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A RIGHT EXACTNESS PROPERTY FOR INTERNAL CATEGORIES

by Dominique BOURN

RÉSUMÉ. Etant donné une catégorie **E** exacte à gauche et Barr-exacte, on établit une propriété d'exactitude à droite pour Cat **E** et plus généralement pour n-Cat **E**, tout à fait analogue à la Barr-exactitude elle-même, mais "relative" à une classe particulière de morphismes Σ. Pour cela, on est amené à démontrer que, si on note Σ_n la classe particulière à n-Cat **E**, la fibration

$$(\ )_n\colon (n+1)\text{-Cat } \mathbf{E} \longrightarrow n\text{-Cat } \mathbf{E}$$

est non seulement un champ pour la topologie des épimorphismes de Σ_n mais possède encore des propriétés plus générales de "descente".

Here is the second of the two papers announced in [5] and concerning right exactness properties of the category Cat **E** of internal categories in a left exact and Barr-exact category **E**.

When **E** is exact in the sense of Barr (Barr-exact, for short) [1], the category Simpl **E** of simplicial objects in **E** is again Barr-exact. It is very disappointing that the category Cat **E** does not seem to behave so well with respect to this kind of exactness property and it is probably the reason why the category Simpl **E** is often prefered to it [7, 13].

Nevertheless the development of a general cohomology theory for an exact category **E** (summarized in [3]), using internal n-groupoids as a non-abelian equivalent to chain complexes of length n, made it necessary to understand precisely what kind of right exactness property does exist in Cat **E** and more generally in n-Cat **E**.

Actually it appeared that some important stability properties can be obtained, in this direction, for Cat **E**, when **E** is left exact and Barr-exact. The first one (vertical stability) is that the functor $(\ )_0$: Cat $\mathbf{E} \to \mathbf{E}$ is a fibred reflexion (i.e., a peculiar kind of fibration) which is a Barr-exact fibration: each fibre is Barr-exact and each change of base functor is Barr-exact [2]. The second one (horizontal stability) is that the fibration $(\ )_0$ is a stack for the regular epimorphism topology in $\mathbf{E}$ [2]. The first result implies that every $(\ )_0$-invertible equivalence relation has a $(\ )_0$-invertible quotient, the second one that every $(\ )_0$-cartesian equivalence relation has a $(\ )_0$-cartesian quotient.

Now, regarding the complementary aspect of the two stability properties, a question naturally arises: is there a class of equivalence relations in Cat $\mathbf{E}$, including the $(\ )_0$-invertible and the $(\ )_0$-cartesian ones, which always have a quotient? Or, equivalently, is there in Cat $\mathbf{E}$ a class Σ of regular epimorphisms, including the $(\ )_0$-invertible and the $(\ )_0$-cartesian ones, towards which the category Cat $\mathbf{E}$ behaves as the category $\mathbf{E}$ behaves towards the class of all regular epimorphisms? In other words, is there a kind of relative Barr-exactness property for Cat $\mathbf{E}$?

The aim of this paper is to give a positive answer to this question. The class Σ_1 in concern is the class of internal functors $f_1\colon X_1 \to Y_1$, having their canonical decomposition $f_1^c.f_1^i$ (where f_1^c is $(\ )_0$-cartesian and f_1^i is $(\ )_0$-invertible) such that f_1^c is a $(\ )_0$-cartesian and f_1^i a $(\ )_0$-invertible regular epimorphism (or equivalently, internally full functors which are epic on objects).

In our mind, such a positive answer is of some interest only if the proposed class has a good stability property with respect to the iterative construction of the categories n-Cat $\mathbf{E}$ of internal n-categories in $\mathbf{E}$. Actually it is the case. Indeed, the functor $(\ )_1\colon$ 2-Cat $\mathbf{E} \to$ Cat $\mathbf{E}$ which is known as a Barr-exact fibration is again a stack for the Σ_1-regular epimorphism topology in Cat $\mathbf{E}$, and this is the beginning of an iteration process.

In fact we shall investigate this question for a general fibred reflexion $c\colon \mathbf{V} \to \mathbf{W}$ which is Barr-exact as a fibration and a stack for a Σ-topology in $\mathbf{W}$. The main difference with the case of the fibred reflexion $(\ )_0$ is that c is no more supposed to be left exact. An equivalent condition for c to be a stack for a Σ-topology is the following one: every c-cartesian equivalence relation in $\mathbf{V}$, above a Σ-exact diagram in $\mathbf{W}$ can be completed in a c-cartesian exact diagram above the given Σ-exact diagram. Then our main result asserts that this property can be extended from c-cartesian equivalence relations to c-full equivalence relations, where a c-full morphism in $\mathbf{V}$ is a morphism whose c-invertible part is a regular epimorphism. Or, more roughly, that something more general than a descent data can even be descended.

One of the interest of taking a general fibred reflexion c, is that this result can be also applied to the quotient functor q: Rel $\mathbf{E} \to \mathbf{E}$ when $\mathbf{E}$ is Barr-exact. Indeed it is a Barr-exact fibred reflexion and a stack for the regular epimorphism topology.

As a by-product, it is shown that this functor q preserves (beside products) a large number of pullbacks, namely those with an edge a q-cartesian morphism, those with an edge a q-invertible regular epimorphism and consequently those with an edge a composite of the two previous ones. The obstruction to the total left exactness of q being only due, for any morphism f_1: $R_1 \to R'_1$ in Rel $\mathbf{E}$, to its q-invertible monic part.

CONTENTS.

I. THE FIBRED REFLEXIONS.

This first section is devoted to some recalls and results about fibred reflexions which are the main tool in this setting, and about the factorization system they produce. A fibred reflexion appears to be, up to equivalence, a fibration with a terminal object in each fiber. The two principal examples are introduced: the functor $(\ )_0$: Cat $\mathbf{E} \to \mathbf{E}$ where $\mathbf{E}$ is left exact, the quotient functor q: Rel $\mathbf{E} \to \mathbf{E}$ where $\mathbf{E}$ is Barr-exact.

1. THE FIBRED REFLEXIONS.

Let us consider the following situation:

$$\mathbf{V} \underset{d}{\overset{c}{\rightleftarrows}} \mathbf{W}$$

where d is fully faithful and c a left adjoint to d. Then c is called a reflexion.

A morphism f: $V \to V'$ in $\mathbf{V}$ is c-invertible if $c(f)$ is an isomorphism and c-cartesian if the following square is a pullback:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & V' \arrow[d] \\ dcV \arrow[r, "dcf"] & dcV' \end{tikzcd}$$

The c-cartesian morphisms are stable under composition. If the morphisms $g.f$ and g are c-cartesian, such is the morphism f. A morphism dh: $dw \to dw'$ is always c-cartesian. The c-invertible morphisms are those which satisfy the diagonality condition of a factorization system [6, 15] with respect to the c-cartesian morphisms [5]. A morphism which is both c-invertible and c-cartesian is invertible. Furthermore, if in a commutative square a parallel pair of edges is c-cartesian and the image of this square is a pullback, then the given square is itself a pullback. It is the case when a parallel pair of edges is c-cartesian and the other one is c-invertible.

The obstruction for c to be a fibration is the lack of an existence condition for cartesian morphisms. This is the meaning of the following definition.

DEFINITION 1. A reflexion c: $\mathbb{V} \to \mathbb{W}$ is called a *fibred reflexion* if the pullback in $\mathbb{V}$ of any c-invertible morphism along a c-cartesian morphism does exist, the parallel edges in this square being in the same classes.

REMARK. A fibred reflexion is, up to equivalence, a fibration: let $c/\mathbb{V}$ be the category whose objects are the triples (X,t,Y) with X an object in $\mathbb{V}$, Y an object in $\mathbb{W}$ and t a morphism $X \to dY$ which is c-invertible. The morphisms are the pairs (f,h) with f: $X \to X'$ and h: $Y \to Y'$ such that $f.t' = t.dh$. There are two functors:

$$c': c/\mathbb{V} \to \mathbb{W} \quad \text{with} \quad c'(X,t,Y) = Y,$$
$$\theta_c: c/\mathbb{V} \to \mathbb{V} \quad \text{with} \quad \theta_c(X,t,Y) = X.$$

Then θ_c is an equivalence of categories and, when c is a fibred reflexion, then c' is a fibration. For any object w in $\mathbb{W}$, we (improperly) denote by $\mathrm{Fib}_c[w]$ the fiber of c' over w. On the other hand, this functor c' has a right adjoint right inverse d'. Consequently each fiber of the fibration c' has a terminal object. So a fibred reflexion appears to be, up to equivalence, a fibration with a terminal object in each fiber. .

If c is a fibred reflexion, we have two important results:

1. Any morphism in $\mathbb{V}$ has a unique, up to isomorphism, decomposition $f^c.f^i$, with f^c c-cartesian and f^i c-invertible, given by the following diagram in which the right hand square is a pullback

$$
\begin{tikzcd}
V \arrow[rr, "f"] \arrow[dr, "f^{i}"] \arrow[dd] & & V' \arrow[dd] \\
& U \arrow[ur, "f^{c}"] \arrow[dl] & \\
dcV \arrow[rr, "dcf"'] & & dcV'
\end{tikzcd}
$$

2. LEMMA 1. The c-cartesian morphisms are stable under pullback whenever they exist, and such pullbacks are preserved by c. (Cf. [5].)

THE MAIN EXAMPLES.

1. A category $\mathbf{E}$ is called *weakly left exact* if it has a terminal object 1, if the kernel pair of a morphism always exists, as well as the pullback of a split epimorphism along any morphism.

An internal category X_1 in $\mathbf{E}$ is a diagram in $\mathbf{E}$:

$$
\begin{tikzcd}
X_0 \arrow[r, "s_0" description] & mX_1 \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_1"] & m_2X_1 \arrow[l, shift right=3, "d_0"'] \arrow[l, "d_1" description] \arrow[l, shift left=3, "d_2"]
\end{tikzcd}
$$

such that m_2X_1 is the vertex of the pullback of d_0 along d_1 and satisfying the usual unitarity and associativity axioms. The internal functors are the natural transformations between such diagrams. We shall denote by Cat $\mathbf{E}$ the category of internal categories in $\mathbf{E}$. It is again weakly left exact and there is a canonical functor $(\ )_0$ associating X_0 to X_1:

$$(\ )_0\colon \mathrm{Cat}\ \mathbf{E} \longrightarrow \mathbf{E}$$

which has a fully faithful right adjoint Gr and a fully faithful left adjoint dis [2]. Hence the functor $(\ )_0$ is both left and right exact.

If $\mathbf{E}$ is left exact (i.e., has a terminal object and pullbacks), then $(\ )_0$ is a fibred reflexion which is moreover left exact. Thus, for any object X in $\mathbf{E}$, GrX and disX are respectively the terminal object and the initial object in the fiber over X.

The $(\ )_0$-cartesian functors are the internally fully faithful functors and the $(\ )_0$-invertible ones are the "bijective on objects" functors [2].

2. An internal category is a groupoid when the following square is a pullback:

$$\begin{tikzcd}
mX_1 \arrow[d, "d_1"'] & m_2X_1 \arrow[l, "d_2"'] \arrow[d, "d_1"] \\
X_0 & mX_1 \arrow[l, "d_1"]
\end{tikzcd}$$

Grd **E** will denote the full subcategory of Cat **E** whose objects are the internal groupoids.

An equivalence relation is an internal groupoid X_1 such that the map $X_1 \to \mathrm{Gr}\ X_0$ is a monomorphism. We shall denote by Rel **E** the full subcategory of Grd **E** whose objects are the equivalence relations, by dis: **E** $\to$ Rel **E** the restriction of the previous dis: **E** $\to$ Cat **E**, and by $(\ )_0$ the composite

$$\mathrm{Rel}\ \mathbf{E} \longrightarrow \mathrm{Cat}\ \mathbf{E} \xrightarrow{(\ )_0} \mathbf{E}$$

Now we suppose that **E** is Barr-exact; it means that **E** is weakly left exact and that every equivalence relation has a quotient (i.e., a coequalizer making this equivalence relation effective) which is universal (i.e., stable under pullbacks along any morphism in **E** which are supposed to exist). Then the quotient functor q: Rel **E** $\to$ **E** determines a left adjoint to dis. It is a fibred reflexion whose q-cartesian morphisms are the discrete fibrations [5].

With these conditions, the functor $(\ )_0$: Rel **E** $\to$ **E** becomes itself a fibred reflexion. For that, let us consider the following diagram

$$\begin{tikzcd}
P \arrow[r, dashed] \arrow[d, dashed, shift right] \arrow[d, dashed, shift left] & mR'_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
V \arrow[r, "f"] \arrow[dr, dashed, "\rho'.f"'] & R'_0 \arrow[d, "\rho'"] \\
 & Q'
\end{tikzcd}$$

If R'_1 is an equivalence relation and f: $V \to R'_0$ a morphism in **E**, then the kernel pair associated to $\rho'.f$ (where ρ': $R'_0 \twoheadrightarrow Q'$ is the quotient morphism of R'_1) determines an equivalence relation R_1 and a functor ϕ_1: $R_1 \to R'_1$ with $\phi_0 = f$ which is internally fully faithful.

Given any morphism f: $V \to V'$, the equivalence relation $R_1[f]$ associated to the kernel pair of f will be called the *kernel equivalence of* f (or shortly the kernel of f). It is all the more justified as the following square is a pullback in Rel **E** and the object dis Y is the initial object in the $(\ )_0$-fiber of Y:

$$\begin{tikzcd} R_1[f] \arrow[r] \arrow[d] & \text{dis } Y \arrow[d] \\ \text{Gr } X \arrow[r, "Gf"'] & \text{Gr } Y \end{tikzcd}$$

REMARK. According to [1], a diagram

$$\begin{tikzcd} \cdot & \cdot \arrow[l] & \cdot \arrow[l, shift left] \arrow[l, shift right] & \cdot \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

is called left exact if the right hand part is the kernel equivalence of the left hand morphism, and exact if, moreover, this morphism is the quotient of this equivalence relation.

2. THE c-DISCRETE CATEGORIES.

The following construction, recalled from [2], is the basic construction allowing the iterative constructive process of the categories n-Cat **E** and n-Grd **E** of internal n-categories and internal n-groupoids in **E**. It is essential for us, keeping in mind that, when **E** = **A** is an abelian category, the categories n-Cat **A** and n-Grd **A** which are then the same, are equivalent to the category C^n(**A**) of abelian chain complexes of length n [4].

