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ESSENTIALLY EQUATIONAL CATEGORIES

by J. ADÁMEK, H. HERRLICH and J. ROSICKÝ

*Dedicated to the memory of  
our friend Evelyn NELSON*

**RÉSUMÉ.** Suivant une idée de P. Freyd, on considère des catégories essentiellement algébriques d'algèbres partielles. Ce sont les algèbres partielles telles que le domaine de chaque opération soit déterminé par des équations utilisant les opérations précédentes. Ce concept est assez général pour englober beaucoup d'exemples importants, en particulier toutes les constructions essentiellement monadiques (i.e., les catégories concrètes sur *Set* dont le foncteur d'oubli est le composé d'une chaîne (éventuellement infinie) de foncteurs monadiques). Par ailleurs, les catégories essentiellement équationnelles ont de bien meilleures propriétés que les catégories quelconques d'algèbres partielles; ainsi, en supposant qu'il y ait génération bornée, on montre que les concepts: essentiellement équationnel, essentiellement monadique et essentiellement algébrique (au sens du second auteur) coïncident, et ceci implique la complétude, la cocomplétude, la solidité, etc., des constructions obtenues.

1. ESSENTIALLY EQUATIONAL CONSTRUCTS.

P. Freyd has remarked in [Fr] that a nice concept of equational theory for partial algebras is obtained by requiring that the domain of definition of each operation be described by equations involving the preceding operations. Example: the category of small categories is given by two unary operations, domain  $d$  and codomain  $c$ , and a partial binary operation  $x.y$ , defined iff  $c(y) = d(x)$ . We are going to formalize the concept of such equational theories of a possibly large type, and to study the structure of the resulting *constructs* (= concrete categories\*) over *Set*, see [A<sub>1</sub>]. P. Freyd used the name essentially algebraic for his concept. We reserve that name for the case where an additional smallness condition (bounded generation) is imposed, see the

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\*) A concrete category over  $\underline{X}$  is a category equipped with a faithful functor to  $\underline{X}$ .



**DEFINITION.** An  $(\Omega, E, \text{Def})$ -algebra is a set  $A$  together with partial operations  $\omega_\alpha: A^n \rightarrow A$  for each  $\omega \in \Omega$  of arity  $n$  such that all  $E$ -equations are satisfied and, given an  $n$ -tuple  $h: n \rightarrow A$ , then  $\omega_\alpha(h)$  is defined iff  $s = t$  in  $\text{Def}(\omega)$  implies  $s_\alpha(h) = t_\alpha(h)$ .

**REMARK 1.** a) Each operation  $\omega$  with  $\text{Def}(\omega)$  empty is total, i.e., everywhere defined. In particular, the first operation of  $\Omega$  is always total.

b) The concept of homomorphism is clear, owing to the special type of partial algebras: given  $(\Omega, E, \text{Def})$ -algebras  $A$  and  $B$ , then a map  $f: A \rightarrow B$  is a *homomorphism* if for each  $\omega \in \Omega$ , whenever  $\omega_\alpha(a_0, a_1, a_2, \dots)$  is defined, then

$$f(\omega_\alpha(a_0, a_1, a_2, \dots)) = \omega_\beta(f(a_0), f(a_1), f(a_2), \dots)$$

(the right-hand side being certainly defined too).

c) The collection of all  $(\Omega, E, \text{Def})$ -algebras need not be legitimate (i.e., equivalent to a class). If it is legitimate, we may consider the construct  $\text{Alg}(\Omega, E, \text{Def})$  of  $(\Omega, E, \text{Def})$ -algebras and homomorphisms.

**EXAMPLES 1.** (i) The construct  $\text{Cat}$  of small categories can be presented by the following essentially equational theory:

$$\Omega = \{c, d, \cdot\} \text{ with } c < d < \cdot \text{ and } \text{ar}(c) = \text{ar}(d) = 1, \text{ar}(\cdot) = 2 \text{ where} \\ \text{Def}(c) = \text{Def}(d) = \emptyset \text{ and } \text{Def}(\cdot) = \{c(x_0) = d(x_1)\}$$

and the set  $E$  expresses that  $\cdot$  is associative, and  $c$  and  $d$  are units of

(ii) Every equational theory is essentially equational. In particular, every variety of algebras has the form  $\text{Alg}(\Omega, E, \text{Def})$ . Also each quasi-variety (presented by implications) is of that form. Consider an implication

$$(\beta_j = \beta'_j, \text{ for } j \in J) \text{ implies } (\gamma = \gamma')$$

between terms using  $\Omega$ -operations and  $m$  variables. We can substitute this by introducing a new operation symbol  $\omega$  of arity  $m$ , inserting  $\omega$  into  $\Omega$  so that all operations involved in  $\beta_j$  and  $\beta'_j$  precede  $\omega$ , and put

$$\text{Def}(\omega) = \{\beta_j = \beta'_j, \mid j \in J\}.$$

Then we expand E by the following two equations:  $\omega = \gamma$  and  $\omega = \gamma'$ .

(iii) Let  $\Omega$  be a chain of unary operations  $\alpha_i$  ( $i \in \text{Ord}$ ) with  $E = \emptyset$  and

$$\text{Def}(\alpha_i) = \{\alpha_0(x) = x\} \text{ for each } i > 0.$$

Observe that there are free algebras of this theory over each set X of variables: form the free algebra on one unary operation  $\alpha_0$  (i.e., the algebra  $X \times \omega$  with  $\alpha_0(x, k) = (x, k+1)$ ) and put  $\alpha_i = \emptyset$  for  $i > 0$ .

