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CAUCHY COMPLETION IN CATEGORY THEORY

by Francis BORCEUX and Dominique DEJEAN

RÉSUMÉ. Une catégorie $\mathbf{C}$ est dite Cauchy complète si tout idempotent se scinde ; toute petite catégorie a une complétion de Cauchy. Ce problème peut être abordé en termes de colimites absolues ou de distributeurs. Nous prouvons un théorème de foncteur adjoint où la condition usuelle de complétion est remplacée par une complétion de Cauchy. Toutes ces notions et résultats peuvent être enrichis dans une "bonne" catégorie fermée. En regardant un anneau comme une catégorie enrichie dans sa catégorie de modules, sa complétion de Cauchy est la catégorie des modules projectifs finiment engendrés. En considérant un espace métrique comme une catégorie enrichie sur $\overline{\mathbf{R}}_+$, sa complétion de Cauchy catégorique est exactement sa complétion en tant qu'espace métrique, à l'aide des suites de Cauchy.

This paper is to be considered as a survey article presenting an original and unified treatment of various results, scattered in the litterature. The reason for such a work is the grewing importance of everything concerned with the splitting of idempotents and the lack of a reference text on the subject. Most of the work devoted to Cauchy completion has been developped in the sophisticated context of bicategories : it's our decision to focuse on direct proofs in the context of classical category theory.

An idempotent endomorphism e splits when it can be written as $e = i \circ r$ with $r \circ i$ an identity. Every small category $\mathbf{C}$ has a completion for the splitting of idempotents (called its Cauchy completion) given by the retracts of its representable functors or, equivalently, by its absolutely presentable presheaves or even by the distributors $\mathbf{1} \multimap \mathbf{C}$ which possess a right adjoint. The Cauchy completeness of a category $\mathbf{C}$ is also equivalent to its completeness for the absolute colimits or to the existence of a right adjoint for every distributor $\mathbf{D} \multimap \mathbf{C}$ ... but this last equivalence is itself equivalent to the axiom of choice.

The classical adjoint functor Theorem involves the assumptions of completeness of the domain category and continuity of the functor. In fact completeness can be replaced by the much weaker assumption of Cauchy completeness and, clearly, the continuity means now the absolute flatness of the functor. We prove the adjoint functor Theorem under those very weak assumptions and, of course, the solution set condition.

We mention how the definition of Cauchy completion in terms of distributors can be easily generalized to the context of category theory based on a closed category. In the case of a ring viewed as a category enriched over its category of modules, the Cauchy completion is the important category of finitely generated projective modules. In the case of a metric space viewed as a category enriched over $\bar{\mathbf{R}}_+$, the Cauchy completion is just the classical completion as a metric space, using Cauchy sequences. This explains the terminology, introduced by F.W. Lawvere (cf. [9]).

1. SPLITTING OF IDEMPOTENTS.

We fix a category $\mathbf{C}$. A morphism $e : C \to C$ is idempotent when $e = e \circ e$. Given a retraction

$$D \underset{i}{\overset{r}{\rightleftarrows}} C, \qquad \text{thus} \qquad r \circ i = \mathrm{id}\,,$$

it follows immediately that $e = i \circ r$ is idempotent. A split idempotent e is one which can be presented as $e = i \circ r$ for some retraction $r \circ i = \mathrm{id}$.

Proposition 1. *The following conditions are equivalent for an idempotent* $e : C \to C$ *:*

(1) e *splits as* $e = i \circ r$ *;*
(2) the equalizer $i = \ker(e, \mathrm{id})$ *exists ;*
(3) the coequalizer $r = \mathrm{coker}(e, \mathrm{id})$ *exists.*

Moreover, the equalizer in (2) and the coequalizer in (3) are absolute.

The equivalences are obvious. We recall that a limit or a colimit is called absolute when it is preserved by every functor. Clearly the property of being a retraction is absolute. ◊

Given a small category $\mathbf{C}$, we use the classical notation $\hat{\mathbf{C}}$ for the category $Funct(\mathbf{C}^{op}, \mathrm{Sets})$ of presheaves on $\mathbf{C}$. We identify $\mathbf{C}$, via the Yoneda embedding $Y : \mathbf{C} \to \hat{\mathbf{C}}$, with the full subcategory of representable functors. We denote by $\bar{\mathbf{C}}$ the full subcategory of $\hat{\mathbf{C}}$ spanned by all the retracts of the representable functors.

