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A GENERALIZED GLOBAL DIFFERENTIAL CALCULUS. II. APPLICATION TO INVARIANCE UNDER A LIE GROUP

by Joseph JOHNSON

RÉSUMÉ. Dans la première partie de cet article (publiée dans le Volume XXVII-3 des "Cahiers", la théorie du Calcul Différentiel sur les variétés différentiables a été généralisée (sans l'affaiblir) en utilisant des fonctions localement définies C^∞ (ou analytiques réelles, ou analytiques complexes) de 1, 2, 3, ... variables comme opérateurs, et en introduisant des relations de commutativité appropriées et des axiomes qui permettent de recoller les informations locales en une information globale. On a montré que les principales propriétés de l'algèbre commutative sont encore valables dans ce cadre, en particulier la possibilité d'ajouter des indéterminées et de résoudre des systèmes d'équations; de plus, on peut prendre des limites et colimites aussi générales (mais petites) que l'on veut.

Dans la seconde partie, cette théorie est appliquée pour généraliser les théorèmes de Lie aux espaces de dimension arbitraire (même infinie) et sans restriction sur la nature des singularités qui peuvent intervenir.

INTRODUCTION.

In the introduction to Part I of this paper (cf. [13], preceding issue of "Cahiers") the subject matter of this concluding statement was briefly explained. Here, for the reader's convenience, a second summary of these concluding pages of the paper is given.

The basic notion used here is that of a "universe group", an object U of K with a given factorization of (U,): $K \to$ **sets** through the category of groups (cf. §12). We then define what it means for a universe group U to "act" on C ϵ K. Then if $c \epsilon$ C and U acts on C, we define what it means for c to be "invariant" (cf. Definition following (12.0.2)) and "locally invariant" (cf. Definition following (12.5)).

The essence of what is proved in the paper can now be stated, confining ourselves, for simplicity, to the case where $U = A_G$, G a Lie group. Our first main result is (12.6), which shows the relationship between the notions "invariant" and "locally invariant". Next, we have 1) of (13.1), which says that c is locally invariant if and only if c is "infinitesimally invariant". Finally, we have 2) of (13.1) which says that the Lie bracket of the Lie algebra $\mathfrak{g}$ of G is compatible with the Lie bracket of derivations associated to elements of $\mathfrak{g}$.

12. ACTIONS OF A GROUP UNIVERSE ON A UNIVERSE.

In these final sections we shall work almost exclusively with K. "Universe" will mean "object of K ", and all limits and colimits of universes will be taken in K except when stated otherwise. The theory will be valid for all senses of the word "admissible", even though we may refer for motivation to the differentiable case. (For the complex-analytic theory, "Lie group"" will mean complex-analytic Lie group.)

A *universe group* consists of a universe U, a functor $F: K \to$ **groups** and an isomorphism $(U, \ )_K \simeq |F|$ (where $| \ |$ denotes underlying set). It will usually be convenient to simply say that "U is a universe group". For instance, if S is any set, $A_0(S)$ is a universe group, since it represents the functor

$$V \mapsto (S, V_0) = \text{a group},$$

as

$$V_0 = \{v \in V \mid 0v = 0\}$$

is a group under + . If U is a universe group, we have a natural transformation $(U, \ )\times(U, \ ) \to (U, \ )$ given by the multiplication, hence $\mu: U \to U \amalg U$. Also we have $1_U: U \to A_0$ such that for each $V \in K$,

$$(1_U, V): (A_0, V) \to (U, V)$$

sends the unique element of (A_0, V) to the identity element of the group (U, V). We also have $r : U \to U$ such that (r, V) is the map of (U, V) into itself that sends each $g \in (U, V)$ to g^{-1}. Conversely, if

$$\mu: U \to U \amalg U, \qquad 1_U: U \to A_0, \qquad r : U \to U$$

are given *a priori* so as to satisfy [*]

$$(U \amalg \mu)\mu = (\mu \amalg U)\mu, \quad (1_U \amalg U)\mu = U \uparrow A_0 \amalg U, \quad (r, U)\mu = (A_0 \uparrow U)1_U,$$

then it is routine to show U has a unique structure of universe group that returns μ, 1_U, r in the manner that was just indicated.

In what follows, G will denote a fixed (admissible) Lie group, e the identity of G. Then multiplication gives us

$$\mu : A_G \to A_{G\times G} = A_G \amalg A_G$$

(cf. (7.9)). We also have $e^\wedge: A_G \to A_0$ $(e^\wedge(f) = f(e))$ and $r : A_G \to A_G$, given by the inverse map of G. The required identities hold, so A_G is a universe group.

