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**DECOMPOSITION OF AUTOMATA AND  
ENRICHED CATEGORY THEORY**  
BY S. KASANGIAN and R. ROSEBRUGH \*

**RÉSUMÉ.** On étend un résultat de la théorie des automates finis concernant la décomposition concaténative de langages réguliers (Paz & Peleg) aux automates à arbres. On utilise dans cet article la théorie catégorielle enrichie des automates, où les automates à arbres se laissent décrire comme catégories enrichies sur une bicatégorie.

**1. INTRODUCTION.**

The study of non-deterministic dynamics viewed as categories enriched in a biclosed monoidal category constructed from the input monoid [1, 2, 4], and its extension to tree automata [3], is here applied to decomposition of the associated behaviours using subsets of the state spaces. Our main result is related to the concatenative decompositions of regular events defined by Paz & Peleg [5]. They showed that the behaviour of a deterministic finite automaton (with a free input monoid) is decomposable exactly when there is a subset of the state set through which every computation passes and which, together with an associated subset, defines a decomposition.

We give a decomposition of the behaviour of (= set of trees accepted by) a deterministic tree automaton in the sense of [3]. The decomposition involves a set of tuples of trees substitutable into a final set of operations of fixed arity. The result of Paz & Peleg reappears essentially as a special case of the result just quoted.

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We recall briefly some definitions relative to tree automata viewed as enriched categories. For further details, see [3].

Given an algebraic theory  $T$  (a category whose objects are finite sets  $[n] = \{1, \dots, n\}$ ,  $n = 0, 1, \dots$  and which admits the category of finite sets as a subcategory, with  $[0] = \emptyset$  as initial object and  $[1] = \{1\}$ , such that  $[m]$  is the  $m$ -fold coproduct of  $[1]$ ), we construct a bi-category  $B(T)$  which has the same objects as  $T$ , the 1-cells from  $[n]$  to  $[m]$  in  $B(T)$  are the subsets of  $T([n], [m])$  and 2-cells are inclusions. Composition of 1-cells and identities are the obvious ones.  $B(T)$  is locally partially ordered, locally complete and cocomplete and also biclosed. The arrows of  $T$  seen as 1-cells of  $B(T)$  are called *atoms*.

Let  $X$  be a  $B(T)$ -category with only one object (say  $*$ ) over  $[0]$ . We call an object  $b$  over  $[n]$  *reachable* if it is the *cotensor* of  $*$  *along an atom*. We call a skeletal  $B(T)$ -category *reachable* if all the objects are reachable.

Reachable  $B(T)$ -categories correspond to *non-deterministic T-algebras* ( $T$ -dynamics). Further, a reachable  $B(T)$ -category  $X$  corresponds to a *deterministic* reachable (i.e., definable)  $T$ -algebra if its underlying category is *discrete* and if it is *cotensored along the atoms*.

We denote by  $X_{[n]}$  the "fibre" of  $X$  over  $[n]$ . Then  $X_{[1]}$  is the carrier of the algebra and  $X_{[0]} = *$ . Denoting by  $[n]$  the trivial, one object category over  $[n]$ , a *tree automaton* (i.e., a  $T$ -dynamics with a subset  $F \subset X_{[1]}$  of *final states*) can be described as a triple  $(X, i, \tau)$  as follows:

- $X$  is a reachable (possibly deterministic)  $B(T)$ -category;
- $i: X \rightarrow [0]$  is the *initial module*, given by  $i(b) = X(b, *)$ , where  $b = (b_1, \dots, b_n)$  is an  $X$ -object over  $[n]$ ;
- $\tau: [1] \rightarrow X$  is the *final module*, given by

$$\tau(b) = \{ g \in T([1], [n]) \mid \text{there exists } a \in F \subset X_{[1]} \text{ and } g \in X(a, b) \}.$$

Notice that if the automaton is deterministic, the definition above can be stated using the cotensor, namely

$$\tau(b) = \{ g \in T([1], [n]) \mid \text{there exists } a \in F \subset X_{[1]} \text{ and } b \otimes g = a \}.$$

Thus, the module  $i$  provides sets of tuples of trees (terms), whereas  $\tau$  selects sets of operations which are "successful" if performed on those trees. The composite module  $i \cdot \tau$  is the *behaviour* of the automaton and consists of the set of trees, i.e., 1-cells at  $T([1], [0])$ , which are recognizable. Henceforth we assume that the tree automata considered are *deterministic*.