Theorem 1. *Let* $\mathbf{C}$ *be a small category and* $\bar{\mathbf{C}}$ *the full subcategory of* $\hat{\mathbf{C}}$ *spanned by the retracts of the representable functors.*

(1) $\bar{\mathbf{C}}$ *contains* $\mathbf{C}$ *as a full subcategoru ;*
(2) every idempotent of $\bar{\mathbf{C}}$ *splits ;*
(3) the inclusion $\mathbf{C} \hookrightarrow \bar{\mathbf{C}}$ *is an equivalence if and only if every idempotent of* $\mathbf{C}$ *splits ;*
(4) the category $\hat{\bar{\mathbf{C}}}$ *of presheaves on* $\bar{\mathbf{C}}$ *is equivalent to the category* $\hat{\mathbf{C}}$ *of presheaves on* $\mathbf{C}$*.*

This small category $\bar{\mathbf{C}}$ *will be called the Cauchy completion of* $\mathbf{C}$ *.*

$\bar{\mathbf{C}}$ is small since $\hat{\mathbf{C}}$ is well-powered. $\bar{\mathbf{C}}$ contains $\mathbf{C}$ as a full subcateogry because the Yoneda embedding is full and faithful. Every idempotent of $\bar{\mathbf{C}}$ splits in $\hat{\mathbf{C}}$ (Proposition 1), thus also in $\bar{\mathbf{C}}$ since the composite of two retractions is again a retraction.

If every idempotent of $\mathbf{C}$ splits, every retract of a representable functor $\mathbf{C}(-, C)$ induces an idempotent on $\mathbf{C}(-, C)$, thus an idempotent e on C. e splits in $\mathbf{C}$, which produces a retraction of C and thus a retraction of $\mathbf{C}(-, C)$, which is necessarily isomorphic to the original retraction (Proposition 1). Thus every retraction of $\mathbf{C}(-, C)$ is itself representable, which proves (3).

To prove (4), it suffices to show that every presheaf F on $\mathbf{C}$ can be uniquely (up to an isomorphism) extended in a presheaf $\bar{F}$ on $\bar{\mathbf{C}}$. From the uniqueness of the splitting in $\bar{\mathbf{C}}$ of an idempotent $e \in \mathbf{C}$ and the Cauchy completeness of the category of sets (Proposition 1), $\bar{F}$ has to map the splitting of e on the splitting of Fe. Now if

$$R \underset{i}{\overset{r}{\leftrightarrows}} \mathbf{C}(-, C) \qquad \text{and} \qquad S \underset{j}{\overset{s}{\leftrightarrows}} \mathbf{C}(-, D)$$

are retractions, every morphism $f : R \to S$ can be written as

$$f = s \circ (j \circ f \circ r) \circ i$$

and $\bar{F}(s)$, $\bar{F}(i)$ are already defined, while $\bar{F}(j \circ f \circ r)$ has to be equal to $F(j \circ f \circ r)$. ◊

The Cauchy completion $\bar{\mathbf{C}}$ of $\mathbf{C}$ has also an interesting description in terms of colimits. We recall that an object $F \in \hat{\mathbf{C}}$ is absolutely presentable when the representable functor $\mathbf{C}(F, -) : \mathbf{C} \to \mathit{Sets}$ preserves all small colimits. Extending the terminology of [5], this is just the notion of an α-presentable object, where $\alpha = 2$ is the only finite regular cardinal.

Proposition 2. *Given a small category* $\mathbf{C}$*, a presheaf* $F \in \hat{\mathbf{C}}$ *lies in the Cauchy completion* $\bar{\mathbf{C}}$ *iff it is absolutely presentable.*

A representable functor is absolutely presentable since $\hat{\mathbf{C}}(\mathbf{C}(-, C), -)$ is just the evaluation at C. Now if R is a retract of $\mathbf{C}(-, C)$, $\hat{\mathbf{C}}(R, -)$ is obtained from $\hat{\mathbf{C}}(\mathbf{C}(-, C), -)$ by a coequalizer diagram; by associativity of colimits, R is absolutely presentable.

Conversely choose $F \in \hat{\mathbf{C}}$ absolutely presentable. F can be presented as the colimit $F = \varinjlim \mathbf{C}(-, C)$ indexed by all the pairs (γ, C), with $\gamma : \mathbf{C}(-, C) \Rightarrow F$. As a consequence

$$\mathbf{C}(F, F) \simeq \mathbf{C}(F, \varinjlim \mathbf{C}(-, C)) \simeq \varinjlim \mathbf{C}(F, \mathbf{C}(-, C)) .$$

Now id_F corresponds to some element in the last colimit and this element is represented by some $\beta : F \Rightarrow C(-, C)$ in the term of index (γ, C) ; in other words, $\alpha \circ \beta = id_F$ and F is a retract of $C(-, C)$. ◊

A colimit is called absolute when it is preserved by every functor. Given a small category **C**, the topos $\hat{\mathbf{C}}$ of presheaves is cocomplete thus, in particular, has all absolute small colimits. We shall say that **C** has all absolute small colimits when it is stable in $\hat{\mathbf{C}}$ under all absolute small colimits. The Cauchy completion can also be seen as the completion for absolute small colimits, as noticed by R. Street (cf. [11]).