Let U be a universe group, C a universe, $\psi : C \to U \amalg C$. We shall say that ψ is an *action* of U on C – even though "co-action" would be more standard terminology – provided the identities [*]

(12.0.1) $$(\mu \amalg C)\psi = (U \amalg \psi)\psi,$$

(12.0.2) $$(1_U \amalg C)\psi = C \uparrow A_0 \amalg C$$

hold. Of course if G acts on the manifold X so that

$$g(hx) = (gh)x, \quad g, h \in G, \quad x \in X \quad \text{and} \quad ex = x,$$

A_X is canonically acted upon by A_G. If $f \in A_X$, $g \in G$, $x \in X$, we have

$$(\psi f)(g,x) = f(gx).$$

One always has the canonical map $C \uparrow C \amalg C$ whatever U and C. It is an action of U on C that we shall refer to as the *trivial action* of U on C. We shall denote it by τ or $\tau_{U,C}$. If ψ is any action of U on C, $c \in C$ will be called an *invariant* of ψ if $\psi c = \tau c$. The main purpose of these final sections is to prove (12.6) and (13.1). The reader should note (12.9) which shows how one can put an admissible structure on the orbit space of a group action.

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[*] The reader should draw the appropriate commutative diagrams.

It helps to develop some notation that is oriented toward a more geometric point of view. Let C ϵ K be given. If U ϵ K, define $S_U C$ = (C, U) (so that SC = $S_{A_0}C$). If $x \in S_U C$, define $c\hat{}(x) = x(c)$ for all $c \in$ C. This definition extends one of our previous uses of "^". Let U be a universe group that acts on C. For any V ϵ K, we can define a "multiplication"

$$S_V U \times S_V C \to S_V C \quad \text{by} \quad gx = (g,x)\psi \in C \quad \text{if} \quad g \in S_V U,\ x \in S_V C.$$

If G acts on the manifold X, $g \in$ G, $x \in$ X, then $g\hat{}x\hat{} = (gx)\hat{}$ since if $c \in A_X$,

$$(g\hat{}x\hat{})(c) = (g\hat{},x\hat{})(\psi c) = (g,x)\hat{}(\psi c) = (\psi c)(g,x) = c(gx) = (gx)\hat{}(c).$$

If g, $h \in S_V U$, define

$$gh \in S_V U \quad \text{by} \quad gh = (g,h)\mu.$$

Of course if g, $h \in$ G, $g\hat{}h\hat{} = (gh)\hat{}$. If A_G acts on C, $g \in$ G, $x \in S_V C$, we shall define

$$gx = g_V x \in S_V C \text{ where } g_V = (A_0 \uparrow V)g\hat{}.$$

We shall use this notation to reformulate (12.0.1) and (12.0.2) (cf. (12.1)).

In what follows, U will denote a universe group that acts on a universe C. In some instances it will be necessary for U to be more specifically A_G itself. For the time being it will generally be considered that the presence of G within a given paragraph will suffice to indicate when that is so. In all instances C remains a general object of K, and it is mainly this generality of C that justifies the often meticulous attention to detail that follows.

The identification $A_0 \amalg C$ = C, though natural, seems to generate confusion and must be resisted for purposes of exposition. When $x \in S_W V$ and the context selects an element h of $(W,Z)_K$ in some unambiguous way, we shall let $x_Z = h \circ x$. For instance, if W = A_0 ($x \in$ SV), the meaning of x_Z is clear since $\#(A_0, Z) = 1$.

Let $e = 1_U$.

***LEMMA* (12.1).** *Let* V ϵ K.

1) $(gh)x = g(hx)$ *if* g, $h \in S_V U$, $x \in S_V C$.

2) $ex = x$ *if* $x \in S_V C$.

Indeed,

$$(gh)x = (gh,x)\psi = ((g,h)\mu,\ x)\psi = (g,h,x)(\mu\amalg C)\ \psi =$$
$$(g,h,x)(U\amalg\psi)\psi = (g,\ (h,x)\psi)\psi = (g,hx)\psi = g(hx).$$

Also

$$ex = (e_V,x)\psi = (A_0\uparrow V,\ x)(e\amalg C)\psi = (A_0\uparrow V,\ x)(C\uparrow A_0\amalg C) = x.$$

COROLLARY (12.1.1). *$(gh)x = g(hx)$ if g, $h \in$ G, $x \in$ C. Also $ex = x$.*

If $g \in S_cU$ and $c \in C$, define $cg = (g,\ C)\psi c \in C$. We note that if x is in S_vC, then $xg = x\circ g \in S_vU$, so $(xg)x \in S_vC$ is defined.

LEMMA (12.2). *$(cg)^\wedge(x) = c^\wedge((xg)x)$ if $c \in C$, $x \in S_vC$, $g \in S_cU$.*

Indeed,

$$(cg)^\wedge(x) = x(cg) = x((g,C)\ \psi c) = (xg,x)\ \psi c = [(xg)x](c) = c^\wedge((xg)x).$$

We shall often write the identity of (12.2) as

$$(cg)^\wedge(x) = c^\wedge(g_V x), \qquad g_V = xg.$$

In particular, when $g \in C$, let $cg = cg_C$. Then we obtain

(12.2.1) $\qquad (cg)^\wedge(x) = c^\wedge(gx), \quad g \in G,\ c \in C,\ x \in S_vC$

since $xg_C = g_V$.