2. THE DECOMPOSITION THEOREM.

**DEFINITION 2.1.** We call a set of states  $R \subset \text{obj } X$  a *decomposition set* for the automaton  $(X, i, \tau)$  if  $i.\tau = \sum_{r \in R} i(r).\tau(r)$ .

**DEFINITION 2.2.** Given a set of states  $S \subset \text{obj } X$ , we call *associated with S* the set of states

$$S^* = \{ s^* \in \text{obj } X \mid \prod_{s \in S} \tau(s) \subset \tau(s^*) \}.$$

**REMARK 2.3.** Notice that if  $b$  is in  $X_m$  and  $c$  is in  $X_n$ , with  $n \neq m$ , it is always the case that  $\tau(b) \cap \tau(c) = \emptyset$ . Hence, the notion of associated set trivializes unless there is an  $n$  with  $S \subset X_n$ . Henceforth we assume this whenever we mention associated sets. Notice also that  $S \subset S^*$  from the definition.

Recall that the behaviour  $i.\tau$  of an automaton  $(X, i, \tau)$  is a module  $i.\tau : [1^*] \rightarrow [0^*]$  and hence a 1-cell from  $[1]$  to  $[0]$  in  $B(T)$ . Notice that it may admit a decomposition into two 1-cells of  $B(T)$ . We have the following:

**DEFINITION 2.4.** Let  $(X, i, \tau)$  be an automaton with behaviour  $i.\tau$ . We say that 1-cells  $D: [1] \rightarrow [n]$  and  $C: [n] \rightarrow [0]$  ( $n \neq 0$  and if  $n = 1$ , then  $D \neq 1_{[1]}$ ) in  $B(T)$  are a *decomposition* if  $i.\tau = C.D$ . Behaviours which admit a decomposition are said to be *decomposable*.

Notice that  $C$  is a set of  $n$ -tuples of trees and  $D$  is a set of  $n$ -ary operations which is performed successfully on these trees. Hence the definition ensures that the last branching of any tree of the behaviour is  $n$ -ary.

We are now able to prove the *Decomposition Theorem*:

**THEOREM 2.5.** Given an automaton  $(X, i, \tau)$  its behaviour is decomposable iff it admits a decomposition set  $S$  such that:

$$i.\tau = \sum_{q \in S^*} i(q) \cdot \prod_{q' \in S} \tau(q').$$

**PROOF.** Observe first that, by Remark 2.3, there exists an  $n$  such that

$$S \subset S^* \subset X_n, \quad \sum_{q \in S^*} i(q) : [n] \rightarrow [0] \quad \text{and} \quad \prod_{q' \in S} \tau(q') : [1] \rightarrow [n],$$

so sufficiency is obvious. To prove necessity, assume the behaviour is decomposable, i.e.,

$$i.\tau = ([1] \xrightarrow{D} [n] \xrightarrow{C} [0]), \quad \text{with } n \neq 0.$$

We define  $S \subset X_n$  as follows:

$$S = \{ s \in X_n \mid i(s) \cap C \neq \emptyset \}.$$

We show first that  $S$  is a decomposition set. For any 1-cell  $h$  in the behaviour  $i.\tau$ , there exists an  $n$ -tuple of trees  $c_1 : [n] \rightarrow [0]$  in  $C$  and an  $n$ -ary operation  $d_1 : [1] \rightarrow [n]$  in  $D$  such that  $h = c_1 d_1$ . Since  $X$  is reachable, there is a  $q = * \phi c_1$  in  $X_n$ , so that  $c_1 \in i(q) \cap C$  and hence  $q \in S$ . Since  $c_1 d_1$  is in  $i.\tau$ ,  $d_1 \in \tau(q)$  and so we have