Proposition 3. *The following conditions are equivalent on a small category* **C** *:*

(1) **C** *is Cauchy complete.*
(2) **C** *has all absolute small colimits.*

Suppose **C** is Cauchy complete and consider an absolute colimit $L = \varinjlim \mathbf{C}(-, C_i)$ in $\hat{\mathbf{C}}$. Since the colimit is absolute

$$\hat{\mathbf{C}}(L, L) = \hat{\mathbf{C}}(L, \varinjlim \mathbf{C}(-, C_i)) = \varinjlim \hat{\mathbf{C}}(L, \mathbf{C}(-, C_i)).$$

So to id_L corresponds the equivalence class of some $\beta : L \to \mathbf{C}(-, C_i)$ and if $s_i : \mathbf{C}(-, C_i) \to L$ is the canonical morphism of the colimit, $s_i \circ \beta = id_L$. Thus L is a retract of $\mathbf{C}(-, C_i)$ and lies therefore in $\bar{\mathbf{C}}$.

The converse is immediate since Cauchy completeness reduces to the existence of some absolute coequalizers (Proposition 1). ◊

2. CAUCHY COMPLETION IN TERMS OF DISTRIBUTORS.

A distributor (also called profunctor, bimodule, module, ...) from a small category **A** to a small category **B** is a functor $F : \mathbf{B}^{op} \times \mathbf{A} \to Sets$; we shall use the notation $F : \mathbf{A} \nrightarrow \mathbf{B}$. If $G : \mathbf{B} \nrightarrow \mathbf{C}$ is another distributor, the composite $G \circ F : \mathbf{A} \nrightarrow \mathbf{C}$ is defined by a colimit

$$(G \circ F)(C, A) = \varinjlim G(C, B) \times F(B, A)$$

indexed by all the morphisms $b : B \to B'$ in **B**, in fact $(x, F(b, A)(y))$ is identified to $(G(C, b)(x), y)$ in the colimit. Morphisms between distributors are just natural transformations ; they are easily equipped with a vertical and an horizontal composition and this yields a corresponding notion of adjoint distributors. Every functor $F : \mathbf{A} \to \mathbf{B}$ induces a distributor $F^* : \mathbf{A} \nrightarrow \mathbf{B}$ given by

$$F^*(B, A) = \mathbf{B}(B, FA).$$

F^* has a right adjoint $F_* : B \to A$ which is just

$$F_*(A, B) = \mathbf{B}(FA, B)$$

(cf. [1, 5]).

We shall denote by $\mathbf{1}$ the category with one single object and one single morphism (= the identity on the object). The topos $\hat{\mathbf{C}}$ of presheaves on a small category $\mathbf{C}$ can be viewed as the category $\mathrm{Dist}(\mathbf{1}, \mathbf{C})$ of distributors from $\mathbf{1}$ to $\mathbf{C}$. On the other hand the category $\mathbf{C}$ itself can be identified with the category $\mathrm{Funct}(\mathbf{1}, \mathbf{C})$ of functors from $\mathbf{1}$ to $\mathbf{C}$. The Yoneda embedding is then just the inclusion

$$\mathrm{Funct}(\mathbf{1}, \mathbf{C}) \to \mathrm{Dist}(\mathbf{1}, \mathbf{C})$$

and the Cauchy completion $\overline{\mathbf{C}}$ appears as a subcategory of $\mathrm{Dist}(\mathbf{1}, \mathbf{C})$.

Proposition 4. *Given a small category $\mathbf{C}$, a distributor $F : \mathbf{1} \dashrightarrow \mathbf{C}$ belongs to the Cauchy completion of $\mathbf{C}$ iff it has a right adjoint.*

Fix a pair $F : \mathbf{1} \dashrightarrow \mathbf{C}$, $G : \mathbf{C} \dashrightarrow \mathbf{1}$ of adjoint distributors, with $F \dashv G$. We have thus :

$$\text{a functor } F : \mathbf{C}^{op} \longrightarrow \mathrm{Sets},$$

$$\text{a functor } G : \mathbf{C} \to \mathrm{Sets}\ .$$

The two composites are just

$$\text{the functor } F \circ G : \mathbf{C}^{op} \times \mathbf{C} \to \mathrm{Sets} : (B, A) \mapsto FB \times GA,$$

$$\text{the set } G \circ F = \frac{\coprod GC \times FC}{(x, F(f)(y)) \sim (G(f)(x), y)}$$

where $f : A \to B$ is an arbitrary morphism of $\mathbf{C}$. The two natural transformations of the adjunction are

$$\text{an element } \alpha = [\,(v, u) \in GC \times FC\,] \in G \circ F,$$

$$\text{a natural transformation } \beta_{B,A} : FB \times GA \to \mathbf{C}(B, A)$$

where C is now a fixed object in $\mathbf{C}$.