LEMMA (12.3). *$(cg)h = c(gh)$ if $c \in C$, g, $h \in G$. Also $ce = c$.*

Let $x = id_C$. Then

$$(cg)h = x((cg)h) = ((cg)h)^\wedge(x) = (cg)^\wedge(hx) =$$
$$c^\wedge(ghx) = (c(gh))^\wedge(x) = c(gh).$$

Also

$$ce = x(ce) = (ce)^\wedge(x) = c^\wedge(ex) = c^\wedge(x) = x(c) = c.$$

Given $g \in G$, define $\rho_g c = cg$ for $c \in C$. Then $\rho_g = (g_C,C)\psi$ is in (C,C). We have

$$\rho_h\rho_g = \rho_{gh}, \quad g,\ h \in G, \quad \rho_e = Id_C.$$

Lemma (12.4) below will allow us to "move around" on SU and SC. We need some preliminary definitions. If $g \in$ SU, let λ_g: U $\to$ U be defined by λ_g = (U, g_U)μ. A proof entirely analogous to that of (12.3) yields

(12.3.1) $g(hu) = (gh)u$ if $g, h \in$ SU, $u \in$ U.

We shall consider λ_g for $g \in$ SU as defining

$$\lambda_g \amalg C: U\amalg C \to U\amalg C,$$

and write $\lambda_g \amalg C$ also as λ_g. Similarly we have

$$\rho_g = U\amalg\rho_g : U\amalg C \to U\amalg C.$$

For the following lemma, one needs to refer to the remark preceding (6.8).

LEMMA (12.4). Let $g, h \in$ SU, $x \in$ SC. Then:
1) $\lambda_g \psi = \rho_g \psi$: C $\to$ U∐C ;
2) $x^{-}\rho_g = (\rho_g)_x (gx)^{-}$;
3) $h^{-}\lambda_g = (\lambda_g)_h (hg)^{-}$.

We get 2 from

$$x\rho_g = x(g_C, C)\psi = (xg_C, x)\psi = (g, x)\psi = gx$$

and 3 from

$$h\lambda_g = h(U, g_U)\mu = (h, g)\mu = hg.$$

For 1 we have

$$(\lambda_g\amalg C)\psi = ((U, g_U)\mu\amalg C)\psi = (U, g_U, C)(\mu\amalg C)\psi =$$
$$(U, g_U, C)(U\amalg\psi)\ \psi = (U\amalg(g_U, C)\psi)\ \psi = (U\amalg\rho_g)\ \psi.$$

If $f \in C_{gx}$, define $fg \in C_x$ if $g \in$ SU by $fg = (\rho_g)_x f$. Similarly, if $g, h \in$ SU, $u \in U_{hg}$, define $gu = (\lambda_g)_h u \in U_g$.

***COROLLARY* (12.4.1).** *Let $c \in$ C, $u \in$ U, $g, h \in$ SU, $x \in$ SC. Then:*

1) $g\psi(c) = \psi(c)g$;
2) $(cg)_x = (c_{gx})g$;
3) $(gu)_h = gu_{hg}$.

The following will turn out to be an important application of (12.4). Recall $\boxtimes = C\uparrow A_G\amalg C$ is the trivial action of A_G on C.

***LEMMA* (12.5).** *Let $c \in C$, $x \in X$, $g \in G$. The following are equivalent:*

1) $(\psi c)_{(g,x)} = (\boxtimes(cg))_{(g,x)}$;
2) $(\psi c)_{(e,gx)} = (\boxtimes c)_{(e,gx)}$.

By using successively 1, 3 of (12.4.1) on $A_G\amalg C$ and 2 of (12.4) we get

$$(\psi c)_{(g,x)} = (g^{-1}(\psi c)g)_{(g,x)} = g^{-1}(((\psi c)g)_{(e,x)}) = g^{-1}((\psi c)_{(e,gx)}g).$$

Similarly we get

$$(\boxtimes(cg))_{(g,x)} = (g^{-1}\boxtimes(cg))_{(g,x)} = g^{-1}((\boxtimes(cg))_{(e,x)}) =$$
$$g^{-1}(((\boxtimes c)g)_{(e,x)}) = g^{-1}((\boxtimes c)_{(e,gx)})g.$$

Therefore clearly $2 \Rightarrow 1$, and conversely, since $\lambda_{g^{-1}}$ and ρ_g are isomorphisms.

If $c \in C$ and $(\psi c)_{(e,x)} = (\boxtimes c)_{(e,x)}$ for every $x \in X$, we shall say that *c is locally invariant*. Obviously c invariant ($\psi c = \boxtimes c$) implies c is locally invariant. In the case of an action of G on a manifold, c locally invariant means c has a constant value on $\mho \cap \text{dom } c$ for every orbit $\mho \subset X$ of the action of G on X. If c is locally invariant, if G is connected and if furthermore for each orbit $\mho$,

$$\mho \cap \text{dom } c \neq \emptyset \Rightarrow \mho \subset \text{dom } c,$$

then c is invariant. Thus, for connected G, we can say that locally invariant with "invariant domain" implies invariant. We now see that this is true in general.