However, the collection of  $(\Omega, E, \text{Def})$ -algebras is illegitimate: for a two-point set  $\{x, y\}$  with  $\alpha_0(x) = x$  and  $\alpha_0(y) = y$  there is more than a proper class of  $(\Omega, E, \text{Def})$ -algebras on  $\{x, y\}$ , since  $\alpha_i$  are arbitrary for all  $i \in \text{Ord} - \{0\}$ .

**DEFINITION.** A construct  $\underline{K}$  is said to be *essentially equational* if there is an essentially equational theory  $(\Omega, E, \text{Def})$  such that the constructs  $\underline{K}$  and  $\text{Alg}(\Omega, E, \text{Def})$  are concretely isomorphic (i.e., there is an isomorphism  $I: \underline{K} \rightarrow \text{Alg}(\Omega, E, \text{Def})$  such that the underlying set of the algebra  $IK$  is the same as that of the object  $K$ , and analogously with underlying maps).

**REMARK 2.** Each essentially equational construct  $(\underline{K}, U)$  has the following properties:

(1)  $\underline{K}$  is complete, moreover the forgetful functor creates limits (including large ones). In fact, let  $D: \underline{D} \rightarrow \text{Alg}(\Omega, E, \text{Def})$  be a diagram with a limit in  $\text{Set}$ :

$$(\pi_d: A \rightarrow \text{UD}d)_{d \in \text{obj}(\underline{D})} = \lim \text{UD}.$$

We define partial operations  $\omega_A$  on A by transfinite induction on the order of  $\Omega$  as follows: if  $\text{ar}(\omega) = n$ , then  $\omega_A$  is defined in  $h: n \rightarrow A$  iff

$$[s = t \text{ in } \text{Def}(\omega)] \text{ implies } s_A(h) = t_A(h),$$

in which case  $\omega_A(h)$  equals to the unique  $a \in A$  with  $\pi_d(a) = \omega_{\text{obj}(\underline{D})}(\pi_d \cdot h)$  for all  $d \in \text{obj}(\underline{D})$ . This gives a partial algebra satisfying all equations in E. Thus,  $A \in \text{Alg}(\Omega, E, \text{Def})$  becomes the only algebra for which all  $\pi_d: A \rightarrow Dd$  are homomorphisms, and it is clear that  $A = \lim D$ .

(ii)  $\underline{K}$  is *fibre-discrete*, i.e., if A and A' are two algebras on the same set X, and if  $\text{id}_X: A \rightarrow A'$  is a homomorphism, then  $A = A'$ . In fact, it is easy to prove by transfinite induction on the order of  $\Omega$  that  $\omega_A = \omega_{A'}$  for each  $\omega \in \Omega$  (moreover, it is a consequence of (1)).

(iii)  $\underline{K}$  is wellpowered (since U creates monos).

(iv) If we consider only a set of operation symbols, i.e., if  $\Omega$  is small, then  $\text{Alg}(\Omega, E, \text{Def})$  has further important properties: it is cowellpowered, cocomplete and has free algebras. This follows from Section III below.

## II. ESSENTIALLY MONADIC CONSTRUCTS.

Recall from [MS] that a functor  $U: \underline{K} \rightarrow \underline{X}$  is *essentially monadic* if it is a (possibly transfinite) compositum of monadic functors. That is, if there exists a chain  $V_{j,i}: \underline{X}_j \rightarrow \underline{X}_i$  of right adjoints (for all  $i \leq j \leq n$ , where  $n$  is an ordinal, or  $n = \infty$  which denotes the ordinal type of Ord) with the following properties:

$$(1) \quad \underline{K} \xrightarrow{U} \underline{X} = \underline{X}_n \xrightarrow{V_{n,0}} \underline{X}_0 ;$$

(2)  $V_{n,i}$  induces a monad  $\pi_i$  over  $X_i$  such that  $\underline{X}_{i+1} = \underline{X}_i^{\pi_i}$  with  $V_{i+1,i}: \underline{X}_i^{\pi_i} \rightarrow \underline{X}_i$  the forgetful functor and  $V_{n,i+1}: \underline{K} \rightarrow \underline{X}_i^{\pi_i}$  the comparison functor (for each  $i \leq n$ );

(3) For each limit ordinal  $i \leq n$ ,  $\underline{X}_i = \lim_{j < i} \underline{X}_j$ .

For example, each quasivariety of algebras is an *essentially monadic construct*, i.e., a construct whose forgetful functor is essentially monadic. The category of small categories is essentially monadic, since it is monadic over the (monadic) construct of graphs (via the forgetful functor forgetting composition).

Unlike monadic constructs (= varietal constructs = varieties of algebras of unbounded types) which are known to be well-behaved, essentially monadic constructs need not be cocomplete nor cowellpowered, as we shall show below. Therefore, essentially equational constructs with free algebras need not be cocomplete nor cowellpowered.

The next result could be deduced from Linton's Theorem (see [L<sub>2</sub>]). We prefer to give an independent proof. Another possibility would be to deduce Theorem 1 from the results of [Ro,] (see our Remark 4 (1)).

