (also called "dense" in [8, 15]) in Ho(**TOP**). The concept of pro-reflection is a weaker form of that of reflection and it allows one to define non-homotopical (also abstract) shape theories [6].

Let now **R** be a pro-epireflective subcategory of **TOP**, then there exists a pro-epireflector p: **TOP** → **Pro-R**, where **Pro-R** is the pro-category over **R**. As for notations, let us recall that p assigns to every topological space X an inverse system $p(X) = (X_a, p_{ab}, A)$ in **R**, and the pro-reflection map for X, denoted p_X: X → $p(X)$, is a natural cone

$$\{p_X{}^a : X \to X_a\}$$

with respect to the bonding morphisms p_{ab} of the system. Moreover, p_X is an epimorphism in **Pro-TOP**, and this means that, for every $a \in A$, there is an index $b \geqslant a$ such that $p_X{}^b : X \to X_b$ is onto [14].

For all matters concerning Shape Theory and pro-categories we refer to the book of S. Mardešić and J. Segal [8], see also [6].

Let us recall also that a morphism f: **X** → Y in **Pro-TOP** is an equivalence class of continuous maps from some X_a, $a \in A$, to Y. f_a: $X_a \to Y$ and f_b: $X_b \to Y$ both represent f if and only if there is a

$$c \in A, \ c \geq a,b, \quad \text{such that} \quad p_c \cdot p_{ac} = p_c \cdot p_{bc} .$$

The usual homotopy functor **TOP** $\to$ Ho(**TOP**) extends in a natural way to a functor from **Pro-TOP** to **Pro-Ho(TOP)** (just replace every continuous map involved by its homotopy class). It follows that two morphisms f,g: X $\to$ Y in TOP give rise to the same morphism from $\mathbf{X}_h$ = $(X_a,[p_{ab}],A)$ to Y, that is $[f] = [g]$, if and only if there is an index $a \in A$ such that f_a, g_a: $X_a \to Y$ are homotopic, say, by means of a homotopy $H^a : X^a \times I \to Y$. This last then defines a homotopy **H**: **X**×I $\to$ Y.

We note that, if $\mathbf{X} = (X_a, p_{ab}, A)$, then **X**×I is the inverse system

$$(X_a \times I, p'_{ab}, A), \text{ where } p'_{ab} = p_{ab} \times \mathrm{id}: X_b \times I \to X_a \times I.$$

The following theorem extends the main result of the first section to the case of a pro-epireflector.

2.1. THEOREM. *Let p:* **TOP** $\to$ **Pro-R** *be a pro-epireflector,* I $\in$ **R**. *The following statements are equivalent:*

(i) $p(X\times I) = p(X)\times I$, *for every space* X.

(ii) Given f,g: X $\to$ R, R $\in$ **R**, *then*

$$f \simeq g \text{ if and only if } [p(f)] = [p(g)].$$

PROOF. The proof is quite similar to that of Theorem 1.2. Only part (ii) $\Rightarrow$ (i) needs some explanation. Call t_X: $p(X\times I) \to p(X)\times I$ the unique morphism in **Pro-R** such that

$$t_X \cdot p_{X\times I} = p_X \times \mathrm{id}.$$

Since **R** is pro-reflective, for every $a \in A$, there is a $b \geq a$ such that $p_X{}^b \times \mathrm{id}$: $X\times I \to X_b \times I$ is onto. Hence the corresponding $t_X{}^b$: $p(X\times I) \to X_b\times I$ is epi in **Pro-R**. Finally, by ([14], Prop. 3.2), t_X is epi.

2.2. COROLLARY. *Let p:* **TOP** $\to$ **Pro-R** *be a pro-epireflector,* I $\in$ **R**. *If either of the conditions (i), (ii) of the theorem is satisfied, then* Ho(**R**) *is pro-epireflective in* Ho(**TOP**).

2.3. EXAMPLES.

(a) Let **R** be one of the following subcategories of **TOP** : (pseudo-)metrizable spaces, first countable spaces,
second countable spaces, separable spaces.

Every such **R** is pro-bireflective [6] in **TOP**. If (X,τ) is any space, its pro-bireflection, $p(X)$ is given by the inverse system $((X,\tau_a),p_{ab},A)$, where, for every $a \in A$, $X_a = X$ as sets, while $\tau_a \leq \tau$, and $(X_a,\tau_a) \in$ **R**; each p_{ab} is the identity on the underlying sets.

