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CONVENIENT VECTOR SPACES EMBED INTO THE CAHIERS TOPOS
by Anders KOCK

RÉSUMÉ. Nous construisons un plongement plein de la catégorie des applications lisses entre espaces vectoriels convenables (Frölicher - Kriegl) dans l'un des topos connu comme un modèle de la Géométrie Différentielle Synthétique. L'étape essentielle consiste à étendre les foncteurs "points proches" de Weil du cas de dimension finie au cas convenable.

We construct a full embedding with good preservation properties of the Frölicher-Kriegl category $F$ (cf. [2, 3, 7, 9]) of "convenient" vector spaces, with all smooth maps, into the fully well-adapted model $C$ for synthetic differential geometry considered by Dubuc in [1], the so-called Cahiers topos (cf. also [4]). Each convenient vector space will, after the embedding, satisfy the vector form of the Axiom 1$^W$ (Kock-Lawvere axiom, cf. [4]) for each Weil algebra $W$, and so the rich calculus of smooth maps in $F$ can be dealt with synthetically in $C$.

The idea of the construction is this: to construct a site of definition for the Cahiers topos, one utilizes that for each Weil algebra $W$, the endofunctor $-aW$ on the category of finite dimensional vector spaces with linear maps extends to an endofunctor on the category $f$ of finite-dimensional vector spaces and smooth maps, a construction which goes back to Weil [10]; the site is then the "semidirect product" $f \times W$ of $f$ and $W$ ($W$ being the category of Weil algebras). We then prove that $-aW$ can also be defined as an endofunctor on the category $F$ of convenient vector spaces and smooth maps. The semidirect product $F \times W$ contains $f \times W$ as well as $F$, and the desired embedding $J : F \to C$ is then simply by "representing from the outside", i.e., utilizing the hom functor of $F \times W$.

1. SOME CALCULUS IN CONVENIENT VECTOR SPACES.

We recall some facts about these, from [2, 3, 7, 8], cf. also [9] and [5].

A convenient vector space is a vector space over $R$ equipped with a linear subspace $X'$ of the full algebraic dual $X^*$, such that $X'$ separates points, and with the following two completeness properties:

1. The bornology induced on $X$ by $X'$ is a complete bornology;
2. any linear \( X \to \mathbb{R} \) which is bounded with respect to this bornology belongs to \( X' \).

In the following \( X, Y, Z, \) etc. always denote convenient vector spaces, \( X = (X, X') \) etc. The vector space \( \mathbb{R}^n \) carries a unique convenient structure, namely the full linear dual.

We recall that a map \( c : \mathbb{R}^n \to X \) is called smooth (or a smooth plot on \( X \)) if for any \( \varphi \in X' \), \( \varphi \circ c : \mathbb{R}^n \to \mathbb{R} \) is smooth (\( = C^\infty \)). And a map \( f : X \to Y \) is called smooth, if \( f \circ c \) is smooth for any smooth plot \( c \) on \( X \).

The smooth linear maps \( X \to \mathbb{R} \) turn out to be exactly the elements of \( X' \).

A main motivation for the notion of convenient vector space is that the vector space \( C^\infty(X, Y) \) of smooth maps from \( X \) to \( Y \) itself carries a canonical convenient structure, making the category of convenient vector spaces and their smooth maps into a cartesian closed category.

A map \( f : X \to Y \) is said to have order \( \geq k \) if there exists a smooth \( f^* : X \times \mathbb{R} \to Y \) with
\[
f(\lambda, x) = \lambda^k \cdot f^*(x, \lambda) \quad \forall x \in X \quad \forall \lambda \in \mathbb{R}.
\]

In \([5]\) (Theorem 2.13), we prove that \( f \) is of order \( \geq k \) iff for any \( x \in X \) and \( \varphi \in Y' \), the map
\[
\mathbb{R} \to \mathbb{R} \quad \text{given by} \quad \lambda \mapsto \varphi(f(\lambda \cdot x))
\]
is of order \( \geq k \).

