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FRAMES IN PRIESTLEY'S DUALITY

by A. PULTR and J. SICHLER *)

Dedicated to Evelyn NELSON

RÉSUMÉ. La dualité de Priestley établit une correspondance biunivoque naturelle entre les (0,1)-treillis distributifs et les espaces compacts ordonnés, et on connaît des caractérisations de différentes sous-classes de treillis en termes des espaces correspondants. Dans cet article, on caractérise ainsi les classes de 'frames', de 'frames' réguliers et de 'frames' compacts.

Celebrated Priestley's articles [3] and [5] established a natural equivalence, now commonly referred to as the Priestley's duality, of a category of ordered compact spaces to the dual of the category of distributive (0,1)-lattices. Numerous results characterizing subclasses of distributive (0,1)-lattices in terms of properties of their corresponding spaces are collected in the survey work [7] by H.A. Priestley.

In this note, we characterize frames (complete $\vee$-completely distributive lattices) through a suitable extremal disconnectedness property of their Priestley spaces, and describe the spaces of regular and of compact frames.

The category of frames is equivalent to the dual of the category of locales (often thought of as "generalized spaces"). Our characterization includes also frame homomorphisms and shows that the category of these "generalized spaces" is equivalent to a category of ordered compact topological spaces.

In the interest of completeness and clarity, we present also proofs of several well-known statements needed along the way.

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1. PRELIMINARIES.

1.1. Conventions and notations.

1.1.1. For any subset M of a poset $(X, \leq)$ denote

$$(M] = \{x \in X \mid \exists\ m \in M\ x \leq m\} \text{ and } [M) = \{x \in X \mid \exists\ m \in M\ m \leq x\}.$$

The set M is *decreasing* (resp. *increasing*) if $(M] = M$ (resp. $[M) = M$).

Let $(X,\tau,\leq)$ be an ordered topological space, that is, let $(X,\leq)$ be a poset and let τ be a topology on X presented as the collection of its open sets. The system of all decreasing (resp. increasing) members of τ will be denoted by $\downarrow\tau$ (resp. $\uparrow\tau$); obviously $\downarrow\tau$ and $\uparrow\tau$ are also topologies on X. The closure operator of the topologies τ, $\downarrow\tau$, $\uparrow\tau$ will be respectively denoted as cl, $\downarrow$cl and $\uparrow$cl.

Since $\uparrow\tau$-closed sets are exactly all τ-closed decreasing sets, for any subset M of X the set $\uparrow\mathrm{cl}(M)$ is easily seen to be the least τ-closed decreasing set containing M.

The system of all decreasing (resp. increasing) τ-clopen sets will be denoted by $\downarrow\mathrm{CO}\tau$ (resp. $\uparrow\mathrm{Co}\tau$). Thus members of $\downarrow\mathrm{CO}\tau$ are subsets of X which are simulaneously $\downarrow\tau$-open and $\uparrow\tau$-closed.

1.1.2. In what follows, clopen sets will always be denoted by lower-case initial letters *a,b,c* ; upper-case letters such as U,V,Y,Z stand for general subsets of the space in question, and *x,y,z* denote its elements.

1.1.3. An ordered topological space $(X,\tau,\leq)$ is said to be *monotonically separated* if for any $x \leq y$ there exist disjoint $U \in \uparrow\tau$ and $V \in \downarrow\tau$ such that $x \in U$ and $y \in V$. The space $(X,\tau,\leq)$ is *totally order disconnected* if for any $x \leq y$ there exists an $a \in \mathrm{CO}\downarrow\tau$ such that $x \in X\setminus a$ and $y \in a$.

Clearly, every totally order disconnected space is monotonically separated.

1.1.4. A *Priestley space* is an ordered totally order disconnected compact space.

1.2. **PROPOSITION** (Priestley [3, 4]). *Let* $(X,\tau,\leq)$ *be a compact monotonically separated space. Then* $\downarrow\tau\cup\uparrow\tau$ *is a subbasis of* τ.