**THEOREM 1.** *Each essentially monadic construct is essentially equational.*

**PROOF.** Each essentially monadic construct can be considered as  $((\text{Set}^{\pi_0})^{\pi_1})^{\pi_2} \dots$  for suitable monads  $\pi_0, \pi_1, \pi_2, \dots$ . The construct  $\text{Set}^{\pi_0}$  is

well-known to be concretely isomorphic to a variety of universal algebras, see [L<sub>1</sub>]. Thus, we can consider  $Set^{\omega}$  as the category  $Alg(\Omega_0, E_0, \emptyset)$  for some type  $\Omega_0$  (carrying an arbitrary well-ordering) and some collection  $E_0$  of equations.

We shall prove that for each monad  $\pi_i = (T, (\mu, \eta))$  over  $Alg(\Omega_0, E_0, \emptyset)$  there is an extension  $(\Omega_i, E_i, Def)$  of the theory  $(\Omega_0, E_0, Def)$  such that the category of  $\pi_i$ -algebras is concretely isomorphic to  $Alg(\Omega_i, E_i, Def)$ . The steps from  $\pi_i$  to  $\pi_{i+1}$  are analogous for each  $i$ , and limit steps cause no difficulties, and thus, the theorem will then be proved. The idea of the extension is to create, for each  $(\Omega_0, E_0)$ -algebra  $A$  and each point  $a$  of the underlying set  $|TA|$  of  $TA$ , an operation symbol  $\xi_a^A$  of arity  $card|A|$ . Each  $\pi_i$ -algebra  $\alpha: TX \rightarrow X$  is equipped with a partial operation  $\xi_a^A$  defined in an  $|A|$ -tuple, i.e., a map  $h: |A| \rightarrow |X|$ , iff  $h$  is an  $\Omega_0$ -homomorphism, and then  $\xi_a^A(h) = \alpha \cdot Th(a)$ .

Thus the definability of  $\xi_a^A$  can be given by equations in  $\Omega_0$ -operations, expressing the fact that  $h$  is an  $\Omega$ -homomorphism, and then new equations are added to  $E_0$  to express (i) the natural transformations  $\eta$  and  $\mu$ , (ii) the functoriality of  $T$  and (iii) the fact that  $\alpha$  is an  $\Omega_0$ -homomorphism.

Formally, let

$$\Omega_1 = \Omega_0 \cup \{ \xi_a^A \mid A \text{ is an } (\Omega_0, E_0)\text{-algebra, } a \in |TA| \}$$

be a well-ordered set extending the ordering of  $\Omega_0$  in any way subject to  $\omega < \xi_a^A$  for all  $\omega \in \Omega_0$ . Put

$$ar(\xi_a^A) = card|A|$$

and

$$Def_1(\xi_a^A) = \text{all equations } \omega(x_0, x_1, \dots) = y \text{ where } \omega \in \Omega_0, \\ (x_0, x_1, \dots) \in |A|^{ar(\omega)} \text{ and } \omega_A(x_0, x_1, \dots) = y \text{ in } A.$$

Finally, let  $E_1$  denote the extension of  $E_0$  by the following equations: for each  $(\Omega_0, E_0)$ -algebra  $A$

$$(i) \quad \xi_{\eta(t)}^A = t \text{ for all } t \in |A|, \text{ and} \\ \xi_{\mu(a)}^A = \xi_{\xi_a^{TA}}^A(\xi_{x_i}^{TA})_{x_i \in |TA|} \text{ for all } a \in |T^2A|;$$

$$(ii) \quad \xi_a^A(f(x))_{x \in |A|} = \xi_{Tf(a)}^B \text{ for all } a \in |TA| \text{ and all} \\ \text{homomorphisms } f: A \rightarrow B \text{ in } Alg(\Omega_0, E_0);$$

$$(iii) \quad \omega(\xi_{a_0}^A, \xi_{a_1}^A, \dots) = \xi_{\omega(a_0, a_1, \dots)}^A \\ \text{for all } \omega \in \Omega_0 \text{ and } (a_0, a_1, \dots) \in |TA|^n,$$

We claim that the category of  $\pi_1$ -algebras is concretely isomorphic to the category of algebras of  $(\Omega_1, E_1, \text{Def}_1)$ . In fact:

A) Each  $\pi_1$ -algebra  $\alpha: TX \rightarrow X$  defines the following partial  $\Omega_1$ -algebra:  $\xi_{\bullet}^{\alpha}(h)$  is defined in an  $|A|$ -tuple  $h: |A| \rightarrow |X|$  iff  $h: A \rightarrow X$  is an  $\Omega_0$ -homomorphism, and then  $\xi_{\bullet}^{\alpha}(h) = \alpha \cdot Th(a)$ . It is clear that  $\xi_{\bullet}^{\alpha}$  is defined in an  $|A|$ -tuple  $h$  iff for each equation  $\omega(x_0, x_1, \dots) = y$  in  $\text{Def}(\xi_{\bullet}^{\alpha})$  we have

$$\omega_x(h(x_0), h(x_1), \dots) = h(x).$$

Furthermore, since  $\alpha\eta = 1$  and  $\alpha T\alpha = \alpha\mu$ , the equations (i) are fulfilled:

$$\xi_{\eta \circ \epsilon}^{\alpha}(h) = \alpha \cdot Th(\eta(t)) = \alpha \cdot \eta \cdot h(t) = h(t),$$

and

$$\begin{aligned} \xi_{\mu \circ \epsilon}^{\alpha}(h) &= \alpha \cdot Th \cdot \mu(a) = \alpha \cdot \mu \cdot T^2 h(a) = \alpha \cdot T(\alpha \cdot Th)(a) \\ &= \xi_{T\alpha}^{\alpha}(x \rightarrow \alpha \cdot Th(x)) = \xi_{T\alpha}^{\alpha}(\xi_{x \circ \epsilon}^{\alpha}(h))_{x \in |TA|}. \end{aligned}$$