A map \( f : X \to Y \) is homogeneous of degree \( i \) if
\[
f(\lambda, x) = \lambda^i \cdot f(x) \quad \forall x \in X \quad \forall \lambda \in \mathbb{R},
\]
and polynomial of degree < \( k \) if it can be written as a sum
\[
f = \sum f_i \quad (i = 0, ..., k-1)
\]
with \( f_i \) homogeneous of degree \( i \). Since \( Y' \) separates points, a map \( f : X \to Y \) is homogeneous (resp. polynomial) with given degree iff for all \( \varphi \in Y' \), \( \varphi \circ f \) has the corresponding property.

One has the following results:

**Theorem 1.1.** Any smooth \( g : X \to Y \) can uniquely be written as a sum of a polynomial map of degree < \( k \), and a map of order \( \geq k \).

In particular, \( g \) is of order \( \geq 1 \) iff \( g(0) = 0 \).

In the light of the above mentioned equivalence of the two def-
initions of order, this is Corollary 1.3 of [5].

The polynomial map in the theorem should be viewed as an approximating Taylor polynomial.

**Theorem 1.2.** Any smooth $i$-homogeneous map $h : X \to Y$ is of form

$$h(x) = H(x, \ldots, x)$$

for some unique symmetric $i$-linear map $H : X^i \to Y$.

This is Corollary 1.4 in [5].

**Theorem 1.3.** Let $f : \mathbb{R}^n \to X$ be smooth. Let $k \geq 0$ be an integer. There exist smooth functions $q_\alpha : \mathbb{R}^n \to X$ and elements $x_\alpha \in X$ such that, for all $t \in \mathbb{R}$,

$$f(t) = \sum_{|\alpha| < k} \frac{t^\alpha}{\alpha!} x_\alpha + \sum_{|\alpha| = k} \frac{t^\alpha}{\alpha!} q_\alpha(t)$$

(with standard conventions about multi-indices $\alpha$). The $x_\alpha$'s are uniquely determined.

Except for the uniqueness assertion, this follows immediately from [5], Theorem 2.12. The uniqueness of the $x_\alpha$'s follows easily from the corresponding result for the case $X = \mathbb{R}$ using that $X'$ separates points.

The $x_\alpha$'s in Theorem 1.3 are of course the "Taylor coefficients"

$$x_\alpha = \frac{1}{|\alpha|!} \frac{\partial^{|\alpha|}}{\partial t^\alpha} f(0) ;$$

however, they do not appear explicitly in the present article.

For any smooth $f : X \to Y$ and $x \in X$, the map

$$x_1 \mapsto f(x + x_1) - f(x)$$

can, by Theorems 1.1 and 1.2, be written as a sum of a smooth linear map $df_x$ and a map of order $\leq 2$. The map

$$X \times X \to Y$$

given by $(x, x_1) \mapsto df_x(x)$

is smooth, and linear in the second variable, cf. e.g. [3]. Thus, it defines a map

$$Df : X \to L(X, Y)$$

where $L(X, Y)$ is the vector space of smooth linear maps $X \to Y$. There is a canonical structure of convenient vector space on $L(X, Y)$ making all the evaluation maps $L(X, Y) \to Y$ smooth and such that $Df$ is smooth.
2. JET CALCULUS AND WEIL PROLONGATIONS.

Let $I \subset C^\infty(R^n)$ be an ideal. For any convenient vector space $X$, we let $I(X)$ be the set of those smooth $f : R^n \to X$ such that for all $\varphi \in X'$, $\varphi \circ f \in I$. We say that

$$f_1 \equiv f_2 \mod I \quad \text{if} \quad f_1 - f_2 \in I(X).$$

This is an equivalence relation. An equivalence class is called a mod 1 jet into $X$. This notion will be proved to have good properties if $I$ is large enough: Let $M \subset C^\infty(R^n)$ denote the (maximal) ideal of functions $i.e., functions of order $\geq 1$. Then $M^r$ is the ideal of functions of order $\geq r$. It is of finite codimension. We shall say that an ideal $I \subset C^\infty(R^n)$ is a Weil ideal if, for some $r$, $M^r \subset I \subset M$. The residue ring $C^\infty(R^n)/I$ is then a Weil algebra (cf. e.g. [4] or [1] for the notion), and any Weil algebra comes about in this way. We shall use the letter $W$ to denote any Weil algebra, but with a given presentation by a Weil ideal $I$, and use "mod-$I$-jet" and "$W$-jet" synonymously.