The two axioms for the adjointness reduce to :

$$G(\beta_{C,A}(u, x))(v) = x \qquad \text{for every } x \in GA,$$

$$F(\beta_{B,C}(y, v))(u) = y \qquad \text{for every } y \in FB.$$

If such an adjoint pair is given, we produce immediately two natural transformations

$$\gamma : F \Rightarrow \mathbf{C}(-, C) : \gamma_B(y) = \beta_{B,C}(y, v)\ ,$$

$$\delta : \mathbf{C}(-, C) \Rightarrow F : \delta_A(g) = F(g)(u)$$

and the second axiom for adjoi-'ness is precisely the relation $\delta \circ \gamma = id_F$. Thus F is a retract of $\mathbf{C}(-, c)$.

Conversely if F is a retract of $\mathbf{C}(-, c)$, with $\delta \circ \gamma = id_F$ as above, the idempotent transformation $\gamma \circ \delta$ on $\mathbf{C}(-, c)$ has the form $\mathbf{C}(-, e)$ with e on idempotent morphism on C. The splitting of the idempotent transformation $\mathbf{C}(e, -)$ on $\mathbf{C}(C, -)$ in Funct($\mathbf{C}$, Sets) produces a retract

$$G \underset{\varphi}{\overset{\psi}{\rightleftarrows}} \mathbf{C}(C, -)\,.$$

It is now sufficient to consider

$$(u, v) = (\psi(id_C), \delta(id_C)) \in GC \times FC,$$

$$\beta_{B,A} : FB \times GA \to \mathbf{C}(B, A) : \; (y, x) \mapsto \varphi_A(x) \circ \gamma_B(y)$$

in order to prove the required adjunction. ◊

The great interest of Cauchy completion in the theory of distributors is described in our next theorem.

Theorem 2. *The following conditions are equivalent on a small category* $\mathbf{C}$.

(1) $\mathbf{C}$ *is Cauchy complete.*

(2) A distributor $1 \overset{\circ}{\rightarrow} \mathbf{C}$ *has a right adjoint iff it is a functor.*

(3) For every small category $\mathbf{A}$ *a distributor* $\mathbf{A} \overset{\circ}{\rightarrow} \mathbf{C}$ *has a right adjoint iff it is a functor.*

(1) <=> (2) is a consequence of Theorem 2 and (3) => (2) is obvious. Let us prove (2) => (3). Consider two distributors F, G with $F \dashv G$ and an object $A \in \mathbf{A}$.

$$\begin{tikzcd} 1 \arrow[r, shift left, "\mathbf{A}(-, A)"', "\circ"] & \mathbf{A} \arrow[l, shift left, "\mathbf{A}(A, -)"', "\circ"] \arrow[r, shift right, "F"', "\circ"] & \mathbf{C} \arrow[l, shift right, "G"', "\circ"] \end{tikzcd} .$$

We obtain the adjunction

$$F \circ \mathbf{A}(-, A) \dashv \mathbf{A}(A, -) \circ G,$$

which proves the existence of some object $H(A) \in \mathbf{C}$ such that

$$F \circ \mathbf{A}(-, A) \simeq \mathbf{C}(-, H(A))$$

(Assumption 2). Now one computes easily the identity

$$(F \circ \mathbf{A}(-, A))(C) \simeq F(C, A)$$

which implies finally

$$F(C, A) \simeq \mathbf{A}(C, H(A)) .$$

The proof extends to the morphisms of $\mathbf{A}$ and therefore F is the distributor associated to the functor H. ◊

The proof of (2) => (3) in the previous theorem uses some choice principle to define H(A). In fact the full strength of the axiom of choice is necessary. This is proved by the following elegant remark, due to A. Carboni and R. Street (cf. [3]). A poset, viewed as small category, is obviously Cauchy complete since the only idempotent morphisms are the identities. If Condition 3 in the previous theorem is now taken as a definition for the Cauchy completeness, it turns out that the Cauchy completeness of posets is equivalent to the axiom of choice.

Proposition 5. *The following conditions are equivalent :*

(1) The axiom of choice.

