Further criteria for totality

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Introduction. This Note is a sequel to the Kelly survey [2] of totality for enriched categories and some familiarity with the latter is assumed. It is supposed throughout that \( V \) is a symmetric monoidal closed category with \( V_o \) admitting all small limits and arbitrary intersections of monics.

Generators and totality.

**Theorem 1.** Any cocomplete category \( A \) is total if it admits arbitrary cointersections of epics and has a small generating set.

**Proof.** It suffices to show that the coend \( \int^a f_a \otimes a \) can be constructed in \( A \) from the generators \( G \) of \( A \). First consider the pushout diagram of the canonical map \( 1 \otimes \epsilon \) and \( k \) with \( 1 \otimes \epsilon \) jointly epic since \( G \) generates \( A \):

\[
\begin{array}{ccc}
fa \otimes (g, a) \otimes g & \xrightarrow{k} & \int f \otimes g \\
\downarrow \otimes \epsilon & & \downarrow e_* \\
1 \otimes \epsilon & \xrightarrow{\text{p.o.}} & q_*
\end{array}
\]

This implies that each \( e_* \) is epic; then the pushout of all those epics over \( a \) in \( A \) is easily seen to be precisely \( \int^a fa \otimes a \) in \( A \), as required.
REMARK. In the above result, epics can be replaced by the maps in $E$ for any $E$-M-factorization system on $A$; a general result concerning limits of $M$-subfunctors can be found in the Lemma of [1], 83.

**The adjoint-functor Theorem and totality.**

**THEOREM 2.** A category $A$ is total iff it is complete with all intersections of [strong] monics and there exists a functor $r$ from $[A^{op}, V]$ to $A$ and a natural [strong] monic $\mu: 1 \rightarrow ry$.

**PROOF.** Necessity is clear. For sufficiency consider the canonical diagram

```
    fa --> A(rya, rf) \rightarrow A(\alpha, r\alpha)
       \downarrow                    \downarrow
       \alpha_1 \downarrow \rightarrow A(rya, ryb) \rightarrow A(1, r\alpha_4)
       \downarrow                    \downarrow
       yb(\alpha) \downarrow \rightarrow A(1, \mu) \rightarrow A(a, ryb)
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The result now follows from the Adjoint-Functor Theorem [1].

As an application, consider the category $A$ of coalgebras for a density comonad on a category $B$. If such an $A$ is complete then it is total with no assumption on $B$.

**References.**