***THEOREM* (12.6).** *Let $c \in C$ be locally invariant with $0c$ invariant, and assume that G is connected. Then c is invariant.*

From (12.5) and local invariance, $(\psi c)_{(g,x)} = (\boxtimes(cg))_{(g,x)}$ for all $g \in G$, $x \in X$. Therefore

$$(\psi c)|(A_G)_g\amalg C = cg|(A_G)_g\amalg C, \quad g \in G.$$

Since $\psi(0c) = \boxtimes(0c)$, $(g^{\wedge}, x)(0(\psi c)) = x(0c)$, $g \in G$. Thus $(\psi c)_{(g,x)}$ not a phantom means c_x not a phantom. Let us identify $G\times X$ with $S(A_G\amalg C)$ for the moment. Then given $x \in X$, we have

$$(g,x) \in \mathrm{dom}\ (\psi c)^\frown \ \Leftrightarrow\ (g',x) \in \mathrm{dom}\ (\psi c)^\frown, \quad g,\ g' \in \mathrm{G}.$$

Let us now use the general result (12.6.1) below with $\mathrm{U} = \mathrm{A_G}$, $w = \psi c$. We obtain $\psi c = \text{¤} c'$ for some $c' \in \mathrm{C}$. Then

$$(e^\sim,\mathrm{C})\ \psi c = c = (e^\sim,\mathrm{C})(\text{¤} c') = c',$$

so $\psi c = \text{¤} c$. Thus we only need to show the following.

***LEMMA* (12.6.1).** *Let* U, C $\in K$ *with* $\mathrm{M} = \mathrm{SU}$ *connected. Let* $w \in \mathrm{U} \amalg \mathrm{C}$, *and assume that* w *satisfies the following conditions:*

1) $m \in \mathrm{M} \Rightarrow w|\mathrm{U}_m \amalg \mathrm{C} = w(m,\ )|\mathrm{U}_m \amalg \mathrm{C}$ *where* $w(m,\ ) = (m_\mathrm{C},\mathrm{C})\,w$.

2) $(mx) \in \mathrm{dom}\ w^\frown \Rightarrow \mathrm{M}\times\{x\} \subset \mathrm{dom}\ w^\frown$, $m \in \mathrm{M}$, $x \in \mathrm{SC}$.

Then $w \in \mathrm{im}\ \text{¤}_{\mathrm{U,C}}$.

This lemma is vacuous when M is, so we shall assume $\mathrm{M} \neq \emptyset$. Let $m_0 \in \mathrm{M}$ and define $c = w(m_0,\ )$. We shall see that $w = \text{¤} c$, where $\text{¤} = \text{¤}_{\mathrm{U,C}}$.

Let $\mathrm{X} = \mathrm{SC}$, $x \in \mathrm{X}$. For $f \in \mathrm{U}\amalg\mathrm{C}$, let $f_x = f|\mathrm{U}\amalg\mathrm{C}_x$. Assume that (12.6.1) is valid when C is local. Let

$$\text{¤}_x = \text{¤}_{\mathrm{U,C}x}\colon \mathrm{C}_x \to \mathrm{U}_m \amalg \mathrm{C}_x, \quad \text{and} \quad w_x = w|\mathrm{U}\amalg\mathrm{C}_x.$$

If $m \in \mathrm{M}$,

$$w_x|\mathrm{U}_m \amalg \mathrm{C}_x = w_x(m,\ )|\mathrm{U}\amalg\mathrm{C}_x.$$

Also dom $(w_x)^\frown = \mathrm{M}$ or it is empty, so 1 and 2 of (12.6.1) hold for w_x. The special case of (12.6.1) where C is local implies that $w_x \in \mathrm{im}\ \text{¤}_x$, so

$$w_x = \text{¤}_x\ w_x(m_0,\ ) = \text{¤}_x c_x = (\text{¤} c)_x.$$

This being so for all $x \in \mathrm{X}$, we shall have $w = \text{¤} c$.

We can now assume that C is local in completing the proof of (12.6.1). From 2, $0w = 0$ or $0w = \emptyset_{\mathrm{U}\amalg\mathrm{C}}$, and so we can assume $0w = 0$. We need to show that $w = \text{¤} c$, i.e., that $w_m = (\text{¤} c)_m$ for every $m \in \mathrm{M}$. Let

$$\mathrm{Z} = \{m \in \mathrm{M} \mid w_m = (\text{¤} c)_m\}.$$

Clearly (by 1) and the definition of c), $m_0 \in \mathrm{Z}$, and Z is open. To prove (12.6.1), we only need to show that Z is closed.

Let $z \in \mathrm{cl}\ \mathrm{Z}$. We have $w_z = [\text{¤}(w(z,\ ))]_z$. Take $p \in 0\mathrm{U}$ such that

$$p^\frown(z) = 0 \quad \text{and} \quad w + p = \text{¤}(w(z,\ )) + p$$

(which is possible since $\partial w = 0 \Rightarrow w_z \neq \emptyset_z$). Choose $m \in Z$ such that $p^\wedge(m) = 0$. Then

$$(\sqcap c)_m = w_m = [\sqcap(w(z,\ ))]_m.$$

As o_m: C $\to$ U_m ΠC is an injection, we see that $w(z,\ ) = c$. Therefore

$$w_z = (\sqcap(w(z,\ ))_z = (\sqcap c)_z,$$

so $z \in Z$ and Z is closed.