Let c be a fibred reflexion. From now on, we suppose that it is a weakly left exact fibred reflexion: the kernel pair of any c-invertible morphism always exists and is c-invertible, in the same way as the pullback of any c-invertible split epimorphism along any c-invertible morphism. Our two main examples are weakly left exact fibred reflexions.

A c-discrete category in **V** is an internal category such that its image by c is discrete, or equivalently such that any structural map of its diagram is c-invertible. We denote by Cat_c**V** the full subcategory of Cat **V** whose objects are the c-discrete categories.

There is a forgetful functor c_0: $\mathrm{Cat}_c\mathbb{V} \to \mathbb{V}$ associating X_0 to X_1. It has a fully faithful right adjoint G_c, given for any object V in $\mathbb{V}$ by the kernel equivalence of $V \to dcV$:

$$dcV \xleftarrow{\lambda V} V \underset{p_1}{\overset{p_0}{\leftleftarrows}} V\times_c V \;\overset{p_0}{\underset{p_2}{\overset{p_1}{\longleftarrow\!\!\longleftarrow\!\!\longleftarrow}}}\; V\times_c V\times_c V$$

which does exist since λV is c-invertible. Then $m(G_cV)$ is nothing but $V\times_cV$, the product of V by itself in the fibre over $c(V)$.

The restriction of the functor dis is again a fully faithful left adjoint to c_0.

The functor $\bar{c} = c.c_0$: $\mathrm{Cat}_c\mathbb{V} \to \mathbb{W}$ has a fully faithful right adjoint $\bar{d} = G_c.d = \mathrm{dis}.d$. It is the "fibration" of internal categories associated to the "fibration" c: $\mathbb{V} \to \mathbb{W}$. The $\bar{c}$-invertible functors f_1: $X_1 \to Y_1$ are such that f_0 and mf_1 are c-invertible.

PROPOSITION 1. *The four following conditions are equivalent:*

1. The functor f_1 is $\bar{c}$-cartesian.

2. The morphism f_0 is c-cartesian and f_1 is a discrete fibration.

3. The morphisms f_0 and mf_1 are c-cartesian.

4. The morphism f_0 is c-cartesian and the functor f_1 is c_0-cartesian.

PROOF. The functor f_1 is $\bar{c}$-cartesian iff the following square (*) is a pullback:

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d] \\ G_c[dcX_0] \arrow[r, "{G_c[dcf_0]}"'] & G_c[dcY_0] \end{tikzcd}$$

Now, its image by the left exact functor c_0 is a pullback:

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d] & Y_0 \arrow[d] \\ dcX_0 \arrow[r, "dcf_0"'] & dcY_0 \end{tikzcd}$$

and consequently f_0 is c-cartesian. The square (*) is a pullback in $\mathrm{Cat}_c\mathbf{V}$, but, c being a fibred reflexion, it is a componentwise pullback. Furthermore $G_c[dcf_0]$, being also $\mathrm{dis}[dcf_0]$ is a discrete fibration. Thus the functor f_1 is a discrete fibration.

If f_1 is a discrete fibration and f_0 c-cartesian, the following square is a pullback and the morphism mf_1 is again c-cartesian:

$$\begin{tikzcd} mX_1 \arrow[r, "mf_1"] \arrow[d, "d_1"'] & mY_1 \arrow[d, "d_1"] \\ X_0 \arrow[r, "f_0"'] & Y_0 \end{tikzcd}$$

Now when f_0 is c-cartesian, $G_c(f_0)$ is a discrete fibration and $f_0\times_c f_0$: $X_0\times_c X_0 \to Y_0\times_c Y_0$ is c-cartesian. If also mf_1 is c-cartesian, then the following square is a pullback:

$$\begin{tikzcd} mX_1 \arrow[r, "mf_1"] \arrow[d] & mY_1 \arrow[d] \\ X_0\times_c X_0 \arrow[r, "f_0\times_c f_0"'] & Y_0\times_c Y_0 \end{tikzcd}$$

since the two horizontal edges are c-cartesian and the two vertical ones c-invertible. Thus the functor f_1 is c_0-cartesian.

Finally if f_0 is c-cartesian and f_1 c_0-cartesian, then the two following squares are pullbacks:

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "f_1"'] & G_cX_0 \arrow[d, "G_cf_0"] \\ Y_1 \arrow[r] & G_cY_0 \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"'] & dcX_0 \arrow[d, "dcf_0"] \\ Y_0 \arrow[r] & dcY_0 \end{tikzcd}$$

Now G_c being left exact, the following one is again a pullback as the composite of two pullbacks:

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "f_1"'] & G_cX_0 \arrow[r] \arrow[d, "G_cf_0"] & G_c[dcX_0] \arrow[d, "G_c[dcf_0]"] \\ Y_1 \arrow[r] & G_cY_0 \arrow[r] & G_c[dcY_0] \end{tikzcd}$$

It is the square (*) and f_1 is $\bar{c}$-cartesian.

PROPOSITION 2. *The functor $\bar{c}$ is a fibred reflexion.*

PROOF. Let Y_1 be a c-discrete category and h: $W \to cX_0$ a morphism in $\mathbb{W}$. Then c being a fibred reflexion, the pullback of λX_0 along dh, as well as the pullback of $\lambda X_0.d_0 = \lambda X_0.d_1$ along dh do exist and they determine a functor h_1: $X_1 \to Y_1$ which is a discrete fibration with h_0 c-cartesian. Hence h_1 is $\bar{c}$-cartesian. •

Let us now consider the following commutative triangle between the two fibred reflexions:

$$\begin{tikzcd} \mathrm{Cat}_c\mathbb{V} \arrow[rr, "c_0"] \arrow[dr, "\bar{c}"'] & & \mathbb{V} \arrow[dl, "c"] \\ & \mathbb{W} & \end{tikzcd}$$

The functor c_0 commutes also with $\bar{d}$ and d. It associates a $\bar{c}$-invertible morphism to a $\bar{c}$-invertible one. Proposition 1 tells us that c_0 preserves the cartesian morphisms.

The same property holds for G_c: $\mathbb{V} \to \mathrm{Cat}_c\mathbb{V}$.

REMARK. We shall denote by $\mathrm{Grd}_c\mathbb{V}$ and $\mathrm{Rel}_c\mathbb{V}$ the full subcategories of $\mathrm{Cat}_c\mathbb{V}$ whose objects are the c-discrete groupoids and the c-discrete equivalence relations.

II. THE BARR-EXACT FIBRED REFLEXIONS.

1. BARR-EXACTNESS.

DEFINITION 2. A fibred reflexion is said to be *Barr-exact* when it is weakly left exact and when every c-invertible (or c-discrete) equivalence relation R_1 has a quotient which is universal.

The functor c being right exact, the quotient morphism ρ: $R_0 \twoheadrightarrow Q$ is c-invertible. The universality condition means, here, that the pullback of any c-invertible exact diagram along any morphism does exist and is a c-invertible exact diagram.

REMARK. In other words, the fibred reflexion c is Barr-exact if its associated fibration c': $c/\mathbb{V} \to \mathbb{W}$ is Barr-exact: each fibre is Barr-exact and each change of base functor is Barr-exact.

EXAMPLES. When $\mathbb{E}$ is Barr-exact, the two main examples are Barr-exact fibred reflexions.

1. That the fibred reflexion $(\ )_0$: Cart $\mathbb{E} \to \mathbb{E}$ is Barr-exact if $\mathbb{E}$ is Barr-exact is shown in [2].

2. We are going to show that, if $\mathbb{E}$ is Barr-exact, the fibred reflexion q: Rel $\mathbb{E} \to \mathbb{E}$ is Barr-exact. First, remark that a q-invertible morphism f_1:$R_1 \to R'_1$ is necessarily an internally fully faithful functor, since the following diagram is a joint pullback, $\rho'.f_0$ being equal to ρ.

$$\begin{tikzcd}
mR_1 \arrow[rr, "mf_1"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & & mR'_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
R_0 \arrow[rr, "f_0"] \arrow[dr, dashed, two heads, "\rho"'] & & R'_0 \arrow[dl, dashed, two heads, "\rho'"] \\
& Q &
\end{tikzcd}$$

Conversely, we have the following result:

LEMMA 2. A morphism f_1: $R_1 \to R'_1$ is internally fully faithful iff qf_1 is a monomorphism.

PROOF. If qf_1 is a monomorphism, then the kernel equivalence of ρ is the kernel equivalence of $q(f_1).\rho$ which is also $\rho'.f_0$. Then the functor f_1 is clearly internally fully faithful.

Conversely let f_1: $R_1 \to R'_1$ be an internally fully faithful functor. We denote by $i.r$ the canonical decomposition of $\rho'.f_0$ as a composite of a monomorphism and a regular epimorphism. f_1 being internally fully faithful, r is necessarily a quotient morphism of R_1 and $q(f_1)$ is, up to isomorphism, the monomorphism i. •

LEMMA 3. A morphism f_1: $R_1 \to R'_1$ is a q-invertible regular epimorphism in Rel $\mathbb{E}$ iff f_1 is internally fully faithful and f_0 is a regular epimorphism. Such morphisms are stable under pullbacks.

PROOF. If f_1 is q-invertible, by the above remark, it is internally fully faithful and, the functor $(\ )_0$: Rel $\mathbb{E} \to \mathbb{E}$ being right exact (it has a right adjoint Gr), the morphism f_0 is a regular epimorphism. Conversely, if f_1 is internally fully faithful, then $q(f_1)$ is a monomorphism (Lemma 2). Furthermore if f_0 is a regular epimorphism then $q(f_1)$ is a regular epimorphism. Thus f_1 is q-invertible. Now f_0 being a regular epimorphism and f_1 being internally fully faithful, f_1 is a componentwise regular epic functor and consequently a regular epimorphism in Rel $\mathbb{E}$. Thus the pullback of f_1 along any morphism g_1 does exist and is componentwise. It is a componentwise regular epimorphism. Moreover, it is clear that the internally fully faithful functors are stable under componentwise pullbacks. Thus the q-invertible regular epimorphisms in Rel $\mathbb{E}$ are stable under pullbacks. •

PROPOSITION 3. *When* $\mathbb{E}$ *is Barr-exact, the fibred reflexion* q*:* Rel $\mathbb{E} \to \mathbb{E}$ *is Barr-exact.*

PROOF. 1. The category $\mathbb{E}$ being weakly left exact, any morphism f_1: $R_1 \to R'_1$ has a kernel pair which is a componentwise kernel pair. Thus if f_1 is internally fully faithful, the kernel pair is fully faithful. But this pair being split, it is a q-invertible pair. Thus any q-invertible morphism has a q-invertible kernel pair.

2. Let us consider a q-invertible equivalence relation R_1 in Rel $\mathbb{E}$ and set $R_0 = R_1$ and $mR_1 = P_1$ for sake of simplicity:

$$\begin{tikzcd} P_1 \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p'_1"'] & R_1 \arrow[l] \end{tikzcd}$$

We denote by Q the common quotient of P_1 and R_1 and by Q_0 the quotient of the image by the functor $(\ )_0$ of the previous diagram:

$$\begin{tikzcd} P_0 \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p'_0"'] \arrow[dr, "\rho_P"'] & R_0 \arrow[l] \arrow[r, two heads, "\rho_0"] \arrow[d, "\rho_R"] & Q_0 \arrow[dl, dashed, "\rho_Q"] \\ & Q & \end{tikzcd}$$

Then $\rho_R.p_0 = \rho_R.p'_0$ and there is a regular epimorphism ρ_Q: $Q_0 \to Q$ such that $\rho_Q.\rho_0 = \rho_R$. The kernel pair of ρ_Q determines an equivalence relation Q_1 which is the componentwise quotient of R_1. The universality of this quotient is given by Lemma 3.

REMARK. By Lemma 2 the canonical mono-epi factorization in $\mathbf{E}$ appears to be, via the functor dis, the image by q of the canonical $(\ )_0$-cartesian-$(\ )_0$-invertible factorization in Rel $\mathbf{E}$.

2. PROPERTIES OF THE BARR-EXACT FIBRED REFLEXIONS.

Let $\mathrm{Rel}_c\mathbf{V}$ be the category of c-discrete equivalence relations in $\mathbf{V}$ and c_0: $\mathrm{Rel}_c\mathbf{V} \to \mathbf{V}$ the restriction of c_0: $\mathrm{Cat}_c\mathbf{V} \to \mathbf{V}$.

LEMMA 4. The reflexion c_0: $\mathit{Rel}_c\mathbf{V} \to \mathbf{V}$ is a fibred reflexion.

PROOF. Let R'_1 be a c-discrete equivalence relation and f: $X \to R'_0$ be a morphism in $\mathbf{V}$. Its canonical decomposition is $f^c.f^i$. We have the diagram:

$$\begin{tikzcd}
\Pi' \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \Pi \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & mR'_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
X \arrow[r, "f^i"'] \arrow[dr, dashed] & Z \arrow[r, "f^c"'] \arrow[d, "\bar{r}"] \arrow[dr, phantom, "(*)"] & R'_0 \arrow[d, "\rho'"] \\
 & Q \arrow[r, "\bar{f}^c"'] & Q'
\end{tikzcd}$$

where $\bar{f}^c.\bar{r}$ *is the canonical decomposition of* $\rho'.f^c$. The square (*) is a pullback (a pair of parallel edges is c-cartesian, the other one c-invertible). Then $\bar{r}$ is a c-invertible regular epimorphism. Π is he vertex of its kernel pair, which determines an equivalence relation Z_1 and a morphism ϕ_1: $Z_1 \to R'_1$ which is a discrete fibration such that $\phi_0 = f^c$ is c-cartesian. It is (Lemma 1) c_0-cartesian. Π' is the vertex of the kernel pair of $\bar{r}.f^i$ which determines an equivalence relation X_1 and a functor ψ_1: $X_1 \to Z_1$ which is internally fully faithful in the fibre $\mathrm{Fib}_c[cQ]$, that is c_0-cartesian.

Now $\bar{c} = c.c_0$: $\mathrm{Rel}_c\mathbf{V} \to \mathbf{W}$ admits $\bar{d} = G_c.d = \mathrm{dis}.d$ as a fully faithful right adjoint. It is a fibred reflexion as a composite of fibred reflexions. The functor dis: $\mathbf{V} \to \mathrm{Rel}_c\mathbf{V}$ is cartesian above $\mathbf{W}$: it preserves cartesian morphisms. Now, if c is Barr-exact, the functor dis has a left adjoint q_c: $\mathrm{Rel}_c\mathbf{V} \to \mathbf{V}$. It is clear that $c.q_c$ is naturally isomorphic to $\bar{c}$.

