CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 29, nº 2 (1988), p. 157-161

<http://www.numdam.org/item?id=CTGDC_1988__29_2_157_0>

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ET GÉOMÉTRIE DIFFÉRENTIELLE
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Vol. XXIX - 2 (1988)

A SIMPLE PROOF THAT LOCALIC SUBGROUPS ARE CLOSED

by P. T. JOHNSTONE

RÉSUMÉ. On donne une démonstration simple du résultat, d'abord établi par Isbell, Kříž, Pultr et Rosický, qui assure qu'un sous-groupe localique d'un groupe localique est nécessairement fermé.

In a recent paper [1], Isbell, Kříž, Pultr and Rosický have established a number of interesting results about localic groups (that is, group objects in the category **Loc** of locales), of which the most surprising (at least initially) is the result that any localic subgroup of a localic group is closed. The proof of this fact in [1], though not in any sense difficult, uses a certain amount of locale-theoretic machinery, in particular the uniformizability (and consequent regularity) of localic groups. The object of this note is to present a very simple proof of the same result, which uses almost no specific facts about locales beyond the result that an intersection of denses sublocales is dense. Moreover, we shall in fact establish a slightly stronger result, which does not appear in [1] (although it could doubtless have been proved by the methods of [1]).

Our starting-point is a seemingly innocuous observation due to G.C. Wraith [3]. Note that we may define a product operation on sublocales of a localic group, similar to that on subsets of a topological group (or indeed a discrete group): if S and T are sublocales of a localic group G, we define S.T to be the image of the sublocale S×T of G×G under the multiplication m: G×G → G. Similarly, we define T^{-1} to be the image of T under the inverse map i: G → G.

LEMMA [3]. *Let* S *and* T *be sublocales of a localic group* G. *Then* S∩T *is nontrivial if and only if the identity element* e *of* G *factors through the sublocale* $S.T^{-1}$.

PROFF. Consider the diagram

$$\begin{array}{ccccc} S\cap T & \longrightarrow & G & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow{\scriptstyle \Delta} & & \big\downarrow{\scriptstyle e} \\ S\times T & \longrightarrow & G\times G & \xrightarrow{m(1\times i)} & G \end{array}$$

in which both squares are pullbacks: the right-hand one because G is a group, and the left-hand one by the definition of $S\cap T$. The image of the bottom composite is by definition $S.T^{-1} \rightarrowtail G$; and if $S\cap T$ is nontrivial then the top composite is an epimorphism, so that the image of the diagonal composite $S\cap T \to G$ is $e\colon 1 \rightarrowtail G$. So the left-to-right implication follows from the functoriality of image factorization in **Loc**. Conversely, if e factors through $S.T^{-1}$ then we have a pullback

$$\begin{array}{ccc} S\cap T & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow{\scriptstyle e} \\ S\times T & \twoheadrightarrow & S.T^{-1} \end{array} \quad ;$$

in general, a pullback of an epimorphism in **Loc** need not be epi, but epis are stable under pullback along closed inclusions (this follows easily from the explicit description of such pullbacks in [2], Lemma II 2.8), and since G, being a localic group, is regular, it follows from [2], Proposition III 1.3, that all its points are closed. So $S\cap T \to 1$ is epi, i.e., $S\cap T$ is nontrivial. •

Note that we used the regularity of localic groups in proving the right-to-left implication of the Lemma; but in what follows we shall use only the left-to-right implication. In this direction, the Lemma is less innocuous than it looks, because the intersection $S\cap T$ may be nontrivial without having any points (indeed, S and T themselves may have no points); so the Lemma manufactures a point of the product $S.T^{-1}$ "out of thin air", from data not involving points of S or T.

We may now state our main result:

THEOREM. *If* S *and* T *are two dense sublocales of a localic group* G, *then* S.T *is the whole of* G.

Before proving the Theorem, we use it to derive:

COROLLARY. *A localic subgroup of a localic group is closed.*

PROOF. As in [1], we may reduce to the case when the subgroup is dense, because the closure of a localic subgroup is a localic subgroup (the proof of this fact is just like that for topological groups). So let S be a dense localic subgroup of a localic group G ; we need to show that S is the whole of G. But $S.S \subseteq S$ since S is a subgroup, and $S.S = G$ by the Theorem. •

The above argument in fact shows that any dense localic subsemigroup of a localic group G must be the whole of G. However, we cannot conclude that an arbitrary localic subsemigroup is closed, since the closure of a subsemigroup is not in general a group.