Let $L_G = (A_G)_e$ (= the localization of A_G at e), and define ψ_e: C $\to$ L_G∐C to be $(e^-$∐C$)\psi$. Let μ_e: L $\to$ L∐L be defined by $\mu_e = (e^-$∐$e^-)\mu$. This defines a "local action" of G on C in the sense of the following definition.

DEFINITION (12.7). A morphism ψ: C $\to$ L_G∐C *is a* local action of G on C *if*

$$(L_G \amalg \psi)\psi = (\mu_e \amalg C) \quad \textit{and} \quad (e^\wedge \amalg C)\psi = C\uparrow A_0 \amalg C.$$

We shall call ψ_e, as defined above, the *local action associated to* ψ. In (12.6) we have seen that when G is connected, $c \in$ C is invariant iff besides $\psi_e c = \sqcap_{L_G,C}c$ we have also that

$$\text{dom } c^\wedge = \{x \in X \mid x(c) \neq \emptyset_0\}$$

is an invariant set under the action of G. Thus the question of the invariance of c decomposes into an analogous question about ψ_e and a purely set-theoretic one about the action of G on X. In what follows, we shall only be looking at local actions ψ: C $\to$ L_G∐C. Of course L_G is a universe group.

The Lie algebra $\mathfrak{g}$ of G is commonly defined as the set of left invariant vector fields on G. We shall find that because we have generalized left actions of G on a manifold, it will be convenient to define $\mathfrak{g}$ to be the set of right invariant vector fields on G with Lie algebra "bracket" given by $[\theta,\chi] = \theta\chi - \chi\theta$ for $\theta, \chi \in \mathfrak{g}$. Thus

$$\mathfrak{g} \subset \text{Ader } A_G, \quad [\theta,\chi]f = \theta(\chi f) - \chi(\theta f) \text{ if } f \in A_G.$$

Since any element θ of g is uniquely determined by its value θ_e at e, where $\theta_e f = (\theta f)(e)$, $f \in A_G$, it follows also that $\mathfrak{g} \subset$ Ader L_G. In the sequel we shall tend to consider that $\mathfrak{g} \subset$ Ader L_G.

We shall need to know, in terms convenient for our theory, how to formulate the idea that a vector field θ is right invariant. Let $f \in A_G$, $h \in G$, with $f(h) \neq \emptyset_0$. Let $g(t)$ be any differentiable curve in G such that $g(0) = e$, $g'(0) = \theta_e$. Then

$$(d/dt)|_{t=0}\, f(g(t)h) = (d/dt)|_{t=0}\, (\mu f)(g(t),h) = [(\theta \amalg 0)(\mu f)](e,h).$$

Because $(e,h)^{\wedge} = h^{\wedge}(e^{\sim},A_G)$, this can be written

$$[(e^{\sim},A_G)((\theta \amalg 0)(\mu f))](h).$$

Using the right invariance of θ, we can calculate this same quantity as

$$((dr_h)(\theta_e))f = \theta_h f = (\theta f)(h).$$

Thus, for right invariant vector fields θ on G, we have the identity

$$(12.7.1) \qquad \theta = (e^{\sim},A_G)(\theta \amalg 0)\mu.$$

It is easy to see that conversely (12.7.1) implies that θ is right invariant.

DEFINITION (12.8). *Let* U *be a universe group. We shall say that* $\theta \in$ Ader U *is right invariant if* $\theta = (e_U,U)(\theta \amalg 0)\mu$.

Let τ: G×X → X be the multiplication map $(g,x) \mapsto gx$ of $S(A_G \amalg C)$ = G×X into SC = X. Let p: G×X → X be such that $p(g,x) = x$. We shall define X/G to be the coequalizer in the category of sober spaces of the pair (τ,p) (cf. §8). Thus if Y is the space of orbits of X with the quotient topology derived from X ↠ Y, then X/G is the soberization of Y. We shall let Inv C denote the set of invariant elements of C.

The following is a routine consequence of the definitions and (8.5).

PROPOSITION (12.9). *For any action of* G *on* C $\in K$ *:*

1) Inv C $\in K$ *;*

2) X/G *is homeomorphic to* $S(\mathrm{Inv}\ C)$.

The many writings on group actions as well as examples from the qualitative theory of ordinary differential equations show us how very perplexing the space X/G can be (cf. [12]). Therein lies the significance of 1 of (12.9).

13. ACTION OF A LOCAL GROUP UNIVERSE ON A UNIVERSE.

We continue the conventions of §12. We let L be a group universe such that L is local. We let $\mu: L \to L\amalg L$ be the comultiplication, and we shall assume that an action $\psi: C \to L\amalg C$ of L on a universe C is given. The goal of the remainder of this paper is to demonstrate Theorem (13.1) below.