Note that $I \in$ **R**, hence $p(I)$ is isomorphic to I in **Pro-R**. From this it follows at once that $p(X\times I) = p(X)\times I$. Then **R** satisfies conditions (i) and (ii) of Theorem 2.1 and Ho(**R**) is pro-reflective in Ho(**TOP**).

(b) Let **R** be as above and let

$$T_0\mathbf{R} = \{X \in \mathbf{TOP} \mid \text{the } T_0\text{-identification of } X \text{ belongs to } \mathbf{R}\}.$$

Then T_0**R** is pro-epireflective in **TOP** ; the pro-epireflection is obtained by composing T_0: **TOP** → $\mathbf{TOP}_0$ with the previous pro-bireflection. By Theorem 2.1 Ho(T_0**R**) is pro-reflective in Ho(**TOP**).

Theorem 2.1 and Corollary 2.2 have indeed an autonomous interest, also they reproduce, for a number of subcategories of **TOP**, the situation one has with the categories Ho(**CW**) and Ho(**TOP**), as recalled at the beginning of the section.

It is known, however, that **Pro-Ho(TOP)** cannot be considered as the homotopy category of **Pro-TOP**. Edwards-Hastings [4] and Porter [11] have described a closed model structure on **Pro-TOP** and have obtained the right homotopy category Ho(**Pro-TOP**), by formally inverting levelwise homotopy equivalences.

Our methods here do not allow us to attach Ho(**Pro-TOP**) directly, but do give information on the related category π(**Pro-TOP**) obtained by passing to homotopy classes of morphisms in **Pro-TOP** with respect to the extended cylinder functor

$$(\ )\times I\colon \mathbf{Pro\text{-}TOP} \to \mathbf{Pro\text{-}TOP}.$$

$f,g \in$ **Pro-TOP**(**X**,**Y**) are homotopic if there is a "homotopy" $H\colon X\times I \to Y$ connecting f and g.

2.4. THEOREM. *Let* R *be a pro-epireflective subcategory of* TOP *with pro-reflector* p: TOP → Pro-R *such that* $p(X\times I) = (X)\times I$, *for every space* X. *Then* π(**Pro-R**) *is reflective in* π(**Pro-TOP**).

PROOF. One has only to recall that (cf. [15]), if **R** is pro-reflective in **TOP** by means of p: **TOP** → **Pro-R**, then the functor

$$p^{*} = \text{invlim. pro-}p\colon \mathbf{Pro\text{-}TOP} \to \mathbf{R}$$

is left adjoint to the embedding **Pro-R** $\subset$ **Pro-TOP** ; in other words, **Pro-R** is reflective in **Pro-TOP**. Since $p^*(X\times I) = p^*(X)\times I$, for every pro-space **X**, then it is clear that one can adapt as well the arguments of Theorem 2.1 to obtain the assertion.

Let us recall briefly the construction of Ho(**Pro-TOP**) as illustrated in [1]. A morphism $i : \mathbf{A} \to \mathbf{X}$ is a trivial cofibration whenever it has the left lifting property with respect to every (Hurewicz) fibration $p: E \to B$ of topological spaces.

A pro-space $Z \in$ **Pro-TOP** is fibrant if, given any trivial cofibration $i : \mathbf{A} \to \mathbf{X}$ and any morphism $f: \mathbf{A} \to Z$, there is an extension

$$f^*: \mathbf{X} \to Z \quad \text{such that} \quad f^*.i = f.$$

If π(**Pro-TOP**)$_f$ denotes the full subcategory of π(**Pro-TOP**) whose objects are all fibrant pro-spaces, then there is a reflector

$$F: \pi(\mathbf{Pro\text{-}TOP}) \to \pi(\mathbf{Pro\text{-}TOP})_f$$

with a trivial cofibration $i_x: \mathbf{X} \to \hat{\mathbf{X}}$ as reflection morphism.

The category Ho(**Pro-TOP**) has the same objects as **Pro-TOP** while morphisms can be defined by means of the bijection

$$\mathrm{Ho}(\mathbf{Pro\text{-}TOP})(\mathbf{X},\mathbf{Y}) \cong [\mathbf{X},\hat{\mathbf{Y}}]$$

induced by composition with $[i_x]$.

From Theorem 2.4 one easily obtains the following

2.5. THEOREM. *Assume the hypothesis of Theorem 2.4. Moreover, let* p^* *take fibrant pro-spaces to fibrant pro-spaces. Then* Ho(**Pro-R**) *is reflective in* Ho(**Pro-TOP**).