(2) Every distributor $F : \mathbf{A} \multimap \mathbf{C}$ *between posets* $\mathbf{A}, \mathbf{C}$ *is a functor when it has a right adjoint.*

By Theorem 2, it suffices to prove (2) => (1). Consider a surjection $g : X \twoheadrightarrow Y$ in Sets and $R \rightarrowtail X \times X$ the equivalence relation defined by f. We define

$$\mathbf{A} = (Y, \Delta_Y) \quad \text{and} \quad \mathbf{C} = (X, R),$$

where Δ_Y denotes the diagonal of Y ; $\mathbf{A}$ and $\mathbf{C}$ are posets. We define also two distributors

$$F : \mathbf{A} \multimap \mathbf{C} : \quad F(x, y) = \begin{cases} 1 & \text{if } y = f(x) \\ \emptyset & \text{if } y \neq f(x) \end{cases}$$

$$G : \mathbf{C} \multimap \mathbf{A} : G(y, x) = \begin{cases} 1 & \text{if } y = f(x) \\ \emptyset & \text{if } y \neq f(x) \end{cases}$$

The two possible composites are given by

$$F \circ G : \mathbf{C} \multimap \mathbf{C} \;:\; F \circ G(x, x') = \begin{cases} 1 & \text{if } (x, x') \in R \\ \emptyset & \text{if } (x, x') \notin R \end{cases}$$

$$G \circ F : \mathbf{A} \multimap \mathbf{A} : \; G \circ F(y, y') = \begin{cases} 1 & \text{if } (y, y') \in \Delta_Y \\ \emptyset & \text{if } (y, y') \notin \Delta_Y \end{cases}$$

It follows immediately that

$$\mathrm{id}_{\mathbf{C}} \leq F \circ G \quad \text{and} \quad G \circ F = \mathrm{id}_{\mathbf{A}}$$

or in other words, G is right adjoint to F. By assumption 2, F is represented by a functor $f : \mathbf{C} \to \mathbf{A}$, which provides a section $f : Y \nrightarrow X$ to g, since $G \circ F = \mathrm{id}_{\mathbf{A}}$. ◊

3. THE MORE GENERAL ADJOINT FUNCTOR THEOREM.

This "more general Adjoint Functor Theorem" is an unpublished result of P. Freyd; it refers to Cauchy completeness instead of usual completeness (cf [8]).

When α is a regular cardinal, an α-limit is the limit of a diagram of cardinality strictly less than α. When A is a category with α-limits, the α-continuity of a functor $F : \mathbf{A} \to \mathbf{B}$ means just the preservation of α-limits; this fact is equivalent to any one of the following conditions :

(1) For **every** $B \in \mathbf{B}$, the functor $\mathbf{B}(B, F-) : \mathbf{A} \to \mathrm{Sets}$ is an α-filtered colimit of representable functors.

(2) For every $B \in \mathbf{B}$, the comma category (B, F) is α-cofiltered, where B stands for the functor $\mathbf{1} \to \mathbf{B}$ choosing the object $B \in \mathbf{B}$.

When A is not necessarily α-complete, the α-continuity of a functor $F : A \to B$ is clearly an uninteresting notion. But conditions (1) and (2) are still equivalent in that case and describe what is called the "α-flatness" of F. Flatness is the correct generalization of the notion of continuity in a context where limits do not necessarily exist (cf. [5], α-Stetigheit). The functor $F : \mathbf{A} \to \mathbf{B}$ will be called absolutely flat when it is α-flat for every regular cardinal α. When **A** is complete, the absolute flatness of F is thus equivalent to the preservation of small limits.

Theorem 3. *Consider an absolutely flat functor $F : \mathbf{A} \to \mathbf{B}$ defined on a Cauchy complete category* **A**. *F has a left adjoint iff the solution set condition holds.*

Fix an object $B \in \mathbf{B}$ and let $S_B \subset |\mathbf{A}|$ be a solution set for B. The Cauchy completeness of **A** implies immediately that of the comma category (B, F).

Consider in (B, F) the following set of objects :

$$\mathcal{S}_B = \{ (A, b) \mid A \in S_B \, ; \, b : B \to FA \}$$

and choose, by absolute flatness of F, an object $Z \in |(B, F)|$ and for every $X \in \mathcal{S}_B$, a morphism $\alpha_X : Z \to X$ in (B, F). Again by absolute flatness of F, choose an object $Y \in |(B, F)|$ and a morphism $u : Y \to Z$ in (B, F) which identifies all the endomorphisms of Z. The solution set condition implies the existence of $X \in \mathcal{S}_B$ and $v : X \to Y$. The endomorphism $u \circ v \circ \alpha_X$ of Z is idempotent since u identifies id_Z and $u \circ v \circ \alpha_X$. By Cauchy completeness of (B, F), $u \circ v \circ \alpha_X$ splits and produces a retract

$$W \underset{i}{\overset{r}{\rightleftarrows}} Z$$

of Z. But then W is a pair (A, b) with $b : B \to FA$ and we shall prove it is the universal reflection of B along F, i.e. the initial object of (B, F).