PROOF. For $T \in \tau$ and $t \in T$, set

$$A = \{x \in X\backslash T \mid x \nleq t\} \quad \text{and} \quad B = \{x \in X\backslash T \mid x \ngeq t\}.$$

Clearly $X\backslash T = A \cup B$. For $x \in A$ (resp. $x \in B$) select $U(t,x) \in \downarrow\tau$ (resp. $U(t,x) \in \uparrow\tau$) and $V(x,t) \in \downarrow\tau$ (resp. $V(x,t) \in \downarrow\tau$) so that

$$t \in U(t,x), \quad x \in V(x,t) \quad \text{and} \quad U(t,x) \cap V(t,x) = \emptyset.$$

We have thus obtained a cover $\{V(x,t) \mid x \in X\backslash T\}$ of the compact set $X\backslash T$; hence there are finite $A' \subset A$ and $B' \subset B$ such that $\{V(x,t) \mid t \in A' \cup B'\}$ covers $X\backslash T$. Therefore

$$t \in \bigcap\{U(t,x) \mid x \in A'\} \cap \bigcap\{U(t,x) \mid x \in B'\} \subset T,$$

where $\bigcap\{U(t,x) \mid x \in A'\} \in \downarrow\tau$ and $\bigcap\{U(t,x) \mid x \in B'\} \in \uparrow\tau$ as required. ∎

1.3. PROPOSITION (Priestley [3,4]). *If* $(X,\tau,\leq)$ *is a Priestley space then* $\downarrow CO\tau$ *is a basis for* $\downarrow\tau$, *and* $\uparrow CO\tau$ *is a basis for* $\uparrow\tau$. *Consequently,* $\downarrow CO\tau \cup \uparrow CO\tau$ *is a subbasis of* τ.

PROOF. If $U \in \downarrow\tau$ and $u \in U$, then $x \nleq u$ for every $x \in X\backslash U$; hence there exists $b(x) \in \uparrow CO\tau$ such that $x \in b(x)$ and $u \in X\backslash b(x)$. Thus $\{b(x) \mid x \in X\backslash U\}$ is a cover of the compact set $X\backslash U$; hence $X\backslash U$ is contained in the τ-clopen increasing set $b = \bigcup\{b(x) \mid x \in F\}$ for some finite $F \subset X\backslash U$. Therefore $u \subset X\backslash b \in \downarrow CO\tau \subset U$. A similar argument shows that $\uparrow CO\tau$ is a basis of $\uparrow\tau$, and the final statement follows easily from 1.2. ∎

1.4. Priestley's duality. The symbol <u>PSp</u> will denote the category of all Priestley spaces and all their order preserving continuous mappings, and <u>DLat</u> will stand for the category of all distributive (0,1)-lattices and all their (0,1)-preserving homomorphisms.

Priestley's duality consists of contravariant functors

$$P: \underline{DLat} \to \underline{PSp} \quad \text{and} \quad D: \underline{PSp} \to \underline{DLat}$$

defined as follows:

For any distributive (0,1)-lattice L, the space $P(L) = (\underline{2}^L,\tau,\leq)$ is formed by all (0,1)-homomorphisms $\alpha: L \to \underline{2}$ onto the two-element lattice $\underline{2}$; the partial order $\leq$ of $P(L)$ is the pointwise order of these morphisms, and its topology τ is inherited from the topology of the product space 2^L with discrete $2 = \{0,1\}$. If φ is a morphism of <u>DLat</u> then $P(\varphi)(\alpha) = \alpha\circ\varphi$.

The complementary functor D is given by

$$D(X,\tau,\leqslant) = (\downarrow CO\tau,\subset) \quad \text{and} \quad D(f) = f^{-1}.$$

THEOREM (Priestley [3,5]). *The composite functors* P∘D *and* D∘P *are naturally equivalent to the respective identity functors of their domains.*

1.5. THEOREM (Priestley [5]). *Let* $L = D(X,\tau,\leqslant)$. *If* $A \subset L$ *then* $\sup(A)$ *exists in* L *if and only if* $\uparrow cl(\cup A)$ *is open in* $(X,\tau,\leqslant)$*; if this is the case, then* $\sup A = \uparrow cl(\cup A)$.

PROOF. If $b = \sup(A)$ in L, then $a \subset b$ holds in $(X,\tau,\leqslant)$ for all $a \in A$ and hence $\cup A \subset b$. The τ-clopen decreasing set b is $\uparrow\tau$-closed, so that $\uparrow cl(\cup A) \subset b$. Now suppose $x \in b$; if $x \in c \in \uparrow CO\tau$ then $b \cap (X\setminus c) \in \downarrow CO\tau$ represents an element of L strictly smaller than $b = \sup(A)$. Thus $X\setminus c$ does not contain $\cup A$, that is, $c \cap \cup A \neq \emptyset$. By 1.3, $x \in \uparrow cl(UA)$.