The aim of this section is to show that q_c is again a Barr-exact fibration and to characterize the q_c-cartesian morphisms.

PROPOSITION 4. *The functor q_c is a fibred reflexion.*

PROOF. Given a c-discrete equivalence relation R'_1 and a morphism h: $V \to q_cR'_1$ in $\mathbb{V}$, the pullback along h in $\mathbb{V}$ does exist by the universality condition and it determines a c-discrete equivalence relation R_1 with a functor h_1: $R_1 \to R'_1$, which, by construction, is q_c-cartesian. •

PROPOSITION 5. *The functor q_c is cartesian between $\bar{c}$ and c: the image by q_c of a $\bar{c}$-cartesian morphism is always c-cartesian. Moreover a $\bar{c}$-cartesian morphism is necessarily a q_c-cartesian morphism.*

PROOF. As the fibration $\bar{c}$ is, up to isomorphism, the composite of the two fibrations $c.q_c$, a $\bar{c}$-cartesian morphism is just a q_c-cartesian morphism above a c-cartesian one. •

PROPOSITION 6. *A morphism f_1: $R_1 \to R'_1$ is q_c-cartesian iff it is a discrete fibration.*

PROOF. For any h: $V \to V'$ in $\mathbb{V}$, the morphism dis$h$ is a discrete fibration. Then if the following diagram is a pullback, f_1 is a discrete fibration:

$$\begin{tikzcd} R_1 \arrow[rr, "f_1"] \arrow[d] & & R'_1 \arrow[d] \\ \mathrm{dis}qR_1 \arrow[rr, "\mathrm{dis}qf_1"'] & & \mathrm{dis}qR'_1 \end{tikzcd}$$

Conversely, let f_1: $R_1 \to R'_1$ be a discrete fibration, and $\psi_1.\phi_1$ its canonical decomposition with ψ_1 $\bar{c}$-cartesian and ϕ_1 $\bar{c}$-invertible. By Proposition 5, the functor ψ_1 is q_c-cartesian and therefore a discrete fibration. Thus ϕ_1 is a discrete fibration, which lies in the Barr-exact fibre $\mathrm{Fib}_c[cR_0]$. Hence ϕ_1 is q_c-cartesian (see [5] Lemma 4) and f_1 as $\psi_1.\phi_1$ is q_c-cartesian. •

REMARK. A q_c-invertible morphism is always a $\bar{c}$-invertible morphism.

PROPOSITION 7. *The functor q_c: $\mathrm{Rel}_c\mathbb{V} \to \mathbb{V}$ is itself a Barr-exact fibred reflexion.*

PROOF. Let us consider the fibration $\bar{c}$: Rel $\mathbf{V} \to \mathbf{W}$. For any object W in $\mathbf{W}$, the fibre $\mathrm{Fib}_{\bar{c}}[W]$ is the category $\mathrm{Rel}(\mathrm{Fib}_c[W])$ and the restriction of q_c to $\mathrm{Fib}_c[W]$ is just the quotient functor

$$q\colon \mathrm{Rel}(\mathrm{Fib}_c[W]) \to \mathrm{Fib}_c[W]$$

relative to the Barr-exact category $\mathrm{Fib}_c[W]$.

Now for any object V of $\mathrm{Fib}_c[W]$,. the fibre $\mathrm{Fib}_{q_c}[V]$ is $\mathrm{Fib}_q[V]$ which is Barr-exact following Proposition 3. Thus the quotients of the q_c-invertible equivalence relations do exist and are componentwise. These q_c-invertible quotients, being componentwise, are preserved by pullbacks because of the universality conditions given by the Barr-exactness of the fibration c. •

REMARK. Thus, by Lemma 1, the functor q_c preserves the pullbacks in which one edge is a discrete fibration.

3. THE FUNCTOR π_c FOR c-DISCRETE GROUPOIDS.

In the same way as in the absolute situation ($\mathbf{E}$ is a Barr-exact category) [5], in the relative case (c a Barr-exact fibration), the functor q_c: $\mathrm{Rel}_c\mathbf{V} \to \mathbf{V}$ can be extended to a functor π_c: $\mathrm{Grd}_c\mathbf{V} \to \mathbf{V}$, left adjoint to the functor dis: $\mathbf{V} \to \mathrm{Grd}_c\mathbf{V}$ where $\mathrm{Grd}_c\mathbf{V}$ is the category of c-discrete groupoids in $\mathbf{V}$. But, the category $\mathbf{V}$ being not supposed left exact, the functor $()_0$: $\mathrm{Grd}_c\mathbf{V} \to \mathbf{V}$ is not, a priori, a fibred reflexion and it is not possible to use the same argument. The aim of this section is to give a construction of π_c and to establish its properties.

The construction of π_c. Let X_1 be a c-discrete groupoid and denote by $\lambda_1 X_1$ the canonical projection $X_1 \to G_c X_0$. Then $(\lambda_1 X_1)_0 = 1_{X_0}$ and $m(\lambda_1 X_1)$: $mX_1 \to X_0 \times_c X_0$ is the factorization of the pair

$$(d_0, d_1)\colon mX_1 \rightrightarrows X_0$$

in the fiber $\mathrm{Fib}_c[cX_0]$. It is a c-invertible morphism. Its canonical decomposition is denoted by $\psi.\phi$, with ϕ a c-invertible regular epimorphism and ψ a c-invertible monomorphism. Whence the following diagram:

$$\begin{tikzcd} mX_1 \arrow[r, two heads, "\phi"] \arrow[dr, shift right, "d_0"'] \arrow[dr, shift left, "d_1"] & T \arrow[r, tail, "\psi"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & X_0 \times_c X_0 \arrow[dl, shift right, "d_0"'] \arrow[dl, shift left, "d_1"] \\ & X_0 & \end{tikzcd}$$

Now if T' is the vertex of the kernel pair of d_1: $T \to X_0$, we get (X_1 and G_cX_0 being two groupoids) two morphisms

$$m_2X_1 \xrightarrow{\phi'} T' \xrightarrow{\psi'} X_0\times_c X_0\times_c X_0$$

with ϕ' a c-invertible regular epimorphism and ψ' a c-invertible monomorphism. It is then possible to complete the following diagram in such a way that the vertical central diagram is a c-discrete groupoid Z_1:

$$\begin{tikzcd}
m_2X_1 \arrow[r, two heads, "\phi'"] \arrow[d, shift right=3, "d_0"'] \arrow[d, "d_1"] \arrow[d, shift left=3, "d_2"] & T' \arrow[r, tail, "\psi'"] \arrow[d, shift right=3, "d_0"'] \arrow[d, "d_1"] \arrow[d, shift left=3, "d_2"] & X_0\times_c X_0\times_c X_0 \arrow[d, shift right=3, "d_0"'] \arrow[d, "d_1"] \arrow[d, shift left=3, "d_2"] \\
mX_1 \arrow[r, two heads, "\phi"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & T \arrow[r, tail, "\psi"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & X_0\times_c X_0 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
X_0 \arrow[r, "1_{X_0}"'] & X_0 \arrow[r, "1_{X_0}"'] & X_0
\end{tikzcd}$$

Now ψ being a monomorphism, Z_1 is an equivalence relation. This construction determines a functor

$$c_0\text{-supp}: \mathrm{Grd}_c\mathbf{V} \to \mathrm{Rel}_c\mathbf{V}$$

(the c_0-support functor) which is a left adjoint to the inclusion i: $\mathrm{Rel}_c\mathbf{V} \to \mathrm{Grd}_c\mathbf{V}$. On the other hand, the fibred reflexion c being Barr-exact and a c-invertible regular epimorphism having a pullback along any morphism in $\mathbf{V}$, the functor c_0-supp is again a fibred reflexion.

REMARK. The functor c_0: $\mathrm{Grd}_c\mathbf{V} \to \mathbf{V}$ being equal to

$$\mathrm{Grd}_c\mathbf{V} \xrightarrow{c_0\text{-supp}} \mathrm{Rel}_c\mathbf{V} \xrightarrow{c_0} \mathbf{V}$$

we can prove, by Lemma 4, that this functor c_0: $\mathrm{Grd}_c\mathbf{V} \to \mathbf{V}$ is again a fibred reflexion. Whence a functor

$$\pi_c = q_c.c_0\text{-supp}: \mathrm{Grd}_c\mathbf{V} \longrightarrow \mathbf{V}$$

left adjoint to dis: $\mathbf{V} \to \mathrm{Grd}_c\mathbf{V}$, which is a fibred reflexion as a composite of fibred reflexions. All the elements of this construction dealing only with c-invertible morphisms, there is a natural isomorphism between $c.\pi_c$ and $\bar{c}$.

We are now going to characterize the π_c-cartesian morphisms.

PROPOSITON 8. *The functor π_c is cartesian between $\bar{c}$ and c: the image by π_c of any $\bar{c}$-cartesian morphism is c-cartesian. Moreover every $\bar{c}$-cartesian morphism is π_c-cartesian.*

PROOF. The functor $c.\pi_c$ is $\bar{c}$ up to isomorphism. All these functors being fibrations, a $\bar{c}$-cartesian morphism f_1 is exactly a π_c-cartesian morphism such that $\pi_c(f_1)$ is c-cartesian. •

PROPOSITION 9. *A functor f_1: $X_1 \to Y_1$ in $\mathrm{Grd}_c\mathsf{V}$ is π_c-cartesian iff f_1 and c_o-supp(f_1) are discrete fibrations.*

PROOF. A π_c-cartesian morphism is exactly a c_o-supp-cartesian morphism such that c_o-supp(f_1) is q_c-cartesian. That means that c_o-supp(f_1) is a discrete fibration and that the following square (*) is a pullback:

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d] \\ c_o\text{-supp}X_1 \arrow[r, "c_o\text{-supp}(f_1)"'] & c_o\text{-supp}Y_1 \end{tikzcd}$$

The lower functor being a discrete fibration, the square (*) is a pullback iff f_1 is a discrete fibration, since the vertical arrows are c_o-invertible. •

Thus, starting from a fibred reflexion c, we have obtained the following commutative diagram of cartesian adjunctions between the fibred reflexions c and $\bar{c}$.

$$\begin{tikzcd} \mathrm{Grd}_c\mathsf{V} \arrow[rr, shift left=4.5, leftarrow, "G_c"] \arrow[rr, shift left=1.5, "C_o"] \arrow[rr, shift right=1.5, leftarrow, "dis"] \arrow[rr, shift right=4.5, "\pi_c"'] \arrow[dr, "\bar{c}"'] & & \mathsf{V} \arrow[dl, "c"] \\ & \mathsf{W} & \end{tikzcd}$$

REMARK. The functor π_c is a fibred reflexion but is no more Barr-exact as it is the case for q_c. It is not even weakly left exact. To see that, we consider the canonical presentation of an internal groupoid X_1 in any Barr-exact category $\mathbf{E}$ [5]:

$$\mathrm{Dec}^2X_1 \underset{\mathrm{Dec}\epsilon X_1}{\overset{\epsilon \mathrm{Dec}X_1}{\rightrightarrows}} \mathrm{Dec}X_1 \xrightarrow{\epsilon X_1} X_1$$

The internal functor ϵX_1 is a discrete fibration. It is π_0-cartesian iff X_1 is an equivalence relation. If not, let us denote by $\tau_1.\sigma_1$ the canonical decomposition of ϵX_1 with τ_1 π_0-cartesian and σ_1 π_0-invertible. As π_0-cartesian, the functor τ_1 is a discrete fibration, then σ_1 is also a discrete fibration. The kernel pair of σ_1 lies in Rel $\mathbf{E}$ since $\mathrm{Dec}X_1$ is in Rel $\mathbf{E}$. Its projections being discrete fibrations, this kernel pair cannot be π_0-invertible (if not X_1 would be certainly an equivalence relation).

III. THE c-FULL MORPHISMS.

1. DEFINITIONS AND FIRST PROPERTIES.

Let c be a Barr-exact fibred reflexion.

DEFINITION 3. A morphism f: $V \to V'$ in $\mathbf{V}$ is said to be *c-faithful* when its c-invertible part f^i is a monomorphism and *c-full* when its c-invertible part f^i is a regular epimorphism.

EXAMPLE. This terminology is suggested by our first main example: if $\mathbf{E}$ is Barr-exact and left exact, the $(\ )_0$-faithful and the $(\ )_0$-full functors are just the internally faithful and the internally full functors.

The class of c-full morphisms will be denoted by c-Full.

Properties of c-Full:

1. An isomorphism is c-full.
2. The composite of two c-full morphisms is c-full.

To see that, we consider the following diagram, where $\overline{f^c}.\overline{g^i}$ is the canonical decomposition of $g^i.f^c$. The square (*) is a pullback since the horizontal edges are c-cartesian and the vertical ones are c-invertible. Consequently $\overline{g^i}$ is a regular epimorphism when g^i is a regular epimorphism and $g.f$ is c-full when g and f are c-full.

$$\begin{tikzcd}
V \arrow[d, "f^i"'] \arrow[dr, "f"] & & \\
U \arrow[r, "f^c"'] \arrow[d, "\bar{g}^i"'] \arrow[dr, phantom, "(*)"] & V' \arrow[d, "g^i"] \arrow[dr, "g"] & \\
U' \arrow[r, "\bar{f}^c"'] & U'' \arrow[r, "g^c"'] & V''
\end{tikzcd}$$

3. **PROPOSITION 10.** *The c-full morphisms are stable under pullbacks whenever they exist. Moreover such pullbacks are preserved by c.*

PROOF. Let us consider the following pullback where $f^c.f^i$ is the canonical decomposition of a c-full morphism f :

$$\begin{tikzcd}
U \arrow[r, "f'^i"] \arrow[d, "h"'] \arrow[dr, phantom, "(1)"] & Y \arrow[r, "f'^c"] \arrow[d, "\ell"] \arrow[dr, phantom, "(2)"] & U' \arrow[d, "k"] \\
V \arrow[r, two heads, "f^i"'] & Z \arrow[r, "f^c"'] & V'
\end{tikzcd}$$

Then if $f'^c.f'^i$ is the canonical decomposition of f', the diagonality condition gives us a morphism ℓ: $Y \to Z$ making the two squares commutative. Now we consider the pullback of f^i along ℓ which does exist since c is Barr-exact and f^i is a c-invertible regular epimorphism:

$$\begin{tikzcd}
U \arrow[dr, "\psi"] \arrow[drr, "f'^i"] \arrow[dd, "h"'] & & \\
& \bar{V} \arrow[r, two heads, "\phi^i"'] \arrow[dl, "\bar{\ell}"] & Y \arrow[d, "\ell"] \\
V \arrow[rr, two heads, "f^i"'] & & Z
\end{tikzcd}$$

Then ϕ^i is a c-invertible regular epimorphism, and f'^i being c-invertible, the factorization ψ: $U \to \bar{V}$ is c-invertible. The above square ((1)+(2)) being a pullback, there is a unique χ: $\bar{V} \to U$ such that

$$h.\chi = \bar{\ell} \quad \text{and} \quad f'^c.f'^i.\chi = f'^c.\phi^i.$$

It is clear that $\chi.\psi = 1$. As ψ is c-invertible, we have $c(\chi) = c(\psi)^{-1}$.

