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FURTHER CRITERIA FOR TOTALITY

by B.J. DAY

Résumé. Cet article est une suite de l'article de Kelly [2] sur l'étude de la totalité pour les catégories enrichies.

Introduction. This Note is a sequel to the Kelly survey [2] of totality for enriched categories and some familiarity with the latter is assumed. It is supposed throughout that V is a symmetric monoidal closed category with V_0 admitting all small limits and arbitrary intersections of monics.

Generators and totality.

THEOREM 1. Any cocomplete category A *is total if it admits arbitrary cointersections of epics and has a small generating set.*

PROOF. It suffices to show that the coend $\int^A fa\otimes a$ can be constructed in A from the generators G of A. First consider the pushout diagram of the canonical map $1\otimes\epsilon$ and k with $1\otimes\epsilon$ jointly epic since G generates A :

$$\begin{tikzcd} fa\otimes A(g,a)\otimes g \arrow[r, "k"] \arrow[d, "1\otimes\epsilon"'] \arrow[dr, phantom, "\text{p.o.}"] & \int^A fg\otimes g \arrow[d, "e_a"] \\ fa\otimes a \arrow[r] & q_a \end{tikzcd}$$

This implies that each e_a is epic; then the pushout of all those epics over a in A is easily seen to be precisely $\int^A fa\otimes a$ in A, as required.

REMARK. In the above result, epics can be replaced by the maps in E for any E-M-factorization system on A; a general result concerning limits of M-subfunctors can be found in the Lemma of [1], §3.

The adjoint-functor Theorem and totality.

***THEOREM** 2. A category* A *is total iff it is complete with all intersections of [strong] monics and there exists a functor r from* $[A^{op}, V]$ *to* A *and a natural [strong] monic* $\mu: 1 \to ry$.

PROOF. Necessity is clear. For sufficiency consider the canonical diagram

$$\begin{tikzcd}
fa \arrow[r] \arrow[dd, "\alpha_a"'] & A(rya,\ rf) \arrow[r, "A(\mu,1)"] \arrow[d, "A(1,\ r(\alpha))"] & A(a,\ rf) \arrow[dd, "A(1,\ r(\alpha_a))"] \\
 & A(rya,\ ryb) \arrow[dr, "A(\mu,1)"] & \\
yb(a) \arrow[ur] \arrow[rr, "A(1,\mu)"'] & & A(a,\ ryb)
\end{tikzcd}$$

The result now follows from the Adjoint-Functor Theorem [1].

As an application, consider the category A of coalgebras for a density comonad on a category B. If such an A is complete then it is total with no assumption on B.

References.

1. B.J. DAY, On adjoint-functor factorization, *Lecture Notes in Math.* 420, Springer (1974), 1-19.
2. G.M. KELLY, A survey of totality for enriched and ordinary categories, *Cahiers Top et Géom. Diff.* XXVII-2 (1986), 109-132 (and references therein).

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