Define $\mathfrak{g}_L$ to be the set of all right invariant elements of Ader L. If $\theta \in \mathfrak{g} = \mathfrak{g}_L$, define

$$D_\theta = (e_C, C)(\theta\amalg 0)\psi \in \mathrm{Ader}\ C.$$

If $\theta \in \mathfrak{g}$ and $c \in C$, define $\theta c = D_\theta(c) \in C$.

THEOREM (13.1). *Let* $c \in C$ *and assume that* $L = L_G$ *acts on* C. *Then:*

1) c *is invariant iff* $\theta c = 0$ *for all* $\theta \in \mathfrak{g}_L$ *;*

2) If $\theta, \chi \in \mathfrak{g}_L$, *then* $D_{[\theta,\chi]} = [D_\theta, D_\chi]$.

If $\theta \in \mathfrak{g}$ and $f \in L\amalg C$, define $\theta f = (\theta\amalg 0)f$, i.e., we just let θ operate in this instance on the first summand of $L\amalg C$.

LEMMA (13.1.1). *Let* $\theta \in \mathfrak{g}$, $c \in C$. *Then* $\theta\psi c = \psi\theta c$.

To prove (13.1.1), we shall first need to establish the following identities:

(13.1.2) $$((e_L, L)\amalg C)(\theta\amalg 0\amalg 0)(\mu\amalg C) = \theta\amalg 0.$$

(13.1.3) $$((e_L, L)\amalg C))(\theta\amalg 0\amalg 0)(L\amalg\psi) = \psi(e_C, C)(\theta\amalg 0).$$

To establish these, we notice first that replacement of $\theta\amalg 0$ and $\theta\amalg 0\amalg 0$ by $L\amalg C$ and $L\amalg L\amalg C$ (considered as identity maps) gives us correct equations

$$((e_L,L)\amalg C)(\mu\amalg C) = L\amalg C \quad \text{and} \quad ((e_L,L)\amalg C)(L\amalg\psi) = \psi(e_C,C).$$

Indeed, the first of these is obvious. Write the second as $\ell = r$. Let $U \in K$, $g \in (L,U)$, $c \in (C,U)$. Then (g,c) is an arbitrary element of $(L\amalg C, U)$ and

$$(g,c)\ell = (g(e_L,L),\ c)(L\amalg\psi) = (ge_L,g,C)(L\amalg\psi) =$$
$$(ge_L,\ (g,c)\psi) = (e_C,cg).$$

Also

$$(g,c)r = (cg)(e_C,C) = (e_U,cg),$$

so $\ell = r$.

It follows that both sides of (13.1.2) are in Ader$(L\amalg C)$ and that both sides of (13.1.3) are in Ader $(\psi(e_C,C))$. Thus to establish these identities, we only need to show that both sides of the equation in question agree on $L \subset L\amalg C$ and on $C \subset L\amalg C$ (cf. (9.8)). On C, the two sides of (13.1.2) are just multiplication by 0. Equality on L amounts to the identity $(e_L,L)(\theta\amalg 0)\mu = \theta$, which is so because θ is assumed to be right invariant. On C, both sides of (13.1.3) are multiplication by 0. On L they give $e_{L\amalg C}\theta = \psi e_C\theta$, which is true since $\psi e_C = e_{L\amalg C}$.

The proof of (13.1.1) is now possible. If $c \in C$,

$$\theta(\psi e) = (\theta\amalg 0)(\psi c) = ((e_L,\ L)\amalg C)(\theta\amalg 0\amalg 0)(\mu\amalg C)(\psi c) =$$
$$((e_L,L)\amalg C(\theta\amalg 0\amalg 0)(L\amalg\psi)(\psi c) = \psi(e_C,C)(\theta\amalg 0)(\psi c) = \psi(\theta c).$$

We can now prove 1 of (13.1). If $\theta c = 0$ for every $\theta \in \mathfrak{g}$, then $\theta\psi c = \psi\theta c = 0$ for every right invariant $\theta \in$ Ader L. Pick coordinates $z_1, \ldots, z_h$ for G in the neighborhood of e, and write

$$L = A_0\langle z_1, \ldots, z_h\rangle.$$

Let R be the local ring of L (so $L = R\amalg\{\emptyset_*\}$). Then $\partial/\partial z_i$ is a linear combination over R of right invariant elements of Ader L. If $r \in R$ and D, E $\in$ Ader L, we have

$$(rD)f = r(Df) \quad \text{and} \quad (D+E)f = Df + Ef$$

for every $f \in L\amalg C$, where D acts on $L\amalg C$ as $D\amalg 0$, E as $E\amalg 0$. Thus

$$(\partial/\partial z_i)(\psi c) = 0\psi c \text{ for } i = 1, \ldots, h.$$

By (10.2.2), this implies $\psi c \in$ C, so c is invariant.