Let us prove that $\psi.\chi = 1$. For that we must prove that $\phi^{i}.\psi.\chi = \phi^{i}$. But

$$f'^{c}.\phi^{i}.\psi.\chi = f'^{c}.f'^{i}.\chi = f'^{c}.\phi^{i}.$$

Then, f'^{c} being c-cartesian, it is sufficent to prove that $c\psi.c\chi = 1$. That is true.

Hence the square (1) is a pullback. f'^{i} a c-invertible regular epimorphism and $f' = f'^{c}.f'^{i}$ a c-full morphism.

Let R_1 and R'_1 be the c-discrete kernel equivalences associated to f^{i} and f'^{i}. The morphisms h and ℓ determine a morphism h_1: $R_1 \to R'_1$ which is a discrete fibration since the square (1) is a pullback. That the square ((1)+(2)) is a pullback implies that the following square is a pullback in $\mathrm{Rel}_c\mathbf{V}$:

$$\begin{tikzcd} R'_1 \arrow[r] \arrow[d, "h_1"'] & \mathrm{dis}U' \arrow[d, "\mathrm{dis}k"] \\ R_1 \arrow[r] & \mathrm{dis}V' \end{tikzcd}$$

where the two vertical edges are discrete fibrations and thus q_c-cartesian morphisms. Consequently, following Proposition 6 and Lemma 1, this pullback is preserved by q_c and the square (2) is a pullback. The pullback (1) is preserved by c since f^{i} and f'^{i} are c-invertible, and the pullback (2) is preserved by c since f^{c} and f'^{c} are c-cartesian (again by Lemma 1). •

REMARK. It is very surprising that, when c is a Barr-exact fibred reflexion, the functor c, although being not supposed to be left exact, preserves such pullbacks. The pullbacks with one edge a c-invertible monomorphism are not preserved in general. The obstruction to the total left exactness of c is thus only due, for any morphism f: $V \to V'$ in $\mathbf{V}$, to the c-invertible monomorphism part of f^{i}.

In particular, this result is true for the quotient functor q: Rel $\mathbf{E} \to \mathbf{E}$ in a Barr-exact category $\mathbf{E}$, which therefore appears to preserve (besides products) a large number of pullbacks.

We are now going to establish a proposition which we need later on and which is a generalization of Proposition 8 and a kind of particular case of Proposition 10.

PROPOSITION 11. *Let f_1: $X_1 \to Y_1$ be an internal functor in* $\mathrm{Grd}_c\mathbf{V}$ *such that f_1 is C_0-cartesian and f_0 c-full. Then $\pi_c(f_1)$ is c-cartesian. Such morphisms are stable under pullbacks (whenever they exist) and such pullbacks are preserved by π_c.*

PROOF. Let $\psi_1.\phi_1$ be the canonical decomposition of f_1 with ϕ_1 a $\bar{c}$-invertible and ψ_1 a $\bar{c}$-cartesian functor. Following Proposition 1, ψ_1 is C_0-cartesian and consequently such is ϕ_1. On the other hand $\pi_c(\psi_1)$ is, following Proposition 8, c-cartesian.

Now ϕ_1 is a C_0-cartesian morphism in the fiber $\mathrm{Fib}_c[cX_0]$, then $\pi_c(\phi_1)$ is a c-invertible monomorphism. The morphism ϕ_0 being a c-invertible regular epimorphism (f_0 c-full), $\pi_c(\phi_1)$ is also a c-invertible regular epimorphism. Thus $\pi_c(\phi_1)$ is an isomorphism and $\pi_c(f_1) = = \pi_c(\psi_1).\pi_c(\phi_1)$ is c-cartesian.

The functor ϕ_1 is π_c-invertible. On the other hand the morphism f_0 being c-full and ϕ_1 being also C_0-cartesian, this functor ϕ_1 is a regular epimorphism in $\mathrm{Grd}_c\mathbf{V}$. Thus, although the fibration π_c is not Barr-exact, the functor f_1 appears to be a π_c-full morphism.

It is then possible to mimic Proposition 10. For that let us consider the following pullback where ϕ'_1 is $\bar{c}$-invertible and ψ'_1 is $\bar{c}$ - cartesian:

$$\begin{tikzcd}
X'_1 \arrow[r, "\phi'_1"] \arrow[d, "k_1"'] \arrow[dr, phantom, "(1)"] & Z'_1 \arrow[r, "\psi'_1"] \arrow[d, "\ell_1"] \arrow[dr, phantom, "(2)"] & Y'_1 \arrow[d, "k_1"] \\
X_1 \arrow[r, "\phi_1"'] & Z_1 \arrow[r, "\psi_1"'] & Y_1
\end{tikzcd}$$

Then, by the diagonality condition, there is a functor ℓ_1: $Z'_1 \to Z_1$ making the two squares commutative. If $f_1 = \psi_1.\phi_1$ is C_0-cartesian, such is $f'_1 = \psi'_1.\phi'_1$. Since ψ_1 and ψ'_1 are again C_0-cartesian (Proposition 1), all the horizontal arrows are C_0-cartesian. The image by C_0 of the given square (1)+(2) is also a pullback with the edge $f_0 = \psi_0.\phi_0$ c-full, hence $f'_0 = \psi'_0.\phi'_0$ is c-full and the functor f'_1 is C_0-cartesian and f'_0 c-full.

On the other hand, following Proposition 10, the image by C_0 of the squares (1) and (2) are pullbacks. Therefore the horizontal arrows being C_0-cartesian, the squares (1) and (2) are themselves pullbacks. The square $\pi_c(2)$ is a pullback (Proposition 8 and Lemma 1). The morphisms $\pi_c(\phi_1)$ and $\pi_c(\phi'_1)$ being isomorphisms, the square $\pi_c(1)$ is a pullback. •

IV. THE MAIN RESULT: c-FULL MORPHISMS AND STACKS.

1. STACKS.

A class Σ of morphisms in a weakly left exact category $\mathbb{W}$ will be called a *proper class* if it satisfies the following conditions:

1. every isomorphism is in Σ,
2. Σ is stable under composition,
3. the pullback of a morphism in Σ along any morphism in $\mathbb{W}$ does exist and is again in Σ.

EXAMPLES. The examples we have in mind are the following:
When c is a left exact fibred reflexion:
1. the class of c-invertible morphisms,
2. the class of c-cartesian morphisms.

When c is a Barr-exact fibred reflexion:
3. the class of c-invertible regular epimorphisms.

When c is a left exact and Barr-exact fibred reflexion:
4. the class c-Full of c-full morphisms.

When $\mathbb{E}$ is left exact:
5. the class of discrete fibrations.

The proper class Γ will be called *topologically proper* when, furthermore, every morphism in Γ is a regular epimorphism (a coequalizer of its kernel pair). This last definition is given to yield a Grothendieck topology in $\mathbb{W}$ (also denoted by Γ).

DEFINITION 4. A Σ-*groupoid* (resp. a Σ-*equivalence relation*) in $\mathbb{W}$ is a groupoid X_1 (resp. an equivalence relation) in $\mathbb{W}$ such that the pair (d_0,d_1): $mX_1 \rightrightarrows X_0$ is in Σ.

A Σ-*exact diagram* is an exact diagram in which every morphism is in Σ.

Given a topologically proper class Γ in $\mathbb{W}$, we recall that an equivalent condition for a fibration c: $\mathbb{V} \to \mathbb{W}$ to be a stack [11,12] for the topology Γ is the conjunction of the two following properties:

1. every c-cartesian diagram above a Γ-exact diagram is exact,
2. every c-cartesian equivalence relation above a Γ-equivalence relation, part of a Γ-exact diagram, can be completed in a c-cartesian diagram above this Γ-exact diagram (see [2]).

The aim of this section is mainly to show that if c is, at the same time, a Barr-exact fibred reflexion and a stack for a topology

Γ, the property 2 for stacks can be extended from c-cartesian equivalence relations to c-full equivalence relations. More roughly: something more general than a descent data can even be descended.

EXAMPLES. Our two main examples are stacks for the regular epimorphism topology (where Γ is the class of all the regular epimorphisms):

1. That, if $\mathbf{E}$ is left exact and Barr-exact, the fibred reflexion $(\ )_0$: Cat $\mathbf{E} \to \mathbf{E}$ is a stack for the regular epimorphism topology is shown in [2].

2. **PROPOSITION 12.** *If* $\mathbf{E}$ *is Barr-exact, the quotient functor* q*:* Rel $\mathbf{E} \to \mathbf{E}$ *is a stack for the regular epimorphism topology.*

PROOF. It is clear that a q-cartesian diagram above an exact diagram is a componentwise exact diagram in Rel $\mathbf{E}$ and consequentely is an exact diagram in Rel $\mathbf{E}$.

Let $\mathbf{R}_1$ be an equivalence relation in Rel $\mathbf{E}$ such that every structural map is q-cartesian and its image by q is an equivalence relation (it is certainly a groupoid, but not in general an equivalence relation). To simplify, we denote $\mathbf{R}_0$ by S_1 and $m\mathrm{R}_1$ by T_1. Whence the following diagram in $\mathbf{E}$:

$$\begin{tikzcd}[column sep=huge, row sep=large]
mT_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & mS_1 \arrow[l] \arrow[l, phantom, shift left=3, "\xrightarrow{m\alpha_1}"] \arrow[l, phantom, shift right=3, "\xrightarrow{m\beta_1}"'] \arrow[r, two heads, "m\rho_1"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & mQ_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
T_0 \arrow[d, "\rho_{T_1}"'] & S_0 \arrow[l] \arrow[l, phantom, shift left=3, "\xrightarrow{\alpha_0}"] \arrow[l, phantom, shift right=3, "\xrightarrow{\beta_0}"'] \arrow[r, two heads] \arrow[d, "\rho_{S_1}"] & Q_0 \arrow[d, "\rho"] \\
qT_1 & qS_1 \arrow[l] \arrow[l, phantom, shift left=3, "\xrightarrow{q\alpha_1}"] \arrow[l, phantom, shift right=3, "\xrightarrow{q\beta_1}"'] \arrow[r, two heads] & K
\end{tikzcd}$$

where K and Q_0 denote the quotient of the equivalence relations, image of $\mathbf{R}_1$ by the functors q and $(\ )_0$. Since β_1 is q-cartesian, the morphism $\overline{\rho_1}$: $(\mathbf{R}_1)_0 \twoheadrightarrow q\mathbf{R}_1$ determined by ρ_{S_1} and ρ_{T_1} is a discrete fibration and consequently q-cartesian. Then its kernel pair is preserved by q and determines an equivalence relation Q_1, by means of the factorizations (d_0,d_1): $mQ_1 \rightrightarrows Q_0$, and a componentwise quotient morphism ρ_1: $S_1 \twoheadrightarrow Q_1$ which is a discrete fibration and thus q-cartesian. ♦

2. *THE* c-FULL MORPHISMS AND THE STACKS.

From now on, $c: \mathbf{V} \to \mathbf{W}$ will be supposed to be a Barr-exact fibred reflexion and a stack for a topology Γ.

DEFINITION 5. A morphism $f: V \to V'$ is called a c-Γ-morphism if f is c-full and $c(f)$ is in Γ; the class of c-Γ-morphisms is denoted c-Γ.

PROPOSITION 13. *A c-Γ-morphism f is a regular epimorphism.*

PROOF. The morphism f being in c-Γ, its c-cartesian part f^c is a regular epimorphism since c is a stack and its c-invertible part f^i is a regular epimorphism, since f is c-full, which is stable under pullbacks since c is Barr-exact; hence $f = f^c.f^i$ is a regular epimorphism. •

PROPOSITION 14. *The class c-Γ is proper. Moreover any pullback with an edge in c-Γ is preserved by c.*

PROOF. Condition 1 is obviously satisfied. Now if f and g are in c-Γ, $g.f$ is c-full and $c(g.f) = cg.cf$ is in Γ. Let $f: V \to V'$ be a c-Γ-morphism and $k: U' \to V'$ any morphism in $\mathbf{V}$. The pullback of $c(f)$ along $c(k)$ does exist in $\mathbf{W}$ since $c(f)$ is in Γ, and consequently the pullback of the c-cartesian morphism f^c above $c(f)$ along k. Since f^i is a c-invertible regular epimorphism, its pullback along any morphism does exist, hence the pullback of f along k exists:

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "h"'] & U' \arrow[d, "k"] \\ V \arrow[r, "f"'] & V' \end{tikzcd}$$

Following Proposition 10, f' is c-full and the image by c of this square is a pullback in $\mathbf{W}$. Then cf' is in Γ according to condition 3, and f' is in c-Γ. •

COROLLARY. *If Γ is a topologically proper class in $\mathbf{W}$ and $c: \mathbf{V} \to \mathbf{W}$ a Barr-exact fibred reflexion which is a stack for the topology Γ, then c-Γ is a topologically proper class in $\mathbf{V}$.*

REMARK. Proposition 13 means that any left exact c-full diagram above a Γ-exact diagram is exact. It can be seen as an extension of the property 1 for a stack from c-cartesian diagrams to left exact

c-full diagrams. The fact that these diagrams must be left exact is only an apparent restriction since any c-cartesian diagram above a left exact diagram is always left exact.

3. THE c-DISCRETE GROUPOID ASSOCIATED TO A c-Γ-GROUPOID.

It is much more difficult, and essential for us, to extend property 2 for a stack from c-cartesian equivalence relations to c-full equivalence relations.