Since any $\uparrow\tau$-closed set is τ-closed and decreasing, if open, the set $\uparrow cl(\cup A)$ represents an element b of L. If $c \in L$ is an upper bound of $A \subset L$, then $c \supset \cup A$ in $(X,\tau,\leqslant)$, and is τ-closed and decreasing. Thus $c \supset \uparrow cl(\cup A) = b$, and $b = \sup(A)$ follows. ∎

2. FRAMES.

2.1. Recall that a *frame* is a complete lattice L satisfying the join distributive law

$$a \wedge (\vee B) = \vee\{a \wedge b \mid b \in B\} \quad \text{for all} \quad a \in L \text{ and } B \subset L.$$

A *frame homomorphism* φ is any morphism from DLat preserving arbitrary joins, that is, such that

$$\varphi(\vee B) = \vee\{\varphi(b) \mid b \in B\} \qquad \text{for any} \quad B \subset L.$$

The category of frames and all frame homomorphism will be denoted by Frm, and its dual, called the category of *locales*, by Loc.

There is a natural functor Ω defined by

$$\Omega(X,\tau) = (\tau, \subset) \quad \text{and} \quad \Omega(f) = f^{-1}$$

that assigns locales to topological spaces. This functor restricts to a full embedding of the subcategory of sober spaces into the category of locales; hence locales can be viewed as a generalization of sober spaces - see Johnstone [2].

2.2. LEMMA. Let $L = D(X,\tau,\leq)$ be a complete distributive lattice. Then the following statements are equivalent:

(1) L is a frame,

(2) $\uparrow cl(U\cap a) = \uparrow cl(U)\cap a$ for all $U \in \downarrow\tau$ and $a \in \downarrow CO\tau$,

(3) $\uparrow cl(U\cap V) = \uparrow cl(U)\cap \uparrow cl(V)$ for all $U,V \in \downarrow\tau$,

(4) $\uparrow cl(U) = cl(U)$ for all $U \in \downarrow\tau$.

REMARK. Thus, in particular, for any frame $L = D(X,\tau,\leq)$ we have $cl(U\cap V) = cl(U)\cap cl(V)$ for all $U,V \in \downarrow\tau$.

PROOF of 2.2. The equivalence of (1) and (2) follows immediately by 1.5, and the implication (3) ⇒ (2) is trivial.

(2) ⇒ (3): If $V \in \downarrow\tau$ then $V = \cup A$ for some $A \subset \downarrow CO\tau$ by 1.3. We have

$$\uparrow cl(U)\cap V = \uparrow cl(U)\cap \cup A = \cup(\uparrow cl(U)\cap a \mid a \in A) =$$
$$\cup(\uparrow cl(U\cap a) \mid a \in A) \subset \uparrow cl(\cup(U\cap a \mid a \in A) = \uparrow cl(U\cap V).$$

By symmetry, $U\cap \uparrow cl(V) \subset \uparrow cl(U\cap V)$; since $\uparrow cl(U) \in \downarrow\tau$ by 1.5,

$$\uparrow cl(U)\cap \uparrow cl(V) \subset \uparrow cl(U\cap V).$$

Therefore

$$\uparrow cl(U)\cap \uparrow cl(V) \subset \uparrow cl(\uparrow cl(U)\cap V) \subset \uparrow cl(\uparrow cl(U\cap V)) =$$
$$\uparrow cl(U\cap V) \subset \uparrow cl(U)\cap \uparrow cl(V).$$

(2) ⇒ (4): Clearly $cl(U) \subset \uparrow cl(U)$. Given $x \in \uparrow cl(U)$, apply 1.3 to obtain $a \in \downarrow CO\tau$ and $b \in \uparrow CO\tau$ such that $x \in a\cap b$. But then

$$x \in \uparrow cl(U)\cap a = \uparrow cl(U\cap a)$$

by (2), and hence $U\cap(a\cap b) = (U\cap a)\cap b \neq \emptyset$. Thus $x \in cl(U)$.

(4) ⇒ (2): Obviously $\uparrow cl(U\cap a) \subset \uparrow cl(U)\cap a$. For any $x \in \uparrow cl(U)\cap a = cl(U)\cap a$, let $b \in \uparrow CO\tau$ contain x ; thus $x \in cl(U)$ and $x \in a\cap b$. Hence

$$(U\cap a)\cap b = U\cap(a\cap b) \neq \emptyset,$$

so that $x \in \uparrow cl(U\cap a)$. ∎

2.3. THEOREM. *Let* $L = D(X,\tau,\leq)$. *Then* L *is a frame if and only if*

$\mathrm{cl}(U) \in \downarrow\tau$ *for every* $U \in \downarrow\tau$.