14. LOW ORDER TERMS IN THE POWER SERIES EXPANSION OF ψ.

Let $x_1, \ldots x_h \in A_G$ be the functions of a coordinate system of G centered at $e \in$ G. Since we shall be working in the neighborhood of 0 in K^h, we shall write e as 0 and assume $x_1(0) = \ldots = x_h(0) = 0$. We have $L_G = A_0\langle x_1, \ldots, x_h\rangle$. Let us apply (11.3) ("Taylor's Theorem") to ψc. We can write

$$\psi(c) = (\psi c)(0) + (x_1 c_{11} + \ldots + x_h c_{1h}) + (1/2)\Sigma_{i,j}\, x_i (c_2)_{ij} x_j + R_2(\psi c)$$

where $R_2(\psi c)$ is an infinitesimal of order > 2. Furthermore, since c_2 is given by partial derivatives, it is a symmetric $h{\times}h$-matrix. Its entries $(c_2)_{ij} \in$ C and satisfy $0(c_2)_{ij} = 0c$. Also $0c_{1j} = 0c$, for $1 \leq j \leq h$.

Until stated otherwise, we shall steadfastly ignore all infinitesimals of order > 2. Setting $x = 0$, we get $(\psi c)(0) = c$. Let us therefore use matrix notation to write

$$\psi c = c + xc_1 + (1/2)xc_2x^T,$$

here considering x as the "row vector" $[x_1 \ldots x_h]$, with x^T the "column vector" transpose of x. For any set S we shall let $M_{u,v}(S)$ denote the set of $u{\times}v$ matrices over S.

We need to do some technical work that will lead to the proof of 2 of (13.1). Let us look at the i-th component c_{1i} of $c_1 \in M_{h,1}(C)$ in

$$\psi c = c + xc_1 + (1/2)xc_2x^T,$$

and apply ψ to it to get

$$\psi(c_{1i}) = c_{1i} + x(c_{1i})_1 + (1/2)x(c_{1i})_2x^T.$$

We shall be needing the specific formula that one can obtain for these $(c_{1i})_1 \in$ C by using the relation $(\mu\amalg C)\psi = (L\amalg\psi)\psi$. For this note that $(c_{1i})_1 \in M_{h,1}(C)$ and so has entries $(c_{1i})_{1j}$, for $1 \leq j \leq h$. It is convenient to define $(c_1)_1$ in $M_{h,h}(C)$ by

$$((c_1)_1)_{ij} = (c_{1i})_{1j},$$

To derive the formula for $(c_1)_1$, we shall first need to consider the Taylor expansion of μ at e (ignoring infinitesimals of order > 2). We shall write

$$L_h \amalg L_h = A_0\langle x_1, \ldots, x_h; y_1, \ldots, y_h\rangle = A_0\langle x; y\rangle.$$

***LEMMA* (14.1).** *For $i = 1, \ldots, h$ we have*

$$\mu x_i = x_i + y_i + x b_i y^T \quad \textit{where} \quad b_i \in M_{h,h}(K).$$

We have

$$(\mu x_i)(x; 0) = x_i \quad \text{and} \quad (\mu x_i)(0; y) = y_i.$$

Thus

$$\mu x_i = x_i + y_i + x a_i x^T + x b_i y^T + y c_i y^T$$

for appropriate $a_i, b_i, c_i \in M_{h,h}(K)$, and a_i and c_i can be taken to be symmetric. But then

$$x a_i x^T = 0 = y c_i y^T \quad \text{for all} \quad x, y \in K^h,$$

so $a_i = 0 = c_i$.

***LEMMA* (14.2).** *Let b_i be defined as in (14.1), $i = 1, \ldots, h$. Then*

$$(c_1)_1 = \Sigma_{i=1}{}^h b_i c_{1i} + c_2.$$

Use the relation $(\mu \amalg C)\psi = (L \amalg \psi)\psi$ on c, writing

$$L \amalg L \amalg C = A_0\langle x\rangle \amalg A_0\langle y\rangle \amalg C.$$

We have

$$(L\amalg\psi)(\psi c) = (L\amalg\psi)(c + x c_1 + (1/2) x c_2 x^T) = \psi c + x\psi c_1 + (1/2) x\psi(c_2) x^T$$
$$= c + y c_1 + (1/2) y c_2 y^T + x(c_1 + (c_1)_1 y^T) + (1/2) x c_2 x^T,$$

as one sees by writing

$$x\psi c_1 = x_1 \psi(c_{11}) + \ldots + x_h \psi(c_{1h})$$

and noting that

$$x_i\psi(c_{1i}) = x_i(c_{1i} + \Sigma_{j=1}{}^h (c_{1i})_{1j}y_j).$$

Let $b_* \in M_{1,h}(M_{h,h}(K))$ be defined by $b_* = [b_1 \ \dots \ b_h]$. Then also

$$(\mu\amalg C)(\psi c) = (\mu\amalg C)(c + xc_1 + (1/2)xc_2x^T) =$$
$$c + (x + y + xb_*y^T)c_1 + (1/2)(x+y)c_2(x+y)^T.$$

In view of (8.18), the expressions we have obtained for $(L\amalg\psi)(\psi c)$ and $(\mu\amalg C)(\psi c)$ must equal each other. Equating the two expressions and performing cancellations we get

$$x(c_1)_1y^T = xb_*y^Tc_1 + (1/2)(yc_2x^T + xc_2y^T).$$

As c_2 is symmetric,

$$(1/2)(yc_2x^T + xc_2y^T) = xc_2y^T.$$

Also

$$xb_*y^Tc_1 = \Sigma_{i=1}{}^h (xb_iy^T)c_{1i} = x(\Sigma_{i=1}{}^h b_ic_{1i})y^T,$$

so (12.2) follows.