By the solution set condition, for every object $V \in |(B, F)|$ there exists an object $U \in S_B$ and a morphism $w : U \to V$. So we obtain a composite

$$W \xrightarrow{i} Z \xrightarrow{\alpha_U} U \xrightarrow{w} V$$

which proves the existence of at least one morphism from W to V.

Next, let us prove that every endomorphism g of W is necessarily the identity. Indeed

$$i \circ g \circ r = i \circ g \circ r \circ u \circ i \circ r = i \circ g \circ r \circ u \circ v \circ \alpha_X$$

$$= i \circ r \circ u \circ v \circ \alpha_X = i \circ r \circ i \circ r = i \circ \mathrm{id}_W \circ r$$

and thus $g = \mathrm{id}_W$ since i is mono and r is epi.

Finally come back to an arbitrary object $V \in |(B, F)|$ and, by absolute flatness of F, choose $T \in |(B, F)|$ and $t : T \to W$ which identifies all the morphisms from W to V. By the solution set condition, choose $S \in S_B$ and $s : S \to T$. The composite

$$W \xrightarrow[i]{} Z \xrightarrow[\alpha_S]{} S \xrightarrow[s]{} T \xrightarrow[t]{} W$$

is necessarily the identity and so by definition of t, all the morphisms from W to V are equal. ◊

4. THE CAUCHY COMPLETION OF AN ENRICHED CATEGORY.

Let us fix a complete and cocomplete symmetric monoidal closed category **V**. If I is the unit of **V**, we denote by **I** the **V**-category with a single object $*$ and $\mathbf{I}(*, A) = I$. Every small **V**-category **C** is equivalent to the category **V**-Funct(**I**, **C**) of **V**-functors from **I** to **C** ; that last category is itself embedded in the category **V**-Dist(**I**, **C**) of **V**-distributors from **I** to **C** (cf. [1]).

Definition 1. *The* **V**-*Cauchy completion of a small* **V**-*category* **C** *is the full* **V**-*subcategory of* **V**-Dist(**I**, **C**) *, whose objects are those* **V**-*distributors with a right* **V**-*adjoint.*

It is possible to generalize with respect to **V** most of the results of our §§ 1-2-3, but this is not our purpose here. We just want to develop some relevant examples of enriched Cauchy completions. We shall start with an example which underlines a striking difference between the classical case and the enriched case. But first of all, a general remark.

A **V**-distributor $F : \mathbf{I} \dashrightarrow \mathbf{I}$ is a **V**-functor $F : \mathbf{I}^{op} \otimes \mathbf{I} \to \mathbf{V}$, i.e. just an object $F \in \mathbf{V}$. If $G : \mathbf{I} \dashrightarrow \mathbf{I}$ is another **V**-distributor, the composite $G \circ F$ is just the object $G \otimes F \in \mathbf{V}$. The adjunction $F \dashv G$ reduces to the existence of morphisms

$$\alpha : I \to G \otimes F, \qquad \beta : F \otimes G \to I$$

which satisfy the equalities

$$(id_G \otimes \beta) \circ (\alpha \otimes id_G) = id_G \; ; \qquad (\beta \otimes id_F) \circ (id_F \otimes \alpha) = id_F \, .$$

Example 1. *When* **V** *is the category of* **V***-complete lattices, the Cauchy completion of* **I** *is no longer small.*

The objects of **V** are thus the complete lattices and the morphisms are the **V**-preserving mappings; the unit $I \in \mathbf{V}$ is just $\{0, 1\}$. **V** is a complete and cocomplete symmetric monoidal closed category in which the product $\prod A_i$ of an arbitrary family of objects A_i coincides with its coproduct $\coprod A_i$; we use therefore the classical notation $\oplus A_i$ (cf. [7]).

Given an arbitrary set K, the object $F = \underset{k \in K}{\oplus} I$ can be viewed as a distributor $F : \mathbf{I} \dashrightarrow \mathbf{I}$ adjoint to itself. Indeed the tensor product $\otimes$ commutes with the direct product $\oplus$ since if has a right adjoint; therefore :

$$F \otimes F \simeq (\underset{k \in K}{\oplus} I) \otimes (\underset{\ell \in K}{\oplus} I) \simeq \underset{(k,\ell) \in K \times K}{\oplus} (I \otimes I) \simeq \underset{(k,\ell) \in K \times K}{\oplus} I$$

and the required canonical morphisms α and β are just the diagonal and the codiagonal. So each object $\underset{k \in K}{\oplus} I$ is in the Cauchy completion $\bar{\mathbf{I}}$ of **I**, which prove that $\bar{\mathbf{I}}$ is not small. ◊

Example 2. *When* **V** *is the category of modules on a commutative ring* R *with a unit, the Cauchy completion of* **I** *is the category of finitely generated projective* R *-modules.*

It is well-known that $\mathbf{V} \simeq Mod_R$ is an abelian complete and cocomplete symmetric monoidal closed category; the unit $I \in \mathbf{V}$ is just the ring R itself.