PROOF. If L is a frame and $U \in \downarrow\tau$ then $\mathrm{cl}(U) = \uparrow\mathrm{vl}(U)$ is open and decreasing by 2.2 and 1.5.

Conversely, let $U \in \downarrow\tau$. Clearly $\mathrm{cl}(U) \subset \uparrow\mathrm{cl}(U)$. On the other hand, $\uparrow\mathrm{cl}(U)$ is the least decreasing closed set containing $U \in \downarrow\tau$; by the hypothesis, $\mathrm{cl}(U) \supset U$ is decreasing, and hence $\mathrm{cl}(U) \supset \uparrow\mathrm{cl}(U)$. Thus $\uparrow\mathrm{cl}(U) = \mathrm{cl}(U)$ is an open set, so that L is complete by 1.5 and, by 2.2, L is a frame. ∎

2.4. A Priestley space $(X,\tau,\leq)$ is called an *f-space* if its closure operator preserves the topology $\downarrow\tau$ (as described by 2.3 above). A continuous order preserving mapping g is an *f-map* if

$$g^{-1}(\mathrm{cl}(U)) = \mathrm{cl}(g^{-1}(U)) \quad \text{for all } U \in \downarrow\tau.$$

2.5. Recalling the form of suprema given by 1.5, we see that a lattice (0,1)-homomorphism $D(g)$ is V-complete if and only if:

$$\text{(a)}\quad g^{-1}(\uparrow\mathrm{cl}(\cup A)) = \uparrow\mathrm{cl}\cup\{\cup g^{-1}(a) \mid a \in A\} \quad \text{for all } A \subset \downarrow\mathrm{CO}\tau.$$

Since every $U \in \downarrow\tau$ has the form $U = \cup A$ for some $A \subset \downarrow\mathrm{CO}\tau$, and because

$$g^{-1}(\cup A) = \cup\{g^{-1}(a) \mid a \in A\},$$

this condition can be rewritten into

$$\text{(b)}\qquad g^{-1}(\uparrow\mathrm{cl}(U)) = \uparrow\mathrm{cl}(g^{-1}(U)) \quad \text{for all } U \in \downarrow\tau.$$

Finally, if the domain and the codomain of g are f-spaces then (b) becomes

$$(*)\qquad g^{-1}(\mathrm{cl}(U)) = \mathrm{cl}(g^{-1}(U)) \quad \text{for all } U \in \downarrow\tau,$$

which characterizes an f-map defined by 2.4. Altogether, we obtain the following result.

COROLLARY. *Priestley's duality induces a natural equivalence of the category* Loc *and the category* F *of all f-spaces and f-maps.*

3. REGULAR AND COMPACT FRAMES.

3.1. Let L be a distributive (0,1)-lattice and let $a \in L$. A *pseudocomplement* $a^* \in L$ of a is defined by the requirement that, for every $b \in L$, $b \leq a^*$ hold if and only if $b \wedge a = 0$. It is easy to see that the pseudocomplement a^* of $a \in L$ is uniquely determined, and that $a^* = \bigvee\{c \in L \mid c \wedge a = 0\}$ in any frame L.

3.2. LEMMA (Priestley [6]). If $a \in L = D(X,\tau,\leq)$ has a pseudocomplement a^* then $a^* = X\backslash[a)$. Conversely, if $[a)$ is open then $X\backslash[a)$ is the pseudocomplement a^* of a. Consequently, if L is a frame, then $[a) \in \uparrow CO\tau$ for every $a \in \downarrow CO\tau$. ∎

3.3. Recall that a frame L is called *regular* if, for every $a \in L$, $a = \bigvee\{b \in L \mid a \vee b^* = 1\}$.

Observe that, by 3.2, $a \vee b^* = 1$ holds in L if and only if $a \cup (X\backslash[b)) = X$, that is, if and only if $[b) \subset a$.

3.4. THEOREM. *The following statements are equivalent for any frame* $L = D(X,\tau,\leq)$:

(1) L *is regular,*

(2) *for every* $a \in \downarrow CO\tau$ *there are* $U \in \downarrow\tau$ *and* $V \in \uparrow\tau$ *such that* $U \subset V$ *and* $\uparrow cl(U) = \uparrow cl(V) = a$,

(3) *for every* $a \in \downarrow CO\tau$ *there are* $U \in \downarrow\tau$ *and* $V \in \uparrow\tau$ *such that* $U \subset V$ *and* $cl(U) = cl(V) = a$.