The fact that  $T$  preserves composition guarantees that

$$(ii) \quad \xi_{\bullet}^{\alpha}(h(f(x))) = \alpha \cdot Th(Tf(a)) = \xi_{T\alpha \circ \epsilon}^{\alpha}(h).$$

Since for each  $\Omega_0$ -homomorphism  $h: A \rightarrow X$  also  $\alpha \cdot Th$  is an  $\Omega_0$ -homomorphism, we have

$$(iii) \quad \omega(\xi_{a_0}^{\alpha}(h), \xi_{a_1}^{\alpha}(h), \dots) = \alpha \cdot Th(\omega(a_0, a_1, \dots)) = \xi_{a_0 \circ \epsilon, a_1 \circ \epsilon, \dots}^{\alpha}(h).$$

Thus, the resulting algebra is a model of  $(\Omega_1, E_1, \text{Def}_1)$ .

B) Each algebra  $X$  of  $\text{Alg}(\Omega_1, E_1, \text{Def}_1)$  defines a map

$$\alpha: TX \rightarrow X, \quad a \mapsto \xi_{\bullet}^X(1_X)$$

which is an  $\Omega_0$ -homomorphism due to (iii). It fulfills  $\alpha\eta = 1$  due to (i). To verify  $\alpha T\alpha = \alpha\mu$ , we use (ii):

$$\alpha T\alpha(a) = \xi_{T\alpha \circ \epsilon}^X(1_X) = \xi_{T\alpha}^X(\alpha)$$

and (i):

$$\alpha\mu(a) = \xi_{\mu \circ \epsilon}^X(1_X) = \xi_{T\alpha}^X(x \rightarrow \xi_{x \circ \epsilon}^X(1_X)) = \xi_{T\alpha}^X(\alpha).$$

Thus,  $(X, \alpha)$  is a  $\pi_1$ -algebra.

C) The passages A and B above are mutually inverse, and they define a concrete isomorphism between  $\text{Alg}(\Omega_0, E_0)^{\pi_1}$  and  $\text{Alg}(\Omega_1, E_1, \text{Def}_1)$ . This is trivial to verify. ■

In particular, any reflective subcategory of a monadic construct

is essentially equational. E.g., Banach spaces are essentially equational via the unit ball functor (see [PR]).

**EXAMPLE 2** of an essentially monadic construct which is not cocomplete. This is an adaptation of the monadic, non-cocomplete category over graphs, presented in [A<sub>2</sub>]. We start with the category  $\mathcal{A}(1,1)$  of algebras with two unary operations (denoted  $\alpha$  and  $\beta$ ). For each such unary algebra  $A$  put

$$A^{(3)} = \{t\} \cup \{(x,y,z) \in A \times A \times A \mid x \neq y \neq z, \beta x = y, \alpha z = \beta z = \alpha y = z\},$$

where  $t$  is a fixed element,  $t \notin A \times A \times A$ ; for each homomorphism  $h: A \rightarrow A'$  in  $\mathcal{A}(1,1)$  put

$$h^{(3)}(t) = t, \quad h^{(3)}(x,y,z) = (h(x), h(y), h(z)) \text{ if } h(x) \neq h(y) \neq h(z), \\ \text{else } h^{(3)}(x,y,z) = t.$$

Now define a functor  $F: \mathcal{A}(1,1) \rightarrow \mathcal{A}(1,1)$  on objects by

$$FA = [PA^{(3)} \cup \{\beta^n b \mid n \in \omega\}] \times \omega$$

where  $PA^{(3)}$  is the power-set of  $A^{(3)}$  and  $\beta^n b$  are pairwise distinct elements not lying in  $PA^{(3)}$ , and define the operations by

$$\alpha(x,n) = \alpha(x,n+1) \\ \text{and } \beta(x,n) = \begin{cases} (x,n) & \text{if } n \neq 0 \\ (b,0) & \text{if } x \in PA^{(3)} \text{ and } n = 0 \\ (\beta^{*1} b, 0) & \text{if } x = \beta^* b \text{ and } n = 0. \end{cases}$$

For morphisms  $f: A \rightarrow A'$  in  $\mathcal{A}(1,1)$ ,

$$Ff = [Pf^{(3)} \cup \text{id}] \times \text{id}.$$

It is easy to verify that  $F(A+FA) = FA$ , and hence,  $F$  generates a free monad. (The free monad  $(T, \mu, \eta)$  is defined by  $TA = A+FA$  with  $\eta_A$  the first injection and  $\mu_A: A+FA+FA \rightarrow A+FA$  merging the two copies of  $FA$ .) Therefore the category  $\text{Alg}(F)$  of  $F$ -algebras (i.e., arrows  $p: FA \rightarrow A$  in  $\mathcal{A}(1,1)$ ) and  $F$ -homomorphisms is monadic over  $\mathcal{A}(1,1)$ , and thus, essentially monadic over  $\text{Set}$ . We shall prove that  $\text{Alg}(F)$  does not have coequalizers.