Let us start with an adjunction $(F \dashv G, \alpha, \beta)$ as described above; we must prove that F is a finitely generated projective module. We can write

$$\alpha(1) = \sum_{i=1}^{k} g_i \otimes f_i \in G \otimes F .$$

Consider the following morphisms u and v :

$$u : F \to \bigoplus_{i=1}^{k} R : x \mapsto (\beta(x \otimes g_i)), \qquad v : \bigoplus_{i=1}^{k} R \to F : (r_i) \mapsto \sum_{i=1}^{k} r_i f_i .$$

The second axiom of adjointness implies, for every $x \in F$:

$$(v \circ u)(x) = \sum_{i=1}^{k} \beta(x \otimes g_i).f_i = x .$$

So F is a retract of a finitely generated free module, thus a finitely generated projective module.

Conversely start with a finitely generated projective module F ; F can be presented as a retract of a finitely generated free module :

$$u : F \to \bigoplus_{i=1}^{k} R , \qquad v : \bigoplus_{i=1}^{k} R \to F, \qquad v \circ u = \mathrm{id}_F .$$

If $e_1, \ldots, e_k$ is the canonical basis of $\bigoplus_{i=1}^{k} R$ and F is identified with a a submodule of $\bigoplus_{i=1}^{k} R$, consider

$$f_j = v(e_j) = \alpha_{1,j} e_1 + \ldots + \alpha_{k,j} e_k, \qquad g_i = \alpha_{i,1} e_1 + \ldots + \alpha_{i,k} e_k .$$

F is the submodule of $\bigoplus_{i=1}^{k} R$ generated by $f_1, \ldots, f_k$ and we define G as the submodule of $\bigoplus_{i=1}^{k} R$ generated by $g_1, \ldots, g_k$. The two required "natural transformations" α , β are then

$$\alpha : R \to G \otimes F, \quad \alpha(1) = \sum_{i=1}^{k} g_i \otimes f_i \quad ; \quad \beta : F \otimes G \to R, \ \beta(f_j \otimes g_i) = \delta_{ij}$$

where δ_{ij} is the Krnoecker symbol. It is now a straightforward computation to verify the two axioms for adjointness. ◊

Our last example will, in particular, justify the terminology, due to F.W. Lawvere, that we have adopted here.

Example 3. *When* $\mathbf{V}$ *is the category* $\bar{\mathrm{R}}_+$ *defined by F.W. Lawvere (cf. [9]) the Cauchy completion of a metric space is its usual completion using Cauchy sequences.*

Let us recall the categorical structure of

$$\bar{\mathrm{R}}_+ = \{ r \mid r \in \mathrm{R} ; r \geq 0\} \cup \{+\infty\}.$$

$\overline{R}_+$ is a poset; we view it as a category, putting a morphism $r \to s$ when $r \geqq s$. The poset $\overline{R}_+$ is complete, thus viewed as a category it is both complete and cocomplete. It becomes a symmetric monoidal closed category when we define

$$r \otimes s = r + s\,, \qquad t^s = \max\{t-s\,,\,0\}\,;$$

as a matter of convention, $\infty - \infty = 0$.

A metric space (E, d) can be seen as a category enriched in $\overline{R}_+$: E is the set of objects and the distance function d is the enriched hom-functor. If (E, d) and (E', d') are metric spaces, an enriched functor $f : (E, d) \to (E', d')$ is just a contracting mapping $f : E \to E'$; in particular, f is continuous. The set of contracting mappings from (E, d) to (E', d') becomes itself an enriched category when we define the distance between $f, g: (E, d) \rightrightarrows (E', d')$ by the formula

$$(f, g) = \bigvee_{x \in E} d'(f(x), g(x))\,.$$

Notice that this new enriched category is generally not a metric space, since (f, g) can become infinite.