Let X_1 be a c-Γ-groupoid in $\mathbf{V}$. Then d_0 and d_1 are c-full, and, following Proposition 10, its image cX_1 by the functor c is again a groupoid.

PROPOSITION 15. *Every c-Γ-groupoid X_1 has an associated c-discrete groupoid $X_1^{\sim}$. If X_1 is an equivalence relation, such is $X_1^{\sim}$.*

PROOF. Consider the following pullback in Grd $\mathbf{V}$:

$$\begin{tikzcd} X_1^{\sim} \arrow[r, tail, "\alpha_1 X_1"] \arrow[d] & X_1 \arrow[d] \\ \mathrm{dis}(dcX_0)=d(\mathrm{dis}\,cX_0) \arrow[r, tail] & dcX_1 \end{tikzcd}$$

It does exist as a componentwise pullback since the internal functor $X_1 \to dcX_1$ is componentwise c-invertible. The $X_1^{\sim}$ is a c-discrete category since $cX_1^{\sim}$ is isomorphic to $\mathrm{dis}(cX_0)$ and it is easy to check that this construction $(\ )^{\sim}$ is a right adjoint to the inclusion i: $\mathrm{Grd}_c\mathbf{V} \to \mathrm{Grd}_{c\text{-}\Gamma}\mathbf{V}$, where $\mathrm{Grd}_{c\text{-}\Gamma}\mathbf{V}$ is the full subcategory of Grd $\mathbf{V}$ whose objects are the c-Γ-groupoids. By construction $m(\alpha_1 X_1)$: $mX_1^{\sim} \to mX_1$ is c-cartesian above $c(s_0)$: $cX_0 \rightarrowtail cmX_1$ and thus it is a monomorphism. If X_1 is an equivalence relation, then the pair (d_0,d_1): $mX_1 \rightrightarrows X_0$ is jointly monic, thus the pair (d_0,d_1): $mX_1^{\sim} \rightrightarrows X_0$ is jointly monic and $X_1^{\sim}$ is an equivalence relation. •

Let us now consider the following commutative triangle:

$$\begin{tikzcd} \mathrm{Grd}_{c\text{-}\Gamma}\mathbf{V} \arrow[rr, "(\ )^{\sim}"] \arrow[dr, "(\ )_0"'] & & \mathrm{Grd}_c\mathbf{V} \arrow[dl, "c_0"] \\ & \mathbf{V} & \end{tikzcd}$$

The functor $(\ )_0$ is no more a reflexion nor a fibration. However there are two classes of morphisms which are of some interest for us in $\mathrm{Grd}_{c\text{-}\Gamma}\mathbf{V}$: the discrete fibrations and the internally fully faithful functors.

PROPOSITION 16. *The functor* $(\ )^\sim$ *preserves the discrete fibrations.*

PROOF. Let $f_1 : X_1 \to Y_1$ be a discrete fibration, then the following square is a pullback:

$$\begin{tikzcd} mX_1 \arrow[r, "mf_1"] \arrow[d, "d_1"'] & mY_1 \arrow[d, "d_1"] \\ X_0 \arrow[r, "f_0"'] & Y_0 \end{tikzcd}$$

d_1 being in $c\text{-}\Gamma$ this pullback is preserved by c and the functor cf_1 is a discrete fibration. Hence the following square is a pullback:

$$\begin{tikzcd} cmX_1 \arrow[r, "cmf_1"] & cmY_1 \\ cX_0 \arrow[r, "cf_0"'] \arrow[u, "cs_0"] & cY_0 \arrow[u, "cs_0"'] \end{tikzcd}$$

and therefore, $m(\alpha_1 X_1)$ and $m(\alpha_1 Y_1)$ being c-cartesian above the morphisms cs_0, the following square is again a pullback, what implies that $f_1^\sim : X_1^\sim \to Y_1^\sim$ is a discrete fibration:

$$\begin{tikzcd} mX_1 \arrow[r, "mf_1"] & mY_1 \\ mX_1^\sim \arrow[r, "mf_1^\sim"'] \arrow[u, "m(\alpha_1 X_1)"] & mY_1^\sim \arrow[u, "m(\alpha_1 Y_1)"'] \end{tikzcd}$$

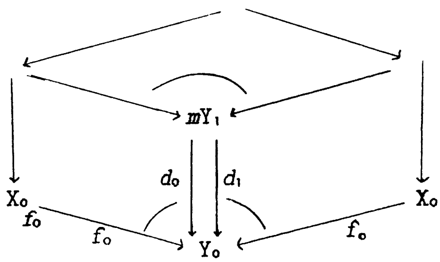
•

PROPOSITION 17. *Let* $f_1 : X_1 \to Y_1$ *be an internally fully faithful functor in* $\mathrm{Grd}_{c\text{-}\Gamma}\mathbf{V}$ *such that* f_0 *is in* $c\text{-}\Gamma$; *then its image by the functor* $(\ )^\sim$ *is* c_0-*cartesian.*

PROOF. That f_1 is internally fully faithful means that the following diagram is a joint pullback:

$$\begin{tikzcd} mX_1 \arrow[r, "mf_1"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & mY_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ X_0 \arrow[r, "f_0"'] & Y_0 \end{tikzcd}$$

We first remark that, the morphism f_0 being in c-Γ, this joint pullback can be constructed by means of three pullbacks in $\mathbf{V}$ with edges in c-Γ:



Therefore mf_1 is in c-Γ. These three pullbacks being preserved by c, the functor cf_1: $cX_1 \to cY_1$ is internally fully faithful in Grd $\mathbf{W}$.

Let $f_0{}^c.f_0{}^i$ be the canonical decomposition of f_0. It determines a decomposition $\psi_1.\phi_1$ of f_1 where ϕ_1: $X_1 \to Z_1$ is internally fully faithful and $\phi_0 = f_0{}^i$ is a c-invertible regular epimorphism and where ψ_1: $Z_1 \to Y_1$ is internally fully faithful and $\psi_0 = f_0{}^c$ is c-cartesian.

α) Let us prove that $\psi_1{}^\sim$ is c_0-cartesian. By our first remark $m\psi_1$ is again c-cartesian. We consider the two following diagrams in $\mathrm{Grd}_{c\text{-}\Gamma}\mathbf{V}$:

$$\begin{tikzcd} Z_1{}^\sim \arrow[r] \arrow[d, "\psi_1{}^\sim"'] \arrow[dr, phantom, "(1)"] & \mathrm{dis}(dcZ_0) \arrow[d, "\mathrm{dis}(dc\psi_0)"] \\ Y_1{}^\sim \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & \mathrm{dis}(dcY_1) \arrow[d] \\ Y_1 \arrow[r] & dcY_1 \end{tikzcd} \qquad \begin{tikzcd} Z_1{}^\sim \arrow[r] \arrow[d] \arrow[dr, phantom, "(3)"] & \mathrm{dis}(dcZ_0) \arrow[d, dashed] \\ Z_1 \arrow[r] \arrow[d, "\psi_1"'] \arrow[dr, phantom, "(4)"] & dcZ_1 \arrow[d, "dc\psi_1"] \\ Y_1 \arrow[r] & dcY_1 \end{tikzcd}$$

The square (1)+(2) is equal to the square (3)+(4). Now the squares (2) and (3) are pullbacks by construction. The square (4) is a componentwise pullback since ψ_0 and $m\psi_1$ are c-cartesian. Then the square (1) is a pullback, what means that $\psi_1{}^\sim$ is $\bar{c}$-cartesian. It is therefore c_0-cartesian (Proposition 1).

β) Let us prove that $\phi_1{}^\sim$ is c_0-cartesian. By our first remark $m\phi_1$ is again a c-invertible regular epimorphism. We consider the two following diagrams in $\mathbf{V}$:

$$\begin{tikzcd} mX_1{}^\sim \arrow[rr, "m\phi_1{}^\sim"] \arrow[d, "{(d_0,d_1)}"'] & (1) & mZ_1{}^\sim \arrow[d, "{(d_0,d_1)}"] \\ X_0\times_c X_0 \arrow[rr, "\phi_0\times_c\phi_0"'] \arrow[d, shift right] \arrow[d, shift left] & (2) & Z_0\times_c Z_0 \arrow[d, shift right] \arrow[d, shift left] \\ X_0 \arrow[rr, "\phi_0"'] & & Z_0 \end{tikzcd} \qquad \begin{tikzcd} mX_1{}^\sim \arrow[rr, "m\phi_1{}^\sim"] \arrow[d] & (3) & mZ_1{}^\sim \arrow[d] \\ mX_1 \arrow[rr, "m\phi_1"'] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & (4) & mZ_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ X_0 \arrow[rr, "\phi_0"'] & & Z_0 \end{tikzcd}$$

The double square (1)+(2) is equal to the double square (3)+(4). The double square (4) is a joint pullback since ϕ_1 is internally fully faithful. The double square (2) is a joint pullback since ϕ_0 is c-invertible. The square (3) is a pullback since its vertical edges are c-cartesian and its horizontal ons are c-invertible. Consequently the square (1) is a pullback and $\phi_1{}^\sim$ is c_0-cartesian. •

4. THE UNIVERSAL REPRESENTATIVE OF THE INTERNAL NATURAL TRANSFORMATIONS.

Let $\mathbf{E}$ be a weakly left exact category and X_1 an internal category in $\mathbf{E}$. The standard simplex [1] is actually a category and it is clear that $X_1{}^{[1]}$ (the cotensor of the internal category X_1 by [1]) is the domain of the universal internal natural transformation with codomain X_1 (see [14]). This internal category will be called the *universal representative* of the natural transformations and denoted by Com X_1. In the category Set of sets, the objects of Com X_1 are the morphisms of X_1, and its morphisms are the commutative squares ("quatuors" in [9]).

Whence the following diagram, with the universal natural transformation γ: $\delta_0 \Rightarrow \delta_1$:

$$\begin{tikzcd} \mathrm{Com}\ X_1 \arrow[r, bend left, "\delta_0"] \arrow[r, bend right, "\delta_1"'] \arrow[r, phantom, "\gamma\Downarrow"] & X_1 \end{tikzcd}$$

The trivial identity natural transformation between the identity morphisms on X_1 and itself yields a σ_0: $X_1 \to \mathrm{Com}\ X_1$ such that

$$\delta_0.\sigma_0 = 1_{X_1} = \delta_1.\sigma_0.$$

Furthermore the universal property of Com X_1 makes δ_0 a left adjoint to σ_0 and δ_1 a right adjoint. On the other hand the construction Com X_1 extends to a 2-functor Com: Cat $\mathbf{E} \to$ Cat $\mathbf{E}$. If the category X_1 is c-discrete, then Com X_1 is c-discrete. If X_1 is a groupoid, then Com X_1 is a groupoid.

In this last case, there is a very strong connexion between the 2-categorical structure of Grd $\mathbf{E}$ and the fibration $(\ )_0$: Grd $\mathbf{E} \to \mathbf{E}$.

PROPOSITION 18. *An internal category* X_1 *is an internal groupoid iff* δ_1: Com $X_1 \to X_1$ *(or equivalently* δ_0*) is* $(\ )_0$*-cartesian above* d_1: $mX_1 \to X_0$ *(resp.* d_0*).*

PROOF. If X_1 is a groupoid, then δ_1 being a right adjoint between two groupoids is an equivalence and thus internally fully faithful, that is $(\ )_0$-cartesian. The converse is pure diagram chasing. •

In the same way, when c: $\mathbf{V} \to \mathbf{W}$ is a weakly left exact fibred reflexion, we have the following result:

COROLLARY. *A c-discrete category* X_1 *is a c-discrete groupoid iff* δ_1: Com $X_1 \to X_1$ *is* c_σ*-cartesian.*

REMARK. If X_1 is an internal groupoid in a weakly left exact category $\mathbf{E}$ then $[\delta_0,\delta_1]$: Com $X_1 \to X_1 \times X_1$ is a discrete fibration.

This result is clearly true in Set and consequently in any weakly left exact category $\mathbf{E}$ via the Yoneda embedding.

5. THE c-CARTESIAN GROUPOID ASSOCIATED TO A c-Γ-GROUPOID.

Let X_1 be a c-Γ-groupoid in $\mathbf{V}$ and let us consider the following internal groupoid in $\mathrm{Grd}_{c\text{-}\Gamma}\mathbf{V}$:

$$\begin{tikzcd} X_1 \arrow[r, "\sigma_0" description] & \mathrm{Com}\ X_1 \arrow[l, shift right=8, "\delta_0"'] \arrow[l, shift left=8, "\delta_1"] & \mathrm{Com}_2 X_1 \arrow[l, shift right=6, "\delta_0"'] \arrow[l, "\delta_1" description] \arrow[l, shift left=6, "\delta_2"] \end{tikzcd}$$

where Com_2X_1 is the universal representative of the triangles of natural transformations (namely $X_1^{[2]}$). The functor $(\ )^{\sim}$ is left exact and yields an internal groupoid in $Grd_c\mathbf{V}$:

$$\begin{tikzcd} X_1^{\sim} \arrow[r, shift left=2, leftarrow, "\delta_0^{\sim}"] \arrow[r] \arrow[r, shift right=2, leftarrow, "\delta_1^{\sim}"'] & (Com\ X_1)^{\sim} \arrow[r, shift left=2, leftarrow] \arrow[r, leftarrow] \arrow[r, shift right=2, leftarrow] & (Com_2X_1)^{\sim} \end{tikzcd}$$

Now δ_0 and δ_1 are internally full and faithful, moreover $(\delta_0)_0 = d_0$ and $(\delta_1)_0 = d_1$ are in c-Γ. Hence, following Proposition 17, the internal functors $\delta_0^{\sim}$ and $\delta_1^{\sim}$ are c_0-cartesian. Then

$$(\delta_0^{\sim})_0 = (\delta_0)_0 = d_0 \quad \text{and} \quad (\delta_1^{\sim})_0 = (\delta_1)_0 = d_1$$

are again in c-Γ; and so, following Proposition 11, the following diagram is a groupoid with every structural map c-cartesian:

$$\begin{tikzcd} \pi_c(X_1^{\sim}) \arrow[r, shift left=2, leftarrow, "\pi_c(\delta_0^{\sim})"] \arrow[r] \arrow[r, shift right=2, leftarrow, "\pi_c(\delta_1^{\sim})"'] & \pi_c((Com\ X_1)^{\sim}) \arrow[r, shift left=2, leftarrow] \arrow[r, leftarrow] \arrow[r, shift right=2, leftarrow] & \pi_c((Com_2X_1)^{\sim}) \end{tikzcd}$$

We call this groupoid the *c-cartesian groupoid associated to* X_1 and denote it by $X_1^{\bullet}$. Now $c\ [\pi_c(\delta_0^{\sim})]$ is, up to isomorphism, $c(d_0)$ and consequently lies in Γ.