First define an algebra  $A$  in  $\mathcal{A}(1,1)$ : it has generators  $p$  and  $q$  and defining relations

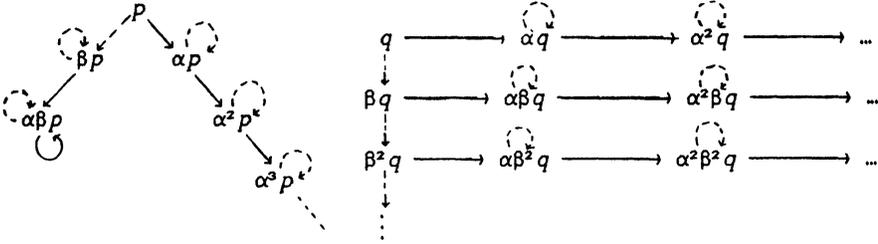
$$\beta\alpha^n p = \alpha^n p \text{ for all } n > 0,$$

$$\beta^2 p = \beta p, \quad \alpha^2 \beta p = \beta \alpha \beta p = \alpha \beta p$$

and

$$\beta\alpha^n \beta^m q = \alpha^n \beta^m q \text{ for all } n > 0, m \geq 0.$$

Here is the resulting algebra (with  $\alpha$  denoted by  $\rightarrow$  and  $\beta$  by  $\dashrightarrow$ )



Next define  $\rho: FA \rightarrow A$  by

$$\rho(x,0) = \beta p \text{ and } \rho(x,n) = \alpha \beta p \text{ for all } x \text{ and all } n > 0.$$

Finally, let  $f, g: A \rightarrow A$  be the unique  $A(1,1)$  homomorphisms such that

$$f(p) = g(p) = p \text{ and } f(q) = \beta p, \quad g(q) = \beta q.$$

We claim that  $f$  and  $g$  are well-defined  $F$ -homomorphisms, i.e.,

$$f \cdot \rho = \rho \cdot Ff \text{ and } g \cdot \rho = \rho \cdot Fg$$

- this is a routine verification, and they do not have a coequalizer in  $\text{Alg}(F)$ . In fact, assume that  $c: (A, \rho) \rightarrow (\bar{A}, \bar{\rho})$  is a coequalizer of  $f$  and  $g$ . Since  $c(\beta p) = c(\beta q)$ , the point  $t_0 = c(q)$  in  $\bar{A}$  has the property that  $\bar{A}^{(3)}$  contains  $t_0^* = (t_0, c(\beta p), c(\alpha \beta b))$ . We have a free choice of what  $\bar{\rho}(\{t_0^*\})$  is in  $\bar{A}$ . It follows that the point  $t_1 = \bar{\rho}(\{t_0^*\})$  has the property that  $\bar{A}^{(3)}$  contains  $t_1^* = (t_1, c(\beta b), c(\alpha \beta p))$ . Etc. Given an ordinal  $i$  such that  $t_j$  are pairwise distinct points of  $\bar{A}$  for all  $j < i$  and that

$$t_j^* = (t_j, c(\beta b), c(\alpha \beta b)) \in \bar{A}^{(3)} \text{ for } j < i,$$

then the point  $t_i = \bar{\rho}(\{t_j^* \mid j < i\})$  is distinct from  $t_j$  ( $j < i$ ) and  $t_i^* \in \bar{A}^{(3)}$ . This shows that  $\bar{A}$  has unbounded cardinality - a contradiction. ■

**REMARK 3.** The following criterion for a concrete category  $(\underline{K}, U)$  over any base-category  $\underline{X}$  (i.e., for a faithful functor  $U: \underline{K} \rightarrow \underline{X}$ ) to be essentially monadic has been proved in [AHT]: Let  $(\underline{K}, U)$  be

(a) solid (=semitopological), i.e., each structured sink has a semi-final lift,

(b) fibre-discrete, i.e., each  $\underline{K}$ -morphism carried by an  $\underline{X}$ -identity is a  $\underline{K}$ -identity,  
 and (c) weakly cowellpowered, i.e., each transfinite chain  $e_{ij}: A_i \rightarrow A_j$  such that all  $e_{i,i+1}$  are regular epis, and all limit ordinals  $j$  fulfill  $(A_j, e_{ij})_{i < j} = \text{colim}_{i < j} A_i$ , is stationary.  
 Then  $U$  is essentially monadic.

We are now going to improve somewhat this criterion by substituting weak cowellpoweredness by a condition more "natural" in the realm of concrete categories: strong fibre-smallness. Two structure maps

$$f: X \rightarrow UK \quad \text{and} \quad f': X \rightarrow UK' \quad (K, K' \in \underline{K})$$

are said to be equivalent if, given any co-structured map  $h: UL \rightarrow X$  then  $f \cdot h$  carries a  $\underline{K}$ -morphism  $L \rightarrow K$  iff  $f' \cdot h$  carries a  $\underline{K}$ -morphism  $L \rightarrow K'$ . A concrete category is said to be *strongly fibre-small* if each  $X \in \underline{X}$  has a small set of representative structured maps with domain  $X$ . Recall from [T] and [AHS] that a concrete category is solid iff it has a reflective MacNeille completion, and it is solid and strongly fibre-small iff it has a fibre-small reflective MacNeille completion.