We denote by $X@W$ or $W@X$ the set of all $W$-jets into $X$. Since $M^r \subset I$, we may choose a finite set of polynomials

$$p_1, \ldots, p_m \in R[t_1, \ldots, t_n]$$

of degree $< r$ which form a basis in $C^\infty(R^n)/I$. It then follows from Theorem 1.3 that any $W$-jet into $X$ has a representative of the form

$$(t_1, \ldots, t_n) \mapsto \sum_{i=1}^{m} h_i(t) x_i$$

for unique $x_i \in X$, and thus $X@W \cong X^m$. This also justifies the $@$ notation, since $W \cong R^m$. Likewise, if $f : X \to Y$ is linear, $f@W : X@W \to Y@W$ may of course be defined. Our aim is to define $f@W$ for any smooth $f : X \to Y$.

Proposition 2.1. If $f_1 \equiv f_2 \mod I$ (where $f_i : R^n \to X$), then we have $g \circ f_1 \equiv g \circ f_2 \mod I$, for any smooth $g : X \to Y$.

Proof. We have $f_1(0) = f_2(0)$ ( = $x_0$, say) since $f_1 \equiv f_2 \mod M$. Since

$$g \circ (f_i - x_0) = \tilde{g} \circ f_i \quad \text{for} \quad \tilde{g}(x) := g(x + x_0),$$

it suffices to prove the result in the case

$$f_1(0) = f_2(0) = 0.$$ 

So $f_1$ and $f_2$ may both be assumed to have order $\geq 1$. 

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To prove $g \circ f_1 \equiv g \circ f_2 \mod I$ means by definition to prove
\[ \varphi \circ g \circ f_1 - \varphi \circ g \circ f_2 \in I, \]
for any smooth linear $\varphi : Y \to \mathbb{R}$, so let such $\varphi$ be given. Change notation and write $g$ for $\varphi \circ g$. Then $g : X \to \mathbb{R}$ may by Theorem 1.1 be written as a sum
\[ \sum_{q=0}^{r-1} h_q + G \]
with $h_q : X \to \mathbb{R}$ smooth homogeneous of degree $q$, and $G$ of order $\geq r$. It suffices to prove that
\[ (2.1) \quad h_q \circ f_1 \equiv h_q \circ f_2 \mod I \quad \forall q = 0, \ldots, r-1 \]
and that
\[ (2.2) \quad G \circ f_1 \equiv G \circ f_2 \mod I. \]
For (2.2), this is trivial; in fact each $G \circ f_i$ ($i = 1, 2$) has itself order $\geq r$ since
\[ \text{order}(f_i) \geq 1 \quad \text{and} \quad \text{order}(G) \geq r. \]
So
\[ G \circ f_1 \in \mathcal{M} \subset I, \quad i = 1, 2. \]
For (2.1), we write, by Theorem 1.2 $h_q$ in the form
\[ h_q(x) = H(x, \ldots, x), \]
where $H : X^q \to \mathbb{R}$ is smooth $q$-linear. For simplicity, let $q = 2$. Then
\[ H(f_1(t), f_1(t)) - H(f_2(t), f_2(t)) = \]
\[ = H(f_1(t), f_1(t)) - H(f_2(t), f_1(t)) + H(f_2(t), f_1(t)) - H(f_2(t), f_2(t)) \]
\[ = H(f_1(t) - f_2(t), f_1(t)) + H(f_2(t), f_1(t) - f_2(t)), \]
and the result follows from