PROOF. (1) ⇒ (2): We have

$$a = \bigvee\{b \mid a \vee b^* = 1\} = \uparrow cl(\bigcup\{b \mid [b) \subset a\}) = \uparrow cl(\bigcup\{[b) \mid [b) \subset a\}).$$

Set

$$U = \bigcup\{b \mid [b) \subset a\} \quad \text{and} \quad V = \bigcup\{[b) \mid [b) \subset a\}.$$

(2) ⇒ (1): If $U \subset V$ are open sets as in (2), then

$$U = \bigcup\{b \in \downarrow CO\tau \mid b \subset U\} \text{ by 1.3,}$$

and

$$a = \uparrow cl(U) = \bigvee\{b \in \downarrow CO\tau \mid b \subset U\}.$$

Since $U \subset V$, any $b \in \downarrow CO\tau$ contained in U is also a subset of the increasing open set V, and hence $[b) \subset V \subset a$; thus $a \vee b^* = 1$ by 3.3.

(2) ⇒ (3): By (2) and 2.2 we have

$$a = \uparrow cl(V) = \uparrow cl(U) = cl(U) ;$$

hence $V \subset cl(U)$. Thus $cl(V) \subset cl(U)$, and $cl(U) \subset cl(V)$ follows from $U \subset V$.

(3) ⇒ (2): Again we have $cl(U) = \uparrow cl(U)$. From

$$V \subset cl(V) = cl(U) = \uparrow cl(U)$$

it follows that $\uparrow cl(V) \subset \uparrow cl(U)$, while the converse inclusion is obtained from $U \subset V$. ∎

3.5. A frame L is *compact* if its unit 1 is compact, that is, if $1 = \vee A$ for $A \subset L$ only when $1 = \vee A'$ for some finite $A' \subset A$.

THEOREM. *A frame* $L = D(X,\tau,\leq)$ *is compact if and only if* X *is the only member of* $\downarrow\tau$ *dense in* X.

PROOF. Let L be compact and let $cl(U) = X$ for some $U \in \downarrow\tau$. Then U is the union of all its clopen decreasing subsets and

$$1 = X = cl(U) = \vee A, \quad \text{where} \quad A = \{a \in \downarrow CO\tau \mid a \subset U\}.$$

By the hypothesis, $1 = \vee A'$ for a finite set $A' \subset A$; since all members of A' are clopen, $X = \vee A' = \cup A'$ is contained in U.

Conversely, let $1 = X = \vee A = cl(\cup A)$ for some $A \subset \downarrow CO\tau$. Thus $\cup A$ is an open decreasing set dense in X, and $\cup A = X$ follows. Since (X,τ) is compact, there is a finite $A' \subset A$ such that $1 = X = \cup A' = \vee A'$. ∎

3.6. If a,b are elements of a complete lattice L, then *a is way below b*, written $a \ll b$, if

$$b \leq \vee C \quad \text{only when} \quad a \leq \vee C' \text{ for some finite } C' \subset C.$$

This relation plays a fundamental role in the theory of continuous lattices; the reader is referred to [8] or [1] for more details. Our final illustration of Priestley's duality interprets this relation for f-spaces.

PROPOSITION. *If* a,b *are elements of a frame* $L = D(X,\tau,\leq)$, *then* $a \ll b$ *in* L *if and only if*

(+) *for every* $U \in \downarrow\tau$, $b \subset cl(U)$ *only when* $a \subset U$.

PROOF. Let $a \ll b$ and $b \subset \mathrm{cl}(U)$. As before,

$$U = \cup C \text{ with } C = \{c \in \downarrow CO\tau \mid c \subset U\},$$

and hence $b \leqslant \vee C$. From $a \ll b$ we obtain the existence of a finite $C' \subset C$ for which $a \leqslant \vee C'$ in L: hence $a \leqslant \cup C'$ by the finiteness of C', and $a \subset U$ follows from $\cup C' \subset U$.

Conversely, let the condition be satisfied, and let $b \leqslant \vee C$ for some $C \subset L$. Set $U = \cup C$. Then $b \subset \mathrm{cl}(U)$, and $a \subset U$ follows by the hypothesis. Since C is an open cover of the compact set $a \subset X$, we have $a \subset \cup C' = \vee C'$ for some finite $C' \subset C$. ∎

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