$Grd_{\Gamma\text{-cart}}\mathbf{V}$ will denote the full subcategory of $Grd_{c\text{-}\Gamma}\mathbf{V}$ whose objects are the internal groupoids in $\mathbf{V}$ such that each structural map is c-cartesian above a map in Γ. It is not difficult to check that the functor $(\ )^{\bullet}$ is a right adjoint to the inclusion

$$i\colon Grd_{\Gamma\text{-cart}}\mathbf{V} \longrightarrow Grd_{c\text{-}\Gamma}\mathbf{V}.$$

6. THE MAIN RESULT.

We are now ready to extend the condition 2 for a stack from c-cartesian equivalence relations to c-Γ-equivalence relations.

Let R_1 be a c-Γ-equivalence relation. First observe that if $c(R_1)$ is certainly a Γ-groupoid, it is not necessarily a Γ-equivalence relation.

PROPOSITION 19. *Every c-Γ-equivalence relation above a Γ-equivalence relation, part of a Γ-exact diagram, can be completed in a left exact c-Γ-diagram above the given Γ-exact diagram.*

REMARK. That means that, under the conditions of Proposition 19, this c-Γ-equivalence relation has a quotient, since a c-Γ-morphism is always a regular epimorphism (Proposition 13).

PROOF. Let R_1 be the given c-Γ-equivalence relation. By hypothesis its image cR_1 is again an equivalence relation and it admits a quotient $r: cR_0 \twoheadrightarrow K$ in $\mathbf{W}$, lying in Γ. We observe that, in our construction of $R_1^{=}$, R_1 and Com R_1 being equivalence relations, such are $R_1^{\sim}$ and $(\mathrm{Com}\ R_1)^{\sim}$. Since $R_1^{=}$ is a c-cartesian groupoid above $c(R_1)$ which is an equivalence relation, it is itself a c-cartesian equivalence relation. The fibred reflexion c is a stack for the topology Γ and consequently $R_1^{=}$ admits a c-cartesian quotient $\rho: R_0^{=} \twoheadrightarrow Q$ above $r: cR_0 \to K$. Whence the following diagram:

$$\begin{tikzcd}
m(\mathrm{Com}\ R_1)^{\sim} \arrow[d, shift left] \arrow[d, shift right] & & \\
mR_1 \arrow[dr, shift left, "d_0"] \arrow[dr, shift right, "d_1"'] \arrow[dd, "\rho_{\mathrm{Com}R_1^{\sim}}"'] & mR_1^{\sim} \arrow[l] \arrow[d, shift left] \arrow[d, shift right] & \\
& R_0 \arrow[d, "\rho_{R_1^{\sim}}"] & \\
mR_1^{=} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & R_0^{=} \arrow[r, two heads, "\rho"'] & Q
\end{tikzcd}$$

The morphism $\rho_{\mathrm{Com}R_1^{\sim}}: mR_1 \twoheadrightarrow mR_1^{=}$ being a regular epimorphism, we see that $\rho.\rho_{R_1^{\sim}}$ is a coequalizer of the pair $(d_0,d_1): mR_1 \rightrightarrows R_0$. It lies in c-Γ since $\rho_{R_1^{\sim}}$ is a c-invertible regular epimorphism and ρ is c-cartesian above r which is in Γ.

Now we must prove that

$$\begin{tikzcd}
R_0 & mR_1 \arrow[l, shift left] \arrow[l, shift right] & m_2R_1 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

is the kernel equivalence of $\rho.\rho_{R_1^{\sim}}$, or equivalently that the functor $\epsilon R_1: R_1 \to R_1^{=}$ in $\mathrm{Grd}_{c\text{-}\Gamma}\mathbf{V}$ defined by the diagram on the next page is internally fully faithful. When the category $\mathbf{V}$ admits products, as it is the case for our two main examples, the proof is straightforward:

$$\epsilon R_1: \quad \begin{tikzcd} R_0 \arrow[d, "\rho_{R_1{}^{\sim}}"'] & mR_1 \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] \arrow[d, "\rho_{\mathrm{Com}R_1{}^{\sim}}"] & m_2R_1 \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] \arrow[d] \\ R_0{}^{*} & mR_1{}^{*} \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] & m_2R_1{}^{*} \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] \end{tikzcd}$$

Indeed, $[\delta_0,\delta_1]$: Com $R_1 \longrightarrow R_1 \times R_1$ is a discrete fibration, and consequently such is

$$[\delta_0,\delta_1]^{\sim}: (\mathrm{Com}\ R_1)^{\sim} \longrightarrow R_1{}^{\sim} \times R_1{}^{\sim};$$

When R_1 is an equivalence relation, it means that $[\delta_0,\delta_1]^{\sim}$ is q_c-cartesian. Now the functor q_c always preserves products when they exist, and thus the following square is a pullback:

$$\begin{tikzcd} R_0 \times R_0 \arrow[d] & mR_1 \arrow[l, "{[d_0,d_1]}"'] \arrow[d] \\ R_0{}^{*} \times R_0{}^{*} & mR_1{}^{*} \arrow[l, "{[d_0,d_1]}"] \end{tikzcd}$$

which implies that ϵR_1 is fully faithful.

There is another but much longer proof when **V** is not supposed to admit products. For that, let us consider the following diagram:

$$\begin{tikzcd} \text{Grd } \mathbf{V} & \mathbf{V} & & \\ K_1: \arrow[d, "\bar{\delta}"'] & mR_1{}^{\sim} \arrow[d, "m(\alpha_1 R_1)"'] & & m(\mathrm{Com}\ R_1)^{\sim} \arrow[ll, shift right, "m(\delta_0{}^{\sim})"'] \arrow[ll, shift left, "m(\delta_1)^{\sim}"] \arrow[d, "m(\alpha_1 \mathrm{Com} R_1)"] \quad \ldots\ldots \\ S_1 \arrow[d, shift right, "\bar{d}_0"'] \arrow[d, shift left, "\bar{d}_1"] & mR_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & & m\mathrm{Com}R_1 \arrow[ll, shift right, "m\delta_0"'] \arrow[ll, shift left, "m\delta_1"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \quad \ldots\ldots \\ R_1 & R_1 & & mR_0 \arrow[ll, shift right, "(\delta_0)_0 = d_0"'] \arrow[ll, shift left, "(\delta_1)_0 = d_1"] \quad \ldots\ldots \end{tikzcd}$$

with horizontal equivalences in **V**, and vertical functors. By construction $R_1{}^{*}$ is the quotient of the componentwise c-invertible equivalence relation in Grd **V**:

$$
\begin{tikzcd}
R_1^{=} & R_1 \arrow[l, "\epsilon R_1"'] \arrow[r] & K_1 \arrow[l, shift right=2, "\bar{d}_0.\bar{j}"'] \arrow[l, shift left=2, "\bar{d}_1.\bar{j}"] & \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2]
\end{tikzcd}
$$

The functors $\bar{d}_0$ and $\bar{d}_1$ are internally fully faithful for symmetrical reasons of the ones which make δ_0 and δ_1 internally fully faithful. Indeed the double diagram in $\mathbf{V}$ giving Com R_1 is symmetrical with respect to the diagonal. The functor $\bar{j}$ is fully faithful as a componentwise a c-cartesian functor above a fully faithful functor in $\mathbf{W}$, namely the image by c of the symmetrical functor of σ_0 (indeed, all our left exact diagrams in $\mathbf{V}$, lying in c-Γ, are preserved by c). Thus $\bar{d}_0.\bar{j}$ and $\bar{d}_1.\bar{j}$ are internally fully faithful.

The morphism $(\epsilon R_1)_0$ being $\rho_{R^{=}}$, and thus a c-invertible regular epimorphism, it is then possible (taking suitable joint pullbacks in $\mathbf{V}$) to factorize ϵR_1 in a $\phi_1.\psi_1$, with ϕ_1 internally fully faithful and ψ_1 $(\ )_0$-invertible (where $(\ )_0$: Rel $\mathbf{V} \to \mathbf{V}$). Let us then consider the following diagram, where (p_0,p_1) is the kernel pair of ϕ_1:

$$
(*)\qquad
\begin{tikzcd}
R_1^{=} \arrow[d, equal] & R_1 \arrow[l, "\epsilon R_1"'] \arrow[r] \arrow[d, "\psi_1"'] & K_1 \arrow[l, shift right=2, "\bar{d}_0.\bar{j}"'] \arrow[l, shift left=2, "\bar{d}_1.\bar{j}"] \arrow[d, "\chi_1"] & \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \\
R_1^{=} & S_1 \arrow[l, "\phi_1"] & P_1 \arrow[l, shift right=1, "p_0"'] \arrow[l, shift left=1, "p_1"] & \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2]
\end{tikzcd}
$$

Since ϕ_1 is fully faithful, such are p_0 and p_1. The functors $1_{R^{=}}$ and ψ_1 being $(\ )_0$-invertible and the diagram $(*)$ being made of componentwise kernel pairs, the functor χ_1 is again $(\ )_0$-invertible. Thus the two following squares are pullbacks, since they have a pair of parallel edges $(\ )_0$-invertible and a pair of parallel edges internally fully faithful:

$$
\begin{tikzcd}
R_1 \arrow[d, "\psi_1"'] & K_1 \arrow[l, shift right=1, "\bar{d}_0.\bar{j}"'] \arrow[l, shift left=1, "\bar{d}_1.\bar{j}"] \arrow[d, "\chi_1"] \\
S_1 & P_1 \arrow[l, shift right=1, "p_0"'] \arrow[l, shift left=1, "p_1"]
\end{tikzcd}
$$

Thus, the pair (ψ_1,χ_1) yields a vertical discrete fibration in Rel(Rel $\mathbf{V}$). Its image by the functor m is a discrete fibration in Rel $\mathbf{V}$:

$$\begin{tikzcd}
m\mathrm{R}_1^{=} \arrow[d, equal] & m\mathrm{R}_1 \arrow[l, dashed] \arrow[d, "m\psi_1"'] & m\mathrm{K}_1 \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "m\chi_1"] \arrow[r, dashed, no head] & {} \\
m\mathrm{R}_1^{=} & m\mathrm{S}_1 \arrow[l, dashed] & m\mathrm{P}_1 \arrow[l, shift left] \arrow[l, shift right] \arrow[r, dashed, no head] & {}
\end{tikzcd}$$

which is also q_c-invertible since $m\mathrm{R}_1^{=}$ is the quotient of the upper line by hypothesis, and the quotient of the lower line since ϕ_1 is fully faithful and $\phi_0 = \rho_{\mathrm{R}_1^{=}}$. A discrete fibration between c-discrete equivalence relations being always q_c-cartesian (Proposition 5), this discrete fibration, which is also q_c-invertible, is an isomorphism. Thus the morphisms $m\psi_1$ and $m\chi_1$ are invertible and consequently ψ_1 and χ_1 are themselves invertible. Then $\epsilon\mathrm{R}_1$ is internally fully faithful. •

REMARK. 1. The quotients given by Proposition 19 are universal since, by Proposition 14, the c-Γ-morphisms are stable under pullbacks.

2. A c-Γ-equivalence relation above a Γ-equivalence relation, part of a Γ-exact diagram, can be seen as a generalized descent data, given by a span $(d_0{}^i, d_1{}^i)$ of c-invertible regular epimorphisms:

$$\begin{tikzcd}
 & m\mathrm{R}_1 \arrow[dl, "d_1{}^i"'] \arrow[dr, "d_0{}^i"] & & \\
\mathbf{V} \quad \cdot \arrow[rrr, "d_1{}^c"'] & & \cdot \arrow[r, "d_0{}^c"] & \mathrm{R}_0 \\
\mathbf{W} \quad c m\mathrm{R}_1 \arrow[rrr, shift left, "c d_0"] \arrow[rrr, shift right, "c d_1"'] & & & c\mathrm{R}_0
\end{tikzcd}$$

Then this Proposition 19 can be interpreted in the following terms: when a stack is Barr-exact, something more general than a descent data can even be descended.

V. THE Σ-EXACTNESS.

From now on, when we shall speak of Cat $\mathbf{E}$, it will be supposed that $\mathbf{E}$ is a left exact and Barr-exact category. Then the functor $(\ )_0$: Cat $\mathbf{E} \to \mathbf{E}$ is a Barr-exact fibred reflexion and is a stack for the regular epimorphism topology. Furthermore it is left exact.

Now, given a $(\ )_0$-full equivalence relation $\mathcal{R}_1$ in Cat $\mathbf{E}$, its image by $(\ )_0$ is again an equivalence relation in $\mathbf{E}$, which consequently admits a quotient. We are thus in the conditions of Proposition 19 and then $\mathcal{R}_1$ admits a $(\ )_0$-full quotient. Consequently every $(\ )_0$-full equivalence relation in Cat $\mathbf{E}$ admits a $(\ )_0$-full quotient. It is a kind of relative Barr-exactness which we are going to establish precisely.

1. DEFINITION OF THE Σ-EXACTNESS PROPERTY.

Let $\mathbf{W}$ be a weakly left exact category, equipped with a proper class Σ.

DEFINITION 6. The category $\mathbf{W}$ will be called Σ-*exact* if furthermore:

1. every Σ-equivalence relation has a quotient (a coequalizer making this equivalence relation effective) which is in Σ and which is universal (the pullback of such a Σ-exact diagram is again exact);

2. if $g.f$ is in Σ and f is a Σ-regular epimorphism then g is in Σ.

EXAMPLES. 1. If c is a Barr-exact fibred reflexion, then $\mathbf{V}$ is Σ-exact for Σ the class of c-invertible regular epimorphisms.

2. When $\mathbf{E}$ is left exact and Barr-exact, then Cat $\mathbf{E}$ is Σ-exact when:

$\Sigma = \Sigma_i$ the class of $(\ )_0$-invertible morphisms,

$\Sigma = \Sigma_0$ the class of $(\ )_0$-cartesian morphisms (since $(\ )_0$ is a stack for the regular epimorphism topology, see [2]).