Let $\theta_i \in \mathfrak{g}$ be defined by $\theta_i f = (\partial f/\partial x_i)(e)$ for $f \in L$. Abbreviate $D_{\theta i}$ by D_i.

LEMMA (14.3). *Let $c \in C$. Then:*

1) $D_ic = c_{1i}$;
2) $D_j(D_ic) = (c_{1i})_{1j}$;
3) $[D_j,D_i](c) = \Sigma_{q=1}{}^h [(b_q)_{ij} - (b_q)_{ji}]c_{1q}$.

For 1,

$$D_ic = (D_i\amalg 0)(\psi c) = (D_i\amalg 0)(c + \Sigma_{j=1}{}^h x_jc_{1j}) = c_{1i}.$$

Then, because of 1, $D_j(D_ic) = (c_{1i})_{1j}$, proving 2. From 2 and (12.2),

$$D_j(D_ic) = [(c_1)_1]_{ij} = \Sigma_{q=1}{}^h (b_q)_{ij}c_{1q} + (c_2)_{ij},$$

so 3 follows immediately from the fact that c_2 is symmetric.

At this point we shall discontinue our practice of ignoring infinitesimals of order > 2. Write $L_\theta = R\amalg\{\emptyset_*\}$.

LEMMA (14.4). *Let $i = 1, \ldots, h$. Then in* Ader L *we have*

$$\theta_i = \partial/\partial x_i + xW_i\,\partial/\partial x$$

where $W_i \in M_{h,h}(R)$ and $\partial/\partial x$ is the "column vector" with entries $\partial/\partial x_1$, $\ldots$, $\partial/\partial x_h$. Furthermore W_i satisfies

$$(W_i)_{vj}(e) = (b_j)_{iv}, \qquad 1 \leq i,\ v \leq h.$$

We can write

$$\psi x_j = x_j + y_j + x\bar{b}_j y^T$$

where $b_j \in M_{h,h}(R)$ and $b_j(e) = b_j$. If θ is any right invariant element of Ader L,

$$\theta x_j = (e\amalg L)(\theta\amalg 0)(x_j + y_j + x\bar{b}_j y^T) = \theta_e x_j + (\theta_e x)\bar{b}_j x^T.$$

In particular,

$$\theta_i x_j = \delta_{i,j} + \Sigma_{v=1}{}^h (\bar{b}_j)_{iv} x_v.$$

Thus

$$\theta_i = \Sigma_{j=1}{}^h (\theta_i x_j)(\partial/\partial x_j) = \partial/\partial x_i + x\, W_i \partial/\partial x,$$

where $(W_i)_{vj} = (\bar{b}_j)_{iv}$. This proves (14.4).

LEMMA (14.5). *For $i = 1, \ldots, h$,*

$$[\theta_i, \theta_j] = \Sigma_{q=1}{}^h ((b_q)_{ji} - (b_q)_{ij})\,\theta_q.$$

From (14.4),

$$[\theta_i,\theta_j] = [\partial/\partial x_i,\ xW_j\partial/\partial x] + [xW_i\,\partial/\partial x,\ \partial/\partial x_j] + [xW_i\,\partial/\partial x,\ xW_j\,\partial/\partial x].$$

Thus, since $x(e) = 0$,

$$[\theta_i,\theta_j]_e = (W_j(e)(\partial/\partial x)_e)_i - (W_i(e)(\partial/\partial x)_e)_j =$$

$$\Sigma_{q=1}{}^h[(W_j(e))_{iq} - (W_i(e))_{jq}](\theta_q)_e = \Sigma_{q=1}{}^h[(b_q)_{ji} - (b_q)_{ij}](\theta_q)_e.$$

Therefore the equation of (14.5) holds, for it is an equation between right invariant vector fields that agree at e.

We can now prove 2 of (13.1). From 3 of (14.3) we have

$$[D_j, D_i]c = \Sigma_{q=1}{}^h ((b_q)_{ij} - (b_q)_{ji})c_{iq},$$

from (14.5) that

$$[\theta_j, \theta_i]c = \Sigma_{q=1}{}^h ((b_q)_{ij} - (b_q)_{ji})\,\theta_q c,$$

and by 1 of (14.3), $\theta_q c = c_{lq}$, so $[D_j, D_i] = D_{[\theta_i, \theta_j]}$. Then 2 of (13.1), $[D_\theta, D_\chi] = D_{[\theta,\chi]}$ is an immediate consequence of the K-bilinearity of this equation.

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