Lemma. Let $H : X^q \to \mathbb{R}$ be $q$-linear smooth, and let $I \supset \mathcal{M}$ be an ideal in $\mathbb{C}^\infty(\mathbb{R}^n)$. If $k : \mathbb{R}^n \to X$ belongs to $I(X)$ then, for any smooth $i_i : \mathbb{R}^n \to X$ ($i = 2, \ldots, q$),
\[ (2.3) \quad H(k(t), i_2(t), \ldots, i_q(t)) \in I. \]

Proof. Again, let $q = 2$ and write
\[ i_2(t) = \sum_{|\alpha| < r} t^{\alpha} \alpha \cdot \zeta + L(t) \]
with \( L(t) \) or order \( \geq r \). Then the function of \( t \) displayed in (2.3) can be written

\[
\sum_{\alpha} t^{\alpha} H(k(t), x_{\alpha}) + H(k(t), L(t)).
\]

The last term here clearly is a function of order \( \geq r \), since \( L \) is, and so is in \( I \). But also each \( H(k(t), x_{\alpha}) \in I \) since they are of form \( \phi \circ k \), \( \phi \in X' \) (namely with \( \phi = H(-, x_{\alpha}) \)), so is in \( I \) since \( k \in I(X) \). The Lemma, and thus the proposition, is proved.

For \( g : X \to Y \) smooth there is thus an evident way of defining \( g \circ \omega : X \circ \omega \to Y \circ \omega \) so as to make \( \omega \) a functor, namely composing with \( g \). If \( f \in X \circ \omega \) is a \( \omega \)-jet represented by \( f : \mathbb{R}^n \to X \), we let \( (g \circ \omega)(f) \) be the \( \omega \)-jet represented by \( g \circ f : \mathbb{R}^n \to Y \). If \( g \) is smooth linear, \( g \circ \omega \) will then be the usual map with this notation.

Our next task is to make \( \omega \) into a functor which \( \omega \)'s so takes values in \( F \). Since \( X \circ \omega \simeq X^m \), \( X \circ \omega \) inherits a structure of convenient vector space from that of \( X^m \). The isomorphism \( X \circ \omega \simeq X^m \) depends on a choice of basis mod \( I \), but any other choice will define an invertible \( m \times m \) matrix, which then defines also a smooth linear isomorphism \( X^m \to X^m \), so the convenient vector space structure on \( X \circ \omega \) is well defined.

**Proposition 2.2.** For \( g : X \to Y \) smooth, the map \( g \circ \omega : X \circ \omega \to Y \circ \omega \) is smooth.

**Proof.** We first do the special case where \( I = M' \subset C^\infty(\mathbb{R}^n) \). As basis mod \( I \), we may choose all monomials in \( t_1, \ldots, t_n \) of degree \( < r \). The statement is then just the fact that, for \( g \) fixed, the \( r \) degree partial derivatives \( \partial^\alpha (g \circ f)/\partial t^\alpha(0) \) depend in a smooth (in fact polynomial) way on the partial derivatives \( \partial^\alpha f / \partial t^\alpha(0) \) ("higher order chain rule"). Since I could not find a reference*, not even an exact statement, of this "evident" fact, I shall be more explicit. Write \( g \) in the form

\[
g = \sum_{q \leq \alpha} h_q + G
\]

with \( h_q : X \to Y \) smooth homogeneous of degree \( q \) and \( G \) of order \( \geq r \). It suffices to prove for each \( h_q \) separately, and for \( G \). Now, since a jet is represented by a function \( f : \mathbb{R} \to X \) or order \( \geq 1 \), \( G \circ f \) has order \( \geq r \), so its partial derivatives of order \( < r \) vanish, so depend smoothly on those of \( f \). Now consider \( h_q \). Write \( h_q(x) = H(x, \ldots, x) \) where \( H : X^q \to Y \) is smooth symmetric \( q \)-linear (Theorem 1.2). Since the partial derivatives of any \( k : \mathbb{R}^n \to \mathbb{Z} \) can be obtained from the \( Di k \)'s, by evaluation at the canonical basis vectors in \( \mathbb{R}^n \), the result

*ADOEDE IN PROOF. I thank the referee for providing the following two references : A. Bastiani, Applications différentiables et variétés différentiables de dimension infinie, J. Analyse Math. Jérusalem XIII (1964), 2-113 ; and P. Ver Eecke, Fondements du Calcul Différentiel, P.U.F., Paris 1984.
can be obtained from the following Lemma (when writing $R^n$ for $X$, $X$ for $Y$ and $Y$ for $Z$).