A strong version of Shape Theory is based on the introduction of the category Ho(**Pro-TOP**). In [1] Cathey and Segal have shown that every topological space admits a "reflection" in Ho(**Pro-ANR**), that is, there exists a pro-reflector, at the homotopical level

$$r : \mathrm{Ho}(\mathrm{TOP}) \to \mathrm{Ho}(\mathbf{Pro\text{-}ANR}).$$

Using the homotopy inverse limit functor

$$\text{holim: Ho(Pro-TOP)} \to \text{Ho(TOP)}$$

as defined in [4] and [11], one can show that the composite

$$r.\text{holim: Ho(Pro-TOP)} \to \text{Ho(Pro-ANR)}$$

is a reflector. This shows that the converse of Theorem 2.4 is not true in general.

We conclude with the following

2.6. REMARK. In this section and in the first one, we have assumed that **R** was a pro-epireflective, resp. epi-reflective, subcategory of **TOP**. We point out that all results are also true if we replace **TOP** by any subcategory **S** which is epireflective in **TOP**. The point is that epimorphisms in **TOP** coincide with the onto maps; hence the reflection map r_X: $X \to r(X)$ of every space X is onto, so that $r_X \times \text{id}$: $X \times I \to r(X) \times I$ is also onto. Similarly in the case of a pro-epireflection morphism. See the proofs of Theorems 1.2 and 2.1.

In [2] the epimorphisms of any subcategory **S** of **TOP** were characterized. Namely, $f \in S(X,Y)$ is epi in **S** if and only if the map f has dense range in Y with respect to **S**-closure. The **S**-closure of a subset $N \subset Y$ is the least regular subobject [7] of Y containing N. Recalling that the product of two regular monomorphisms is again regular, it follows at once that if f: $X \to Y$ is epi in **S**, then $f \times \text{id}$: $X \times Z \to Y \times Z$ is also epi for every Z in **S**.

ACKNOWLEDGMENT. I wish to express my gratitude to Tim Porter for his very valuable suggestions about the arrangement of the paper.

REFERENCES.

1. F. CATHEY & J. SEGAL, Strong Shape Theory and resolutions, *Top. Appl.* 15 (1983), 119-130.
2. D. DIKRANJAN & E. GIULI, Epimorphisms and cowellpoweredness of epireflective subcategories of TOP, *Rend. Circ. Mat. Palermo*, Suppl. 6 (1984), 121-136.
3. D. DIKRANJAN & E. GIULI, Closure operators induced by topological epireflections, *Coll. Math. Soc. J. Bolyai*, Eger 1983 (to appear).
4. D.A. EDWARDS & H.M. HASTINGS, Čech and Steenrod homotopy theories with applications to Geometric Topology, *Lecture Notes in Math.* 542, Springer (1976).
5. R. ENGELKING, *General Topology, Monogr. Math.* 60, Warsawa, 1977.
6. E. GIULI, Pro-reflective subcategories, *J. Pure Appl. Alg.* 33 (1984), 19-29.
7. H. HERRLICH & G. STRECKER, *Category Theory*, Heldermann, Berlin 1979.
8. S. MARDEŠIĆ & J. SEGAL, *Shape Theory*, North Holland, 1982.
9. K. MORITA, On shapes of topological spaces, *Fund. Math.* 86 (1975), 251-259.
10. J.R. PORTER & C. VOTAW, S(α)-spaces and regular Hausdorff extensions, *Pacific J. Math.* 45 (1973), 327-375.
11. T. PORTER & J.-M. CORDIER, Homotopy limits and homotopy coherence, Mimeographed Notes, Perugia, 1984.
12. R. PUPIER, La complétion universelle d'un produit d'espaces complètement réguliers, *Publ. Dept. Math. Lyon* 6-2 (1969).
13. F. SCHWARZ, Product compatible reflectors and exponentiability, *Proc. Int. Conf. Categorical Topology*, Toledo 1983, Heldermann (1984), 505-522.
14. L. STRAMACCIA, Monomorphisms and epimorphisms of inverse systems, *Comm. Math. Univ. Carolinae* 24 (1983), 495-505.
15. L. STRAMACCIA, Reflective subcategories and dense subcategories, *Rend. Sem. Mat. Univ. Padova* 67 (1982), 181-198.

Dipartimento di Matematica
Università di Perugia
Via Pascoli
06100 PERUGIA, ITALY