PROOF OF THE THEOREM. Consider first the case when G has enough points (i.e., G is an LT-group, in the terminology of [1]). In this case the result follows directly from the Lemma, for all we have to do is to show that every point of G factors through S.T. Let $g\colon 1 \to G$ be an arbitrary point of G. Since inversion and left multiplication by g are both locale isomorphisms $G \to G$, we know that $g.T^{-1}$ is a dense sublocale of G ; hence $S \cap g.T^{-1}$ is dense in G by [2], Lemma II 2.4, and in particular nontrivial. So by the Lemma e factors through $S.(g.T^{-1})^{-1} = S.T.g^{-1}$; but this is equivalent to saying that g factors through S.T.

In the general case, we would like to apply the above argument to the generic point of G, i.e., to carry it through in the slice category **Loc**/G, or equivalently the category of internal locales in the topos **Sh**(G) of sheaves on G. However, we cannot do this simple-mindedly, because the Lemma is not constructively valid (it relies on the fact that the terminal locale A has no proper sublocales other than the initial locale 0, so that any nontrivial locale maps epimorphically to it), and so cannot be applied in **Sh**(G). We shall therefore have to be more explicit.

Consider the diagram

$$\begin{array}{ccccc}
P & \longrightarrow & G\times G & \xrightarrow{\ \pi_2\ } & G \\
\big\downarrow & & \big\downarrow{\scriptstyle \langle \pi_1,\, m(i\times 1),\, \pi_2\rangle} & & \big\downarrow{\scriptstyle \Delta} \\
S\times T\times G & \xrightarrow{\ u\times v\times 1\ } & G\times G\times G & \xrightarrow{\ m\times 1\ } & G\times G
\end{array}$$

in which both squares are pullbacks (the left-hand one being the definition of P) and u, v are the inclusion maps of S, T into G. The image of the bottom composite may not be precisely S.T×G, but it is surely contained in the latter; so if we can show that the top composite is epimorphic then we may conclude as in the Lemma that the diagonal Δ factors through S.T×G, and hence that S.T is the whole of G. Now S×T×G is the intersection of the sublocales S×G×G and G×T×G of G×G×G ; pulling these back along the vertical map in the middle, we see that P is the intersection of

$$u\times 1\colon S\times G \to G\times G \quad \text{and} \quad \langle m(iv\times 1),\pi_2\rangle\colon T\times G \to G\times G.$$

Both these sublocales are dense in G×G, so their intersection is dense; but that is not enough to show that the composite

$$P \longrightarrow G\times G \xrightarrow{\pi_2} G$$

is epimorphic. For this, we need to show that the corresponding frame homomorphism G → P is injective, i.e., that if x and y are elements (i.e., open sets) of G with $x \neq y$, then their inverse images in P are distinct. There is no loss of generality in assuming that $x \nleq y$; then we need to show that the inverse image in P of the locally closed sublocale of G which is the intersection of (the open sublocale) x with the closed complement of y is nontrivial. But if $w\colon Z \to G$ is any sublocale of G whatever, we obtain a pullback square

$$\begin{array}{ccc}
P\times_G Z & \longrightarrow & T\times Z \\
\Big\downarrow & & \Big\downarrow {\scriptstyle \langle m(iv\times w),\pi_2\rangle} \\
S\times Z & \xrightarrow{u\times 1} & G\times Z
\end{array}$$

on pulling back the square expressing P as the intersection of S×G and T×G along $1\times w\colon G\times Z \to G\times G$. And both S×Z and T×Z are dense in G×Z, so we conclude that $P\times_G Z$ is dense in G×Z, and hence nontrivial if Z is. This completes the proof. •

ACKNOWLEDGEMENT. The author is indebted to the Australian Research Grants Committee for making it possible for both himself and Gavin Wraith to visit the University of Sydney in September 1987. Without those visits, this paper would probably not have been written.

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