**Lemma 2.3.** Let $H : Y^q \rightarrow Z$ be symmetric smooth $q$-linear. Then there is a fixed formula

$$D^k (H (f, \ldots, f)) = \Sigma H (D^k f, \ldots, D^k f)$$

valid for all smooth $f : X \rightarrow Y$.

**Proof** and more precise statement. Let

$$k(x) := H(f(x), \ldots, f(x)).$$

Then $D^k (x ; x_1, \ldots, x_p)$ equals the following finite sum (2.4), whose index set is the set of partitionings of $p = \{1, 2, \ldots, p\}$ into $\leq q$ disjoint subsets $\pi(1), \ldots, \pi(s(\pi))$

$$(2.4) \sum_{s(\pi)} H(\prod_{\pi(1)} f(x; x_{\pi(1)}), \ldots, \prod_{\pi(s(\pi))} f(x; x_{\pi(s(\pi))}), f(x), \ldots, f(x))$$

$$(q - s(\pi)) \text{ f(x)'s};$$

here $[q]_{\pi}$ denotes $q, (q-1), \ldots, (q-r+1)$,

and if $B \subset p$ is a subset, with $b$ elements $i_1, \ldots, i_b$, then we have put

$$D^k (x ; x_B) := D^k f(x ; x_{i_1}, \ldots, x_{i_b}).$$

This formula is easily verified by induction, and the Lemma is proved.

Now let $I \subset M^X$ be a general Weil ideal. Choosing a basis $h_1, \ldots, h_n$ mod $I$ amounts to an $R$-linear splitting $\sigma$ of the projection

$$C^\infty (R^n)/M^X \rightarrow C^\infty (R^n)/I = W.$$ 

It induces a smooth linear splitting $X \circ \sigma$ of

$$X^m := X \circ (C^\infty (R^n)/M^X) \rightarrow X \circ W \simeq X^m.$$ 

By the well-definedness result (Proposition 2.1), for $g : X \rightarrow Y$ smooth, $g \circ W$ equals the composite

$$X \circ W \xrightarrow{X \circ \sigma} X \circ (C^\infty (R^n)/M^X) \xrightarrow{g \circ \ldots} Y \circ (C^\infty (R^n)/M^X) \rightarrow Y \circ W,$$

where the middle map is smooth by the special case already proved. Thus, the composite is smooth.

This proves the Proposition. Thus each Weil algebra $W$ defines an endofunctor $- \circ W : F \rightarrow F$. 

3. TRANSITIVITY OF PROLONGATIONS.

For any vector space $X$ and Weil algebras $W_1, W_2$ we have of course

$$(3.1) \quad X \mathbin{\otimes} (W_1 \mathbin{\otimes} W_2) \cong (X \mathbin{\otimes} W_1) \mathbin{\otimes} W_2$$

naturally in $X$ with respect to linear maps. Our aim in this section is to prove that for convenient vector spaces $X$, this isomorphism is natural in $X$ with respect to smooth maps.

Recall that we may consider as a subring

$$R[t_1, \ldots, t_n] \subset C^\infty(R^n).$$

Let $I \subset C^\infty(R^n)$ be a Weil ideal representing the Weil algebra $W$. In the following commutative diagram with exact rows, $I'$ is defined as intersection (pullback):

$$
\begin{array}{cccc}
0 & \rightarrow & I & \rightarrow & C^\infty(R^n) & \rightarrow & C^\infty(R^n)/I = W & \rightarrow & 0 \\
& \uparrow & & \uparrow & & \uparrow \alpha & & \\
0 & \rightarrow & I'(t_1, \ldots, t_n) & \rightarrow & R[t_1, \ldots, t_n] & \rightarrow & R[t_1, \ldots, t_n]/I' & \rightarrow & 0.
\end{array}
$$