$$i.\tau \subset \sum_{q' \in S} i(q') . \tau(q').$$

Hence  $S$  is a decomposition set. Further,

$$C \subset \sum_{q' \in S} i(q') \subset \sum_{q \in S} i(q)$$

since  $S \subset S^\wedge$ . Next we show that  $D \subset \prod_{q' \in S} \tau(q')$ . Given  $d \in D$ , we know that, for all  $c \in C$ ,  $cd \in i.\tau$  and there is a  $q = * \phi c$  in  $S$  such that  $c \in i(q)$  and  $d \in \tau(q)$ . Since  $S$  is a decomposition set, for all  $q' \in S$ , there exists a  $c^- \in i(q')$  such that  $q' = * \phi c^-$  and  $d \in \tau(q')$ . Thus it follows that  $d \in \prod_{q' \in S} \tau(q')$ . Therefore  $\sum_{q \in S} i(q)$  and  $\prod_{q' \in S} \tau(q')$  are non-empty and moreover

$$\sum_{q \in S} i(q) . \prod_{q' \in S} \tau(q') \supset C.D = i.\tau.$$

To finish to show the reverse inclusion, let

$$z = xy, \quad \text{with } x \in \sum_{q \in S} i(q) \quad \text{and } y \in \prod_{q' \in S} \tau(q').$$

Now there exists a  $q^-$  in  $S^\wedge$  with  $x \in i(q^-)$ , i.e.,  $q^- = * \phi x$ . Further, by the definition of an associated set,  $y \in \prod_{q' \in S} \tau(q')$  implies  $y \in \tau(q^-)$  for all  $q \in S^\wedge$ . Hence

$$y \in \tau(q^-) \quad \text{and} \quad z = xy \in i(q^-) . \tau(q^-) = \sum_{q \in S} i(q) . \tau(q) = i.\tau.$$

**REMARK 2.6.** Observe that the proof of Theorem 2.5 ensures that the decomposition described above is *maximal* with respect to the obvious partial order on the set of pairs of 1-cells which decompose the behaviour. Recall that, for any  $n$ ,  $B(T)([n], [n])$  is a monoid, with identity  $1_{[n]}$  and multiplication given by composition of 1-cells.

**PROPOSITION 2.7.** Let  $(X, i, \tau)$  be an automaton and  $S \subset X_n$  a decomposition set. Define

$$L = \prod_{q' \in S} \sum_{q \in S} X(q', q).$$

$L$  is a submonoid of  $B(T)([n], [n])$ .

**PROOF.** It is immediate that  $1_{[n]} \in L$ , since  $S \subset S^{\wedge}$  and  $1_{[n]} \subset X(s, s)$  for all  $s \in S$ . Notice also that, since the automaton is deterministic, an equivalent definition of the associated set of states (Definition 2.2) is

$$S^{\wedge} = \{ s^{\wedge} \in \text{obj } X \mid s\phi z \in F \text{ for all } s \in S \text{ implies } s^{\wedge}\phi z \in F \}.$$

Now we need to show that if  $x, y \in L$ , then  $xy \in L$ . Given a  $z : [n] \rightarrow [n]$  such that  $s\phi z \in F$  for all  $s \in S$  and observing that  $s\phi y \in S^{\wedge}$ , we have that

$$(s\phi y)\phi z = s\phi(yz) \quad \text{for all } s \in S.$$

By the same argument,  $(s\phi x)\phi yz \in F$  for all  $s \in S$ . But

$$(s\phi x)\phi yz = s\phi(xyz) = (s\phi xy)\phi z,$$

so that  $s\phi xy \in S^{\wedge}$  for all  $s \in S$ . Thus  $xy \in L$ .

The 1-cells of  $B(T)([n], [n])$  are  $n$ -tuples of  $n$ -ary operations and composition is substitution. If we take  $n = 1$ , the 1-cells of  $B(T)([1], [1])$  are unary operations so that giving a decomposition set  $S \subset X_1$  amounts to considering actions of the monoid  $L$  above (a submonoid of  $B(T)([1], [1])$ ) on a set of trees.

In the next section we will see an interpretation of these results in the more special context of sequential automata.