**THEOREM 2.** *Each solid, fibre-discrete and strongly fibre-small concrete category is essentially monadic.*

**PROOF.** Let  $U: K \rightarrow X$  fulfill the hypotheses above. Define a chain  $V_{ij}: X_i \rightarrow X_j$  of essentially monadic functors for arbitrary ordinals  $i \in j$ , and right adjoints  $U_i: K \rightarrow X_i$ , as follows:

- (1)  $X_0 = X$  and  $U_0 = U$ ,
- (2) given  $U_i$ , then the monad  $\pi_i = (T_i, \mu^i, \eta^i)$  it induces fulfills:  $X_{i+1} = X_i^{\pi^i}$ ,  $V_{i+1, i}$  is the forgetful functor and  $U_{i+1}$  is the comparison functor,
- and (3) given a limit ordinal  $i$ , then  $X_i = \lim_{j < i} X_j$ ,  $U_i = \lim_{j < i} U_j$ .

It is proved in [AHT] that since  $U$  is solid, the above chain is well-defined (Corollary 2.7). For each object  $A$  of  $\underline{K}$  we have adjunctions as follows:

$$\begin{array}{ccc} U_i A & \xrightarrow{V_{j, i} \eta^j U_i A} & U_j F_j U_j A \\ \hline F_i A & \xrightarrow{\delta^{i, j}_A} & F_j A \end{array}$$

where  $F_j$  denotes the left adjoint of  $U_j$ . Following the argument of [AHT], 2.9, it is sufficient to show that for each  $A$  in  $\underline{K}$ , the chain  $\delta^{i, j}_A$  is stationary: then

$$\underline{K} = \lim_{j \in \text{Ord}} \underline{X}_j \quad \text{and} \quad U = \lim_{j \in \text{Ord}} U_j .$$

The universal maps  $\eta^j$  are shown in [AHT] to be semifinal lifts of appropriate structured sinks with codomain  $UA$  (see Propositions 1.4 and 2.7). Since  $\underline{K}$  is strongly fibre-small, these universal structured maps have a small set of representatives with respect to the above equivalence. However, if two *semi-universal* maps [cf. HS,]  $f: X \rightarrow UK$  and  $f': X \rightarrow UK'$  (of whatever structured sinks) are equivalent, then  $K$  and  $K'$  are isomorphic in  $\underline{K}$ , with an isomorphism  $i: K \rightarrow K'$  satisfying  $if = f'$ . Hence, the chain  $\delta^{j,j}_A$  is stationary, and the proof is concluded. ■

**COROLLARY.** *Each solid, fibre-discrete and strongly fibre-small construct is essentially equational.*

**REMARK 4.** (i) Any solid strongly fibre-small construct is isomorphic to the construct of models of a theory more general than an essentially equational one (see [Ro<sub>1</sub>] and [Ro<sub>2</sub>]). We mean a theory written in a language  $\Omega$  including operation and relation symbols which is given by axioms

$$(*) \quad (\forall x)(\varphi(x) \rightarrow (\exists! y)\psi(x,y)),$$

where  $\varphi$  and  $\psi$  are conjunctions of atomic formulas and  $x$  and  $y$  strings of variables. These theories were introduced by M. Coste [C]. If  $x = (x_i)_{i < n}$  and  $y = (y_j)_{j < m}$  then (\*) yields an  $m$ -tuple of partial  $n$ -ary operations with the domain described by  $\bigwedge_{i \in I} \varphi_i(x)$ . Fibre-discreteness eliminates relation symbols.

(ii) Without fibre-discreteness, we have the following more general concept, introduced in [MS] and studied in [AHT]: a functor  $U: \underline{K} \rightarrow \underline{X}$  is said to have a *monadic decomposition* if it factors as a right adjoint functor  $\bar{U}: \underline{K} \rightarrow \bar{K}$  inducing the trivial monad  $1_{\bar{K}}$  followed by an essentially monadic functor  $V: \bar{K} \rightarrow \underline{K}$ . The above theorem can be formulated (and proved) more generally as follows:

*Each solid and strongly fibre-small concrete category has a monadic decomposition.*

This result was proved for weakly cocomplete categories in [AHT]. The following example shows that the present formulation is sharper:

**EXAMPLE 3** of an essentially equational construct, which is not weakly cocomplete, although it is solid and strongly fibre-small. Let  $\Omega$  consist of a unary operation symbol  $\psi$  and, for each ordinal  $n$ , an  $n$ -ary operation symbol  $\varphi_n$ , ordered by  $\psi < \varphi_n < \varphi_m$  (for all  $n < m$ ). Let  $E$  be the following set of equations:

$$\varphi_n(x_0, x_1, x_2, \dots) = x_0 \quad \text{whenever } x_i = x_j \text{ for some } i \neq j$$

(for all ordinals  $n$ )

and let  $\text{Def}(\varphi_n)$  consist of the equations  $\psi(x) = x$  for all variables  $x$ . Thus,  $\text{Alg}(\Omega, E, \text{Def})$  is the essentially equational construct of partial algebras  $A$  with a unary, everywhere defined, operation  $\psi$ , and of  $n$ -ary operations  $\varphi_n$  defined on  $\text{fix } \psi$  (and equal to the first projection whenever  $n > \text{card } A$ , owing to the equations in  $E$ ).

(a)  $\text{Alg}(\Omega, E, \text{Def})$  is solid and strongly fibre-small. In fact, let  $\underline{K}_n$  be the essentially equational construct obtained from the given construct by forgetting all the operations  $\varphi_k$  for  $k \geq n$ . It is easy to verify that  $\underline{K}_n$  is solid and strongly fibre-small.