3. When $\mathbf{E}$ is left exact and Barr-exact, then Cat $\mathbf{E}$ is Σ-exact, for Σ the class of discrete fibrations (cf. [5], Proposition 5).

REMARK. The class of Σ-regular epimorphisms yields a Grothendieck topology, called the Σ-topology. Indeed:

- an isomorphism is in Σ and is a regular epimorphism;
- the Σ-regular epimorphisms are stable under pullback because of the universality condition of the Σ-exactness;
- the composite of two Σ-regular epimorphisms is in Σ. Moreover the composite $g.f$ of two regular epimorphisms is again a regular epimorphism, provided the morphism f is stable under pullback as a regular epimorphism. Thus the composite of two Σ-regular epimorphisms is a Σ-regular epimorphism.

2. FIRST PROPERTIES OF THE Σ-EXACTNESS.

$\mathrm{Rel}_\Sigma\mathbb{W}$ will denote the subcategory of $\mathrm{Rel}\,\mathbb{W}$ whose objects are the equivalence relations such that the pair $(d_0,d_1)\colon mR_1 \rightrightarrows R_0$ is in Σ. That Σ contains the class of isomorphisms yields a fully faithful functor

$$\mathrm{dis}\colon \mathbb{W} \longrightarrow \mathrm{Rel}_\Sigma\mathbb{W}.$$

The Σ-exactness condition implies that this functor has a left adjoint $q_\Sigma\colon \mathrm{Rel}_\Sigma\mathbb{W} \to \mathbb{W}$.

PROPOSITION 20. *A morphism $f_1\colon R_1 \to R'_1$ in $\mathrm{Rel}_\Sigma\mathbb{W}$ is q_Σ-cartesian iff it is a discrete fibration.*

PROOF. Let f_1 be a q_Σ-cartesian morphism; then the following diagram is a pullback:

$$\begin{tikzcd}
R_1 \arrow[r, "f_1"] \arrow[d] & R'_1 \arrow[d] \\
\mathrm{dis}\,q_\Sigma R_1 \arrow[r, "\mathrm{dis}\,q_\Sigma f_1"'] & \mathrm{dis}\,q_\Sigma R'_1
\end{tikzcd}$$

$\mathrm{dis}\,q_\Sigma f_1$ being a discrete fibration, such is f_1.

The converse is more difficult. In the absolute situation ($\mathbb{W}$ Barr-exact), it is a consequence of the Example ([1], p. 73) which is obtained by the metatheorem. Here we must find a direct proof.

Let $f_1\colon R_1 \to R'_1$ be a discrete fibration and consider the following diagram:

$$\begin{tikzcd}
R_1 \arrow[rr, "f_1"] \arrow[dr, "f'_1"] \arrow[dd] & & R'_1 \arrow[dd] \\
& R''_1 \arrow[ur, "f''_1"'] \arrow[dl] \arrow[dr, phantom, "(*)"] & \\
\mathrm{dis}\,q_\Sigma R_1 \arrow[rr, "\mathrm{dis}\,q_\Sigma f_1"'] & & \mathrm{dis}\,q_\Sigma R'_1
\end{tikzcd}$$

where the square $(*)$ is a pullback (it does exist thanks to the universality condition). Then f''_1 is a discrete fibration, and consequently such is f'_1. The proof will be completed by the following Lemma.

LEMMA 5. A q_Σ-invertible discrete fibration f'_1 is an isomorphism.

PROOF. ρ and ρ'' denote the quotient morphisms of R_1 and R''_1.

1. Let us show that f'_0 is a monomorphism. The kernel equivalence of f'_0 is denoted by $R_1[f'_0]$. That $\rho''.f'_0 = \rho$ implies that the following diagram in Rel $\mathbb{W}$ is a componentwise pullback:

$$\begin{tikzcd}
R_1[f'_0] \arrow[r, "\phi_1"] \arrow[d] & \mathrm{dis}\, R''_0 \arrow[d] \\
R_1 \arrow[r, "f'_1"'] & R''_1
\end{tikzcd}$$

If f'_1 is a discrete fibration, then ϕ_1 is a discrete fibration and, disR''_0 being discrete, $R_1[f'_0]$ is discrete and f'_0 is a monomorphism.

2. Let us show that f'_0 is a regular epimorphism. For that, consider the two following diagrams:

$$\begin{tikzcd}
mR_1 \arrow[r, "mf'_1"] \arrow[d, "d_1"'] & mR''_1 \arrow[r, "d_0"] \arrow[d, "d_1"'] & R''_0 \arrow[d, "\rho''"] \\
R_0 \arrow[r, "f'_0"'] & R''_0 \arrow[r, "\rho''"'] & Q
\end{tikzcd}$$

$$\begin{tikzcd}
mR_1 \arrow[r, "d_0"] \arrow[d, "d_1"'] & R_0 \arrow[r, "f'_0"] & R''_0 \arrow[d, "\rho''"] \\
R_0 \arrow[rr, "\rho"'] & & Q
\end{tikzcd}$$

They are globally equal. The first one is a pullback since f'_1 is a discrete fibration; hence the second one is also a pullback and $f'_0.d_0$ is a Σ-regular epimorphism since ρ is a Σ-regular epimorphism. d_0 being split, f'_0 is a regular epimorphism. Thus f'_0 is an isomorphism and f'_1, being a discrete fibration, is an isomorphism. •

PROPOSITION 21. *The functor q_Σ is a fibred reflexion.*

PROOF. It is a consequence of the universality condition. •

Later on, we shall need the following result about some particular q_Σ-invertible morphisms.

LEMMA 6. Let f_1: $R_1 \to R'_1$ be an internally fully faithful morphism between two Σ-equivalence relations such that f_0 is a Σ-regular epimorphism. Then f_1 is a q_Σ-invertible morphism. Such q_Σ-invertible morphisms are stable under pullbacks and these pullbacks are preserved by q_Σ.

PROOF. The morphism f_0 being a Σ-regular epimorphism, $q_\Sigma(f_1)$ is certainly a Σ-regular epimorphism. We consider the following diagram:

$$\begin{tikzcd}
 & mR_1 \arrow[r, "mf_1"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \arrow[ddl, dashed, "\rho''"'] & mR'_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
 & R_0 \arrow[r, "f_0"] \arrow[d, "\rho"'] & R'_0 \arrow[d, "\rho'"] \\
\Pi \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & Q \arrow[r, "q_\Sigma(f_1)"'] & Q'
\end{tikzcd}$$

If f_1 is internally fully faithful, the pair (d_0,d_1): $mR_1 \rightrightarrows R_0$ is the kernel pair of $\rho'.f_0$ and therefore of $q_\Sigma(f_1).\rho$. Thus, if (p_0,p_1): $\Pi \rightrightarrows Q$ is the kernel pair of $q_\Sigma(f_1)$, then ρ and ρ'' determine a joint pullback. Hence ρ'' is a Σ-regular epimorphism and p_0 is equal to p_1. Then $q_\Sigma(f_1)$ is also a monomorphism, and so an isomorphism. It follows from condition 2 that such q_Σ-invertible morphisms are stable under pullback, and these pullbacks are preserved by q_Σ, two parallel edges being q_Σ-invertible. •

3. A STABILITY PROPERTY FOR Σ-EXACTNESS.

We are now in a position to prove that Cat $\mathbf{E}$ is Σ_1-exact, with Σ_1=0-Full.

Let c: $\mathbf{V} \to \mathbf{W}$ be a fibred reflexion; we say that c is a *left exact fibred reflexion* if $\mathbf{V}$ is left exact and c is a left exact functor. If Σ is a class of morphisms in $\mathbf{W}$ and if c is Barr-exact, c-Σ will denote the class of morphisms f in $\mathbf{V}$ such that f is c-full and $c(f)$ in Σ.

PROPOSITION 22. *Let* $\mathbf{W}$ *be a* Σ*-exact category and* c *a left exact and Barr-exact fibred reflexion which is a stack for the* Σ*-topology. Then* $\mathbf{V}$ *is* c-Σ*-exact.*

PROOF. Mimicking Proposition 14, it is clear that c-Σ is a proper class in $\mathbf{V}$. Every c-Σ-equivalence relation R_1 is such that $c(R_1)$ is an equivalence relation since c is left exact. It is then a Σ-equivalence relation, and thus it admits a quotient in Σ. By Proposition 19, c being a stack for the Σ-topology, R_1 has a quotient in c-Σ, which is universal (Remark following Proposition 19). This is the condition 1 for the c-Σ-exactness.

To prove the condition 2, let $g.f$ in c-Σ, with f a c-Γ-regular epimorphism. Then $c(g).c(f)$ is in Σ, with $c(f)$ a Σ-regular epimorphism, and thus $c(g)$ is in Σ. We must prove that g is c-full. For that, we consider the following diagram:

$$\begin{tikzcd}
V \arrow[d, two heads, "f^i"'] \arrow[dr, dashed, "f"] & & \\
U \arrow[r, "f^c"'] \arrow[d, two heads, "\overline{g^i}"'] & V' \arrow[d, "g^i"] \arrow[dr, "g"] & \\
U' \arrow[r, dashed, "\overline{f^c}"'] & U'' \arrow[r, "g^c"'] & V''
\end{tikzcd}$$

where $\overline{f^c}.\overline{g^i}$ is the canonical decomposition of $g^i.f^c$. That $g.f$ is in c-Σ implies that $\overline{g^i}.f^i$ is a c-invertible regular epimorphism. The morphism f^i being also a c-invertible regular epimorphism (f in c-Σ), $\overline{g^i}$ is a c-invertible regular epimorphism. Now $c(\overline{f^c})$ is, up to isomorphism, equal to $c(f)$, and thus is a Σ-regular epimorphism. Then c being a stack for the Σ-topology and by condition 1 for stacks, f^c and $\overline{f^c}$ are c-cartesian regular epimorphisms. In particular f^c is a regular epimorphism stable under pullback. As $g^i.f^c = \overline{f^c}.\overline{g^i}$ is a regular epimorphism, such is g^i, and g is in c-Σ. •

4. THE c-Σ-REGULAR EPIMORPHISMS.

A c-invertible regular epimorphism is always a c-Σ-regular epimorphism. Now, c being a stack, any c-cartesian f morphism above a Σ-regular epimorphism is a c-Σ-regular epimorphism (f will be called a c-Σ-cartesian regular epimorphism).

More generally a c-Σ-regular epimorphism f is just a c-full morphism such that $c(f)$ is a Σ-regular epimorphism.

Indeed, if f is a c-Σ-regular epimorphism, then, c being right exact, cf is a Σ-regular epimorphism. On the other hand, f being in c-Σ, it is c-full.

Conversely, let $f^c.f^i$ be the canonical decomposition of f. If f is c-full, f^i is a c-invertible regular epimorphism. Now f^c is c-cartesian above $c(f)$. If $c(f)$ is a Σ-regular epimorphism, then f^c is a c-Σ-cartesian regular epimorphism. Thus $f = f^c.f^i$ is a c-Σ-regular epimorphism as a composite of two c-Σ-regular epimorphisms.

5. A STABILITY PROPERTY FOR STACKS.

When c: $\mathbf{V} \to \mathbf{W}$ is a left exact fibred reflexion, such is c_0: $\mathrm{Cat}_c\mathbf{V} \to \mathbf{V}$. If furthermore c is Barr-exact, c is again Barr-exact [2]. Our present aim is to prove that, when c is also a stack for a Σ-topology in $\mathbf{W}$, then c_0 is a stack for the c-Σ-topology in $\mathbf{V}$.

For that, we begin by the following lemmas.

LEMMA 7. Let f: $V \to V'$ be a c-Σ-morphism; then $G_c(f)$: $G_cV \to G_cV'$ is an internal functor in $\mathrm{Cat}_c\mathbf{V}$ which is componentwise a c-Σ-morphism. If f is also a c-Σ-regular epimorphism, $G_c(f)$ is a regular epimorphism in $\mathrm{Cat}_c\mathbf{V}$.

PROOF. Let $f^c.f^i$ be the canonical decomposition of f. Then $G_c(f^c)$ is $\bar{c}$-cartesian. Thus $m[G_c(f^c)] = f^c\times_c f^c$, in the same way as f^c, is c-cartsian above $c(f)$ which is in Σ and $G_c(f^c)$ is a functor which is componentwise a c-Σ-cartesian morphism. On the other hand $G_c(f^i)$ is $\bar{c}$-invertible. The morphism $m[G_c(f^i)] = f^i\times_c f^i$ reduces to the product $f^i\times f^i$ in the left exact and Barr-exact fiber $\mathrm{Fib}_c[c(V)]$. Now if f^i is a regular epimorphism, such is $f^i\times_c f^i$ and $G_c(f^i)$ is a functor which is componentwise a c-invertible regular epimorphism. Thus $G_c(f)$ is componentwise a c-Σ-morphism. If furthermore $c(f)$ is a Σ-regular epimorphism, then f^c and $f^c\times_c f^c$ are c-Σ-cartesian regular epimorphisms and $G_c(f)$ is a functor which is componentwise a regular epimorphism, and therefore is a regular epimorphism in $\mathrm{Cat}_c\mathbf{V}$. •

LEMMA 8. If f_1: $X_1 \to Y_1$ is a c_0-cartesian functor such that f_0 is in c-Σ, then f_1 is componentwise in c-Σ. If f_0 is also a c-Σ-regular epimorphism, then f_1 is a regular epimorphism in $\mathrm{Cat}_c\mathbf{V}$.

PROOF. If f_1 is c_0-cartesian, then the following square is a pullback, and, since $\mathbf{V}$ is left exact, it is a componentwise pullback.

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d] \\ G_cX_0 \arrow[r, "G_c(f_0)"'] & G_cY_0 \end{tikzcd}$$

If f_0 is in c-Σ, $G_c(f_0)$ is componentwise in c-Σ, and thus f_1 is componentwise in c-Σ. The proof is exactly the same for the second part of this lemma. •

PROPOSITION 23. *Let c: $\mathbb{V} \to \mathbb{W}$ be a left exact and Barr-exact fibred reflexion. If $\mathbb{W}$ is Σ-exact and c a stack for the Σ-topology, then c_0: $\mathrm{Cat}_c\mathbb{V} \to \mathbb{V}$ is a stack for the c-Σ-topology.*

PROOF. Let the following diagram be a c_0-cartesian diagram above a c-Σ-exact diagram:

$$\begin{tikzcd} U_1 \arrow[r, shift left] \arrow[r, shift right] & X_1 \arrow[r, "f_1"] & Y_1 \end{tikzcd}$$

It is left exact as a cartesian diagram above a left exact diagram. Since f_0 is a c-Σ-regular epimorphism (the c_0-underlying diagram being c-Σ-exact), then, following Lemma 8, f_1 is a regular epimorphism and our diagram is exact. This is the condition 1 for stacks.