### 3. APPLICATIONS TO SEQUENTIAL AUTOMATA.

The B-categorical approach to tree automata admits a straightforward specialization to sequential automata. However, we will follow the lines of [1, 2, 4] giving a slightly different (though obviously equivalent) description in terms of categories enriched in a monoidal biclosed category, i.e., in a biclosed category with one object.

The input monoid  $X$  yields a monoidal biclosed category  $X^{\sim} = 2^X$ , where the tensor product is just the Frobenius product of subsets of  $X$  and the internal homs are given by left and right quotients. A (not necessarily deterministic) dynamics is then an  $X^{\sim}$ -category  $Q$  where objects  $q, q'$  in  $Q$  are the states and  $Q(q, q')$  is the set of monoid elements which act on  $q$  (possibly in a non-deterministic way), carrying it to  $q'$ . A deterministic dynamics is an  $X^{\sim}$ -category which is tensored along the "atoms", i.e., the elements of  $X$ , and whose underlying category is discrete. An  $X$ -automaton is then a triple  $(Q, i, \tau)$  as in the following diagram

$$1 \xrightarrow{\tau} Q \xrightarrow{i} 1$$

where  $1$  is the trivial, one-object  $X^{\sim}$ -category and  $i$  and  $\tau$  are the *initial* and *final* modules. The behaviour is again  $i, \tau$  and it is the subset of  $X$  (i.e., a *language*) recognized by the automaton. Modulo a "normalization" described in [4], we can think of these modules, as given by

$$i(q) = \Sigma_{j \in J} Q(j, q) \quad \text{and} \quad \tau(q) = \Sigma_{t \in F} Q(q, t),$$

where  $J$  and  $F$  are the sets of initial and final states. A deterministic automaton has a deterministic dynamics and further the initial module is required to be  $I_*$  for some  $X^{\sim}$ -functor  $I$  from  $1$  to  $Q$ , i.e.,  $Q(i, q) = I_*(q)$ .

As for reachability, here it means that for all  $q$  in  $Q$ ,  $i(q) \neq \emptyset$ . The definitions of decomposition set (2.1) and associated set (2.2) apply straightforwardly to this context. The decomposition of a behaviour still exhibits it as the composite of two 1-cells of the (one-object) bicategory  $X^{\sim}$ . Given a language  $A$  in  $X^{\sim}$ , a decomposition for it is a pair of languages  $B, C$  such that  $A = B.C$ ,  $B \neq \{e\}$ ,  $C \neq \{e\}$ . This definition applies of course to behaviours and yields the notion of *decomposable behaviour*. The following is the analogue of Proposition 2.5.

**PROPOSITION 3.1.** *Given an automaton  $(Q, i, \tau)$  with deterministic dynamics, its behaviour is decomposable iff it admits a decomposition set  $S$  such that  $i, \tau = \Sigma_{q \in S} i(q) \amalg_{q' \in S} \tau(q')$ .*

The proof of Proposition 3.1 is nearly identical to that of Proposition 2.5 provided some attention is paid to different interpretations. In particular, recall the different meanings of  $i$  and  $\tau$  and that now the decomposition set and its associated set are obviously

constructed without the restrictions of Remark 2.3:  $Q$  is all in one "fibre". Further, in the proof the tensor in  $Q$  (rather than the cotensor) is used because no contravariance is involved. The same interchange of tensor and cotensor provides the adjustments necessary to prove the analogue of Proposition 2.7.

**PROPOSITION 3.2.** *Let  $\langle Q, i, \tau \rangle$  be an automaton with deterministic dynamics and  $S \subset Q$  a decomposition set. Define*

$$L = \prod_{q' \in S} \Sigma_{q \in S} Q(q', q).$$

*L is a monoid.*

Restricting ourselves to deterministic automata (that is with one initial state) and observing that the initial state is a decomposition set, we get the following:

**PROPOSITION 3.3.** *Let  $\langle Q, i, \tau \rangle$  be a deterministic reachable automaton. Then the behaviour is a monoid iff  $i_0^* = F$ , where  $i_0$  is the initial state.*

Notice finally that by specializing further to finite state deterministic automata on a free monoid  $X$ , we get the results of Paz & Peleg (see [5], Theorem 1, Lemma 3, Theorem 3).

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