Consider a structured sink  $A = \langle f_i: UA_i \rightarrow X \rangle$  in  $\text{Alg}(\Omega, E, \text{Def})$ . For each cardinal  $n$  we have the corresponding semifinal lift  $h_n: X \rightarrow UK_n$  of the sink with the forgotten operations in  $\underline{K}_n$ . The structure of the algebra  $K_n$  is obvious: for each  $k < n$  the operation  $\varphi_k$  is defined "freely" in each  $k$ -tuple of elements of  $\text{fix } \psi$  in which its definition is not forced by the maps  $f_i$  or the equations in  $E$ . With increasing  $n$  the number of these operations  $\varphi_k$  increases; however, the set  $\text{fix } \psi$  remains the same. Consequently, there is a cardinal  $\kappa_0$  such that the algebras  $K_n$  are the same for all  $n \geq \kappa_0$  (except that new operations  $\varphi_k$  are added defined by  $\varphi_k(x_0, x_1, \dots) = x_0$  for all  $k \geq \kappa_0$ ). This algebra  $K_{\kappa_0}$  thus yields the semifinal lift of the given sink in  $\text{Alg}(\Omega, E, \text{Def})$ .

Analogously with strong fibre-smallness: two structured maps  $f_i: X \rightarrow UK_i$  ( $i = 1, 2$ ) are equivalent in  $\text{Alg}(\Omega, E, \text{Def})$  iff the corresponding structured maps in  $\underline{K}_n$  are equivalent, for  $n = \text{card } X$ . Since  $\underline{K}_n$  is strongly fibre-small, so is  $\text{Alg}(\Omega, E, \text{Def})$ .

(b)  $\text{Alg}(\Omega, E, \text{Def})$  is not weakly cowellpowered.

We shall define a non-stationary chain  $e_{i,j}: A_i \rightarrow A_j$  ( $i, j \in \text{Ord}$ ,  $i \leq j$ ) such that each  $e_{i,i+1}$  is a regular epi and the limit steps are colimits of the preceding chain. The algebras  $A_i$  will have distinguished, pairwise distinct elements  $a_k \in A_i$  for all  $k \leq i$  with the following properties:

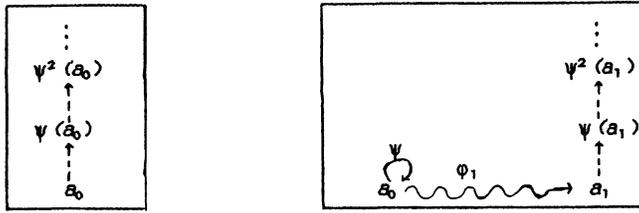
$$\psi(a_k) = a_k \text{ and } \psi^s(a_i) \neq a_k \text{ for all } k < i \text{ and all } s \in \omega,$$

and

$$e_{i,j}(a_k) = a_k \text{ for all } k \leq i \leq j.$$

*First step:*

Here  $e_{0,1}$  is a coequalizer of the homomorphisms  $p_0, q_0: A_0 \rightarrow A_0$  given by  $p(a_0) = a_0$  and  $q(a_0) = \psi(a_0)$ .



$$A_0 \xrightarrow{e_{0,1} = \text{const } a_0} A_1$$

Isolated step: Given  $A_i$ , define homomorphisms  $p_i, q_i: A_0 \rightarrow A_i$  by

$$p_i(a_0) = a_i \quad \text{and} \quad q_i(a_0) = \psi^i(a_0).$$

The coequalizer  $e_{i, i+1}: A_i \rightarrow A_{i+1}$  of  $p_i$  and  $q_i$  merges all points  $\psi^i(a_0)$  of  $A_i$  to a point, denoted by  $a_{i+1}$ , in  $A_{i+1}$ . All other equivalence classes of  $\ker e_{i, i+1}$  are singletons (denoted by  $x$  instead of  $\{x\}$ , with  $x \in A_i - \{\psi^i(a_0)\}_{i \in \mathbb{N}}$ ). Finally, denote

$$a_{i+1} = \varphi_{i+1}(a_0, a_1, \dots, a_j, \dots)_{j \leq i}.$$

Limited steps: in  $A_i = \text{colim}_{j < i} A_j$  we have points  $a_j$ , for each  $j < i$ , since  $e_{j,k}(a_j) = a_j$  for all  $k$  with  $j < k < i$ . Put

$$a_i = \varphi_i(a_0, a_1, \dots, a_j, \dots)_{j < i}.$$

OPEN PROBLEM. Is each solid, weakly cowellpowered concrete category strongly fibre-small?

### III. ESSENTIALLY ALGEBRAIC CONSTRUCTS.

We have seen that each essentially monadic construct is essentially equational. It is clear that the converse does not hold in general: not even all equational constructs are essentially monadic (e.g., the equational construct of complete lattices does not have free objects). It is well-known that, for equational constructs, the existence of free algebras is the "right" additional condition: equational constructs with free algebras are precisely the varietal, or the monadic constructs [L,]. For essentially equational constructs this, however, is not sufficient. We have presented an example of an essentially equational theory which has free algebras but is not legi-

timate (Ex. 1 (iii)).