Let $\mathcal{R}_1$ be a c_0-cartesian equivalence relation in $\mathrm{Cat}_c\mathbb{V}$, above a c-Σ-equivalence relation in $\mathbb{V}$, part of a c-Σ-exact diagram. If we denote $\mathcal{R}_0$ by X_1 and $m\mathcal{R}_1$ by U_1, we obtain the following diagram in $\mathbb{V}$:

$$\begin{tikzcd}[column sep=large, row sep=huge] mU_1 \arrow[r, shift left, "m\delta_0"] \arrow[r, shift right, "m\delta_1"'] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & mX_1 \arrow[r, two heads] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & mQ_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ U_0 \arrow[r, shift left, "(\delta_0)_0"] \arrow[r, shift right, "(\delta_1)_0"'] & X_0 \arrow[r, two heads, "\rho_0"'] & Q_0 \end{tikzcd}$$

where the lower line is a c-Σ-exact diagram. δ_0 and δ_1 being c_0-cartesian, and $(\delta_0)_0$ and $(\delta_1)_0$ being in c-Σ, the morphisms $m\delta_0$ and $m\delta_1$ are in c-Σ and the upper line is a c-Σ-equivalence relation. We denote by $m\rho_1$: $mX_1 \twoheadrightarrow mQ_1$ its quotient morphism which lies in c-Σ (following Proposition 22).

Now we consider the following diagram:

$$\begin{tikzcd}[column sep=large] mU_1 \arrow[r, shift left, "m\delta_0"] \arrow[r, shift right, "m\delta_1"'] \arrow[d, "{[d_0,d_1]}"'] & mX_1 \arrow[r, two heads, "m\rho_1"] \arrow[d, "{[d_0,d_1]}"] & mQ_1 \arrow[d, "{[d_0,d_1]}"] \\ U_0 \times_c U_0 \arrow[r, shift left, "(\delta_0)_0 \times_c (\delta_0)_0"] \arrow[r, shift right, "(\delta_1)_0 \times_c (\delta_1)_0"'] & X_0 \times_c X_0 \arrow[r, two heads, "\rho_0 \times_c \rho_0"'] & Q_0 \times_c Q_0 \end{tikzcd}$$

The lower line is c-Σ-exact following Lemma 7. That δ_0 and δ_1 are c_0-cartesian means exactly that the two left hand commutative squares are pullbacks. Thus the morphisms $[d_0,d_1]$ yield a vertical discrete fibration between two c-Σ-equivalence relations. Following Propositions 22 and 20, the right hand square is a pullback. We must prove that

$$\begin{tikzcd}[column sep=huge] mQ_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & Q_0 \end{tikzcd}$$

is underlying to a c-discrete category. If it is the case, the quotient morphism $\rho_1 : X_1 \to Q_1$ will be c_0-cartesian, following our last remark.

Now we consider the following c-Σ-exact diagram:

$$m_2\mathbb{R}_1 : \quad \begin{tikzcd}[column sep=huge] m_2U_1 \arrow[r, shift left, "m_2\delta_0"] \arrow[r, shift right, "m_2\delta_1"'] & m_2X_1 \arrow[r, "m_2\rho_1"] & m_2Q_1 \end{tikzcd}$$

and we denote by $\mathbb{R}_0$, $m\mathbb{R}_1$, $m_2\mathbb{R}_1$ the c-Σ-equivalence relations, images of $\mathbb{R}_1$ by the functors c_0, m, m_2 ($m_2\mathbb{R}_1$ is just given by our last diagram).

We have the following square in $\mathrm{Rel}_{c\text{-}\Sigma}\mathbf{V}$:

$$\begin{tikzcd}[column sep=huge] m_2\mathbb{R}_1 \arrow[rr, "d_0"] \arrow[d, "d_2"'] & & m\mathbb{R}_1 \arrow[d, "d_1"] \\ m\mathbb{R}_1 \arrow[r, "{[d_0,d_1]}"] \arrow[rr, bend right, "d_0"'] & \mathbb{R}_0 \times_c \mathbb{R}_0 \arrow[r, "p_0"] & \mathbb{R}_0 \end{tikzcd}$$

It is a pullback since X_1 and U_1 are internal categories and we are going to prove that it is preserved by $q_{c\text{-}\Sigma}$.

Let us consider the following diagram:

$$\begin{tikzcd}
U_0\times_c U_0 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \cdot \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & U_0 \arrow[d, shift left] \arrow[d, shift right] \\
X_0\times_c X_0 \arrow[r, "X_0\times_c\rho_0^{\,i}"] \arrow[dr, dashed, "\rho_0\times_c\rho_0"'] & X_0\times_c Z_0 \arrow[r, "p_0"] \arrow[d, dashed, "\rho_0\times_c\rho_0^{\,c}"] \arrow[dr, phantom, "(*)"] & X_0 \arrow[d, "\rho_0"] \\
& Q_0\times_c Q_0 \arrow[r, "p_0"'] & Q_0
\end{tikzcd}$$

where the square (*) is a pullback and

$$\rho_0^{\,c}.\rho_0^{\,i}\colon X_0 \longrightarrow Z_0 \longrightarrow Q_0$$

the canonical decomposition. It upper part determines the decomposition of the functor p_0: $\mathbb{R}_0\times_{R0} \to \mathbb{R}_0$ in a $q_{c\text{-}\Sigma}$-cartesian and a $q_{c\text{-}\Sigma}$-invertible functors. The morphism $X_0\times_c\rho_0^{\,i}$ is a c-invertible regular epimorphism (since ρ_0 is in c-Σ) and consequently a c-Σ-regular epimorphism. Then, following Lemma 6 and Lemma 1, the functor $q_{c\text{-}\Sigma}$ preserves the pullbacks along p_0: $\mathbb{R}_0\times_c\mathbb{R}_0 \to \mathbb{R}_0$.

Furthermore the functor $[d_0,d_1]$: $m\mathbb{R}_1 \to \mathbb{R}_0\times_c\mathbb{R}_0$, being a discrete fibration, is $q_{c\text{-}\Sigma}$-cartesian and thus $q_{c\text{-}\Sigma}$ preserves pullbacks along $[d_0,d_1]$. Hence our previous pullback is preserved by $q_{c\text{-}\Sigma}$ and determines a c-discrete category:

$$\begin{tikzcd}
Q_0 \arrow[r] & mQ_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] & m_2Q_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift left=2, "d_2"]
\end{tikzcd}$$

which is the componentwise quotient of $\mathcal{R}_1$. •

VI. THE Σ_n-EXACTNESS PROPERTY FOR THE CATEGORY n-Cat E OF INTERNAL n-CATEGORIES IN E.

We are now ready to apply our results to the tower of Barr-exact fibrations of n-categories [2]:

$$1 \longleftarrow \mathbf{E} \longleftarrow \mathrm{Cat}\ \mathbf{E} \quad \dots \quad (n\text{-}1)\text{-}\mathrm{Cat}\ \mathbf{E} \longleftarrow n\text{-}\mathrm{Cat}\ \mathbf{E} \ \dots$$

Here is the first step:

1. A RIGHT EXACTNESS PROPERTY FOR INTERNAL CATEGORIES.

Let $\mathbf{E}$ be a left exact and Barr-exact category. We recall that

$$(\ )_0 \colon \mathrm{Cat}\,\mathbf{E} \longrightarrow \mathbf{E}$$

is a left exact and Barr-exact fibred reflexion which is also a stack for the regular epimorphism topology. Then starting from the proper class $\Sigma_0 = \mathbf{E}$, the category $\mathbf{E}$ is Σ_0-exact.

The proper class $(\ )_0$-Σ_0 in Cat $\mathbf{E}$ is just the class of 0-full functors (or shortly full functors) in Cat $\mathbf{E}$. We denote this class by Σ_1. By Proposition 22, the category Cat $\mathbf{E}$ is again Σ_1-exact.

The class of Σ_1-regular epimorphisms is then the class of full functors $f_1\colon X_1 \to Y_1$ such that f_0 is a regular epimorphism. They will be called the fully regular epimorphisms of Cat $\mathbf{E}$. These fully regular epimorphisms are componentwise regular epimorphisms in Cat $\mathbf{E}$.

REMARK. A componentwise regular epimorphism functor is clearly a regular epimorphism in Cat $\mathbf{E}$. However the class of such morphisms is obviously too large with respect to a right exactness property: every equivalence relation R_1 in Cat $\mathbf{E}$ has its $d_0, d_1\colon mR_1 \rightrightarrows R_0$ componentwise regular epimorphisms, but has not always a quotient (take $\mathbf{E}$ = Set).

It is easy to show that, in general, a componentwise regular epimorphism functor in Cat $\mathbf{E}$ is not a fully regular epimorphism: take a discrete fibration $f_1\colon X_1 \to Y_1$ with f_0 a regular epimorphism; it is then a componentwise regular epimorphism. But as a discrete fibration, it is always internally faithful, that means $(\ )_0$-faithful.

2. THE TOWER OF INTERNAL n-CATEGORIES.

We recalled that, if $c\colon \mathbf{V} \to \mathbf{W}$ is a left exact fibred reflexion, then $c_0\colon \mathrm{Cat}_c\mathbf{V} \to \mathbf{V}$ is again a left exact fibred reflexion. Furthermore if c is Barr-exact, c_0 is Barr-exact.

It is clearly the beginning of an iteration process. Starting from $(\ )_0\colon \mathrm{Cat}\,\mathbf{E} \to \mathbf{E}$, we denote as follows the first step of this process

$$(\ )_1 \colon 2\text{-}\mathrm{Cat}\,\mathbf{E} \longrightarrow \mathrm{Cat}\,\mathbf{E}$$

and we call this new category the *category of internal 2-categories in* $\mathbf{E}$, since, if $\mathbf{E}$ = Set, this construction actually produces the category of 2-categories.

Let us denote by $(n+1)$-Cat $\mathbf{E}$ the n-th step of the process:

$$(\ )_n\colon (n+1)\text{-Cat } \mathbf{E} \longrightarrow n\text{-Cat } \mathbf{E}$$

and call it the *category of internal $(n+1)$-categories in* $\mathbf{E}$, as it is the case if $\mathbf{E}$ = Set [2].

When $\mathbf{E} = \mathbf{A}$ is an abelian category, then n-Cat $\mathbf{A}$ and n-Grd $\mathbf{A}$ are identical, and they are equivalent to the category $C^n(\mathbf{A})$ of positive chain complexes of length n in $\mathbf{A}$ [4].

3. A RIGHT EXACTNESS PROPERTY FOR INTERNAL 2-CATEGORIES.

When $\mathbf{E}$ is left exact and Barr-exact, our fibred reflexion

$$(\ )_1\colon 2\text{-Cat } \mathbf{E} \longrightarrow \text{Cat } \mathbf{E}$$

is again left exact and Barr-exact. Following Proposition 23, this functor $(\ )_1$ is a stack for the Σ_1-topology and, by Proposition 22, the category 2-Cat $\mathbf{E}$ is $(\ )_1$-Σ_1-exact.

We denote by Σ_2 the class $(\ )_1$-Σ_1. It is the class of 2-functors $f_2\colon X_2 \to Y_2$ which are $(\ )_1$-full and such that f_1 is full. A Σ_2-regular epimorphism is moreover such that f_0 is also a regular epimorphism. We shall call such a 2-functor a *fully regular epimorphic* 2-functor. In the case $\mathbf{E}$ = Set, a fully regular epimorphic 2-functor is a 2-functor $f_2\colon X_2 \to Y_2$ epimorphic on objects, such that its underlying functor $f_1\colon X_1 \to Y_1$ is full and that, for each pair $(\phi,\psi)\colon x \to x'$ of 1-morphisms in X_2, with a 2-cell $\bar{\gamma}\colon f_2(\phi) \Rightarrow f_2(\psi)$ in Y_2, there is a 2-cell $\gamma\colon \phi \Rightarrow \psi$ in X_2, satisfying $f_2(\gamma) = \bar{\gamma}$.

4. A RIGHT EXACTNESS PROPERTY FOR INTERNAL n-CATEGORIES.

The proper class Σ_n in n-Cat $\mathbf{E}$ is defined by induction, by

$$\Sigma_n = (\ )_{n-1}\text{-}\Sigma_{n-1}.$$

A n-functor $f_n\colon X_n \to Y_n$ is in Σ_n iff, for each i, $1 \leqslant i \leqslant n$, $f_i\colon X_i \to Y_i$ is $(i-1)$-full.

By Proposition 22, the category n-Cat $\mathbf{E}$ is Σ_n-exact. The Σ_n-regular epimorphisms in n-Cat $\mathbf{E}$ are those n-functors in Σ_n such that, moreover, f_0 is a regular epimorphism. We call them the *fully regular epimorphic n-functors*.

By Proposition 23, the functor

$$(\ )_n\colon (n{+}1)\text{-Cat}\ \mathbf{E} \longrightarrow n\text{-Cat}\ \mathbf{E}$$

is a stack for the Σ_n-topology, and that makes possible to iterate our process.

Thus we have established a precise and strong exactness property for n-Cat $\mathbf{E}$, mimicking strictly the Barr-exactness. This property is again satisfied in the category n-Grd $\mathbf{E}$, the full subcategory of n-Cat $\mathbf{E}$ whose objects are the internal n-groupoids. It is thus possible, always mimicking the absolute case, to define the *first cohomology group of n-Grd $\mathbf{E}$ with values in an internal abelian group* A in $\mathbf{E}$. It is easy to check (and will be published later on) that:

The n-th cohomology group of $\mathbf{E}$ with values in A, *as defined in [3], is the first cohomology group of n-Grd $\mathbf{E}$.*

Indeed, what was called an aspherical n-groupoid in [3] is just a n-groupoid X_n such that the terminal map $X_n \to 1$ is a fully regular epimorphic n-functor, that is a n-groupoid with a fully global support.

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U.F.R. de Mathématiques
33 rue Saint-Leu
80039 AMIENS Cedex. FRANCE