The enriched category $\mathbf{I}$ is just the singleton viewed as a metric space. A distributor $f : \mathbf{I} \xrightarrow{\ o\ } (E, d)$ with (E, d) a metric space, is then a mapping $f : E \to \overline{R}_+$ which satisfies the condition :

$$\forall\ x, y \in E \qquad d(x, y) \geqq \max\{f(y) - f(x), 0\}\,.$$

From the symmetry of d, we deduce immediately

$$\forall\ x, y \in E \qquad d(x, y) \geqq |f(x) - f(y)|$$

which proves that f is just a contracting mapping. Now when (E, d) is a metric space, the symmetry of d implies that a distributor g: $(E, d) \xrightarrow{\ o\ } \mathbf{I}$ is also just a contracting mapping $g: E \to \overline{R}_+$. The distributor is right adjoint to f when

$$\mathrm{id}_{\mathbf{I}} \geqq g \circ f \quad \text{and} \quad f \circ g \geqq \mathrm{id}_E$$

which means

(1) $$0 = \bigwedge \{\, f(x) + g(x) \mid x \in E \,\}\,,$$

(2) $$\forall\ x, y \in E \qquad f(x) + g(y) \geqq d(x, y)\,.$$

From the first condition we deduce that for some $x \in E$, $f(x) < \infty$. But as f is contracting, for every $y \in E$

$$|f(y) - f(x)| \leqq d(y, x) < \infty$$

which proves $f(y) < \infty$. An analogous argument holds for g . As a conclusion, a pair

$$f : \mathbf{I} \xrightarrow{\ \circ\ } (E, d), \qquad g : (E, d) \xrightarrow{\ \circ\ } \mathbf{I}$$

of adjoint distributors is a pair $f, g : E \rightrightarrows \mathsf{R}_+$ of contracting mappings which satisfy the axioms (1) and (2).

Denote by $(\overline{E}, \overline{d})$ the usual completion of the metric space (E, d); we shall prove that $(\overline{E}, \overline{d})$ is just the $\overline{\mathsf{R}}_+$-category of those enriched distributors $\mathbf{I} \xrightarrow{\ \circ\ } (E, d)$ which possess a right adjoint. Choose an element $a \in \overline{E}$ and define

$$f_a : E \to \overline{\mathsf{R}}_+ , \qquad f_a(x) = \overline{d}(a, x) .$$

It follows immediately that for every x, y in E

$$\left| f_a(x) - f_a(y) \right| \leqq d(x, y) \leqq f_a(x) + f_a(x) .$$

Moreover choosing a sequence (a_n) in E converging to a,

$$\bigwedge_{i=0}^{\infty} (f_a(a_n) + f_a(a_n)) = 0 .$$

These relations indicate that putting $f_a = g_a$ we have defined two distributors

$$f_a : \mathbf{I} \xrightarrow{\ \circ\ } (E, d) \qquad \text{and} \qquad g_a : (E, d) \xrightarrow{\ \circ\ } \mathbf{I}$$

with $f_a \dashv g_a$. This correspondance $a \mapsto f_a$ is injective since $a \neq b$ in E implies

$$f_a(a_n) \to 0 \quad \text{and} \quad f_b(a_n) \to d(a, b) \neq 0 .$$

To prove the surjectivity of our construction, start with an adjoint pair $f \dashv g$ of distributors, as described above. For every $n \in \mathbf{N}$ choose $a_n \in E$ such that

$$f(a_n) + g(a_n) < \frac{1}{n} .$$

This implies, for $k, \ell \geqq n$:

$$\begin{aligned} d(a_k, a_\ell) \leqq f(a_k) + g(a_\ell) &\leqq f(a_k) + g(a_k) + f(a_\ell) + g(a_\ell) \\ &\leqq 1/k + 1/\ell \leqq 2/n . \end{aligned}$$

Thus (a_n) is a Cauchy sequence in E and we define a as its limit in $\overline{E}$. For every $x \in E$ we deduce

$$\left| f(x) - f(a_n) \right| \leqq d(x, a_n) \leqq f(x) + g(a_n)$$

and thus, computing the various limits

$$f(x) = \lim_{n \to \infty} d(x, a_n) = \bar{d}(x, a) = f_a(x) .$$

Finally we have to prove that the distance $\bar{d}$ on $\bar{E}$ coincides with that between the corresponding distributors. Choose a, b in E with (a_n) a sequence in E converging to a.

$$\delta(f_a, f_b) = \bigvee \{ \max \{ f_b(x) - f_a(x), 0 \} \mid x \in E \}$$

$$= \bigvee \{ \max \{ \bar{d}(b, x) - \bar{d}(a, x), 0 \} \mid x \in E \} \leqq \bar{d}(b, a) .$$

When $a = b$, both distances are thus equal to zero. When $a \neq b$, the difference $\bar{d}(b, a_n) - \bar{d}(a, a_n)$ becomes positive for n sufficiently big and moreover converges to $\bar{d}(b, a)$; this proves the converse inequality. ◊

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