There certainly are various possibilities of what additional condition to choose. We have decided for the following  $[A_1]$ : a construct is said to have *bounded generation* if for each cardinal  $n$  the collection of all objects  $A$  on  $n$  generators (i.e., having a set  $X \subset UA$  of cardinality  $\leq n$  such that no proper initial subobject of  $A$  contains  $X$ ) has a set of representatives with respect to isomorphism. We define this concept for a general base-category now:

**DEFINITION.** A concrete category  $U: \underline{K} \rightarrow \underline{X}$  is said to have *bounded generation* if for each object  $Z$  of  $\underline{X}$  there is, up to isomorphism in  $\underline{K}$ , only a set of structured maps  $f: Z \rightarrow UK$ ,  $K \in \underline{K}$ , which are extremally generating (i.e.,  $f$  does not factor through any initial subobject of  $K$  which is not an isomorphism).

**REMARK 5.** (i) Every essentially equational construct  $\text{Alg}(\Omega, E, \text{Def})$ , with  $\Omega$  being a set only, has bounded generation.

(ii) An equational construct has bounded generation iff it has free algebras.

(iii) An essentially equational construct which is fibre-small and has free algebras need not have bounded generation (see Ex. 3; the algebras  $A_i$  are extremally generated by  $a_0$ ).

We are now going to prove that for constructs with bounded generation, "essentially monadic" is equivalent to "essentially equational". We first recall a closely related concept, which will also turn out to be equivalent. A structured map  $f: X \rightarrow UK$  in a concrete category  $(\underline{K}, U)$  is said to be *generating* if for  $p, q: K \rightarrow L$  in  $\underline{K}$ ,  $f \cdot p = f \cdot q$  implies  $p = q$ .

**DEFINITION**  $[H_2]$ . A concrete category is said to be *essentially algebraic* if it is fibre-discrete and each structured source factors as a generating map followed by a monosource.

**THEOREM 3.** Let  $\underline{X}$  be a complete base-category with large intersections. For each concrete category  $(\underline{K}, U)$  with bounded generation, the following are equivalent:

- (i)  $(\underline{K}, U)$  is essentially algebraic;
- (ii)  $(\underline{K}, U)$  is solid and fibre-discrete;
- (iii)  $U$  creates limits.

PROOF.  $i \Rightarrow ii$ . Each essentially algebraic category is topologically algebraic  $[H_2]$  and hence, solid  $([HS]$  or  $[T])$ .

$ii \Rightarrow iii$ . Easy and well-known  $[T]$ .

$iii \Rightarrow i$ . We prove that each structured source has a generating-monosource factorization in three steps.

(a) Each structured map  $f: X \rightarrow UK$  has an extremally generating-mono-factorization. In fact, since  $U$  creates limits,  $\underline{K}$  has large intersections and  $U$  preserves them. Let  $m: K_0 \rightarrow K$  be the intersection of all  $\underline{K}$ -monos through which  $f$  factors in  $\underline{X}$ . Then  $f$  factors through  $m$ , and the factorizing structured map is extremally generating.

(b) Each small structured source  $(f_i: X \rightarrow UK_i)_{i \in I}$  has a generating-monosource factorization obtained by factoring the canonical map

$$X \rightarrow \prod_{i \in I} UK_i = U(\prod_{i \in I} K_i).$$

(c) Each large structured source  $(f_i: X \rightarrow UK_i)_{i \in I}$  has first extremally generating-mono factorizations

$$X \xrightarrow{f_i} UK_i = X \xrightarrow{g_i} UL_i \xrightarrow{Um_i} UK_i$$

for each  $i \in I$  due to (a) above, and since bounded generation guarantees that the source  $(g_i: X \rightarrow UL_i)_{i \in I}$  is essentially small, the latter source has a factorization as a generating map  $g: X \rightarrow UL$  followed by a monosource  $(n_i: L \rightarrow L_i)_{i \in I}$ . Then the given structured source factors as  $g$  followed by the monosource  $(m_i n_i: L \rightarrow K_i)_{i \in I}$ . ■

**THEOREM 4.** For each construct  $(\underline{K}, U)$  having bounded generation, the following are equivalent:

- (i)  $(\underline{K}, U)$  is essentially equational;
- (ii)  $(\underline{K}, U)$  is essentially monadic;
- (iii)  $(\underline{K}, U)$  is essentially algebraic;
- (iv)  $(\underline{K}, U)$  is solid and fibre-discrete;
- (v)  $U$  creates limits.

Each construct with the above properties is cocomplete, cowellpowered and has free objects.

In fact, each solid construct with bounded generation is clearly strongly fibre-small. Thus (iv)  $\Rightarrow$  (ii) follows from Theorem 2, and (ii)  $\Rightarrow$  (i) is Theorem 1, (i)  $\Rightarrow$  (v) is in Remark 2, and hence, the equivalence follows from Theorem 3. Cocompleteness and free objects follow from solidness (see  $[T]$ ), and bounded generation implies that free objects (and thus all objects) have only a set of quotients. ■

**REMARK 6.** For an essentially equational construct  $\langle \underline{K}, U \rangle$  with bounded generation the following hold:

(i)  $\langle \underline{K}, U \rangle$  is isomorphic to a quasivariety (i.e., an SP-class of total algebras) iff  $\langle \underline{K}, U \rangle$  is an implicationally defined construct of total algebras, iff  $U$  preserves and reflects regular epimorphisms (cf. [F, BH, H<sub>1</sub>]).

(ii)  $\langle \underline{K}, U \rangle$  is isomorphic to a variety (i.e., an HSP-class of total algebras) iff  $\langle \underline{K}, U \rangle$  is an equationally defined construct of total algebras, iff  $U$  is monadic, iff  $U$  preserves regular epimorphisms and reflects congruence relations (cf., e.g., [F, L<sub>1</sub>]).

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