Since there is a basis mod $I$ consisting of polynomials, it follows that

$$C^\infty(R^n) = R[t_1, \ldots, t_n] + I;$$

thus from the Noether isomorphism

$$P/P\cap I \cong (P+I)/I,$$

it follows that $\alpha$ is an isomorphism. More generally, if $X$ is a convenient vector space, the subspace of $C^\infty(R^n, X)$ consisting of smooth polynomial functions may be identified with $X \mathbin{\otimes} R[t_1, \ldots, t_n]$ (Theorem 1.3). So if we denote by $I(X)$ the subspace of functions $R^n \rightarrow X$ which are $\equiv 0$ mod $I$, and $I'(X)$ the polynomial functions among them, we have a commutative diagram with exact rows and with the left hand square a pullback:

$$
\begin{array}{cccc}
0 & \rightarrow & I(X) & \rightarrow & C^\infty(R^n, X) & \rightarrow & C^\infty(R^n, X)/I(X) = X \mathbin{\otimes} W & \rightarrow & 0 \\
& \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & I'(X) = X \mathbin{\otimes} I' & \rightarrow & X \mathbin{\otimes} R[t_1, \ldots, t_n] & \rightarrow & X \mathbin{\otimes} W & \rightarrow & 0
\end{array}
$$

Henceforth, we shall write $I$ instead of $I(X)$ when the context (diagram) will inform us about $X$.

For the proof of naturality of $(3.1)$ with respect to smooth maps,
we shall make essential use of the cartesian closedness of the category $F$ of convenient vector spaces with smooth maps: for $X$, $Y$ convenient vector spaces, the vector space $C^\infty(X, Y)$ of smooth maps $X \to Y$ carries a natural structure of convenient vector space making it the exponential object $Y^X$ in $F$. In particular

$$C^\infty(\mathbb{R}^{n+m}, X) \simeq C^\infty(\mathbb{R}^m, C^\infty(\mathbb{R}^n, X)),$$

natural in $X \in F$, and this will be the essence in the proof. Let $W_1$, $W_2$ be Weil algebras with presentation $C^\infty(\mathbb{R}^n)/I_1$ and $C^\infty(\mathbb{R}^m)/I_2$, respectively. Then $W_1 \otimes W_2$ has presentation $C^\infty(\mathbb{R}^{n+m})/(I_1, I_2)$, where $(I_1, I_2)$ is the ideal generated by functions $h(s), g(s, t)$ with $h \in I_1$ and functions $h(s, t), g(t)$ with $g \in I_2$ (where $s = (s_1, ..., s_n)$ etc.). Consider the following commutative diagram (in which the two bottom corners represent the two sides of (3.1)):

$$\begin{align*}
\begin{array}{ccc}
\mathbb{R}[s, t]/(I_1, I_2) & \otimes & \mathbb{R}[s, t] \\
\alpha_X & \otimes & \beta_X \\
\mathbb{R}[s, t]/I_2 \otimes \mathbb{R}[s]/I_1 & \otimes & \mathbb{R}[s, t]/I_2 \otimes \mathbb{R}[s]/I_1
\end{array}
\end{align*}$$

$$\begin{align*}
\begin{array}{ccc}
C^\infty(\mathbb{R}^{n+m}, X)/(I_1, I_2) & \otimes & C^\infty(\mathbb{R}^{n+m}, X)
\alpha_X & \otimes & \beta_X \\
C^\infty(\mathbb{R}^m, C^\infty(\mathbb{R}^n, X))/I_2 & \otimes & C^\infty(\mathbb{R}^m, C^\infty(\mathbb{R}^n, X))/I_2
\end{array}
\end{align*}$$

Here $\alpha_X$ and $\alpha_X \otimes X$ are evident, whereas $\beta_X$ utilizes (3.2) and $\beta_X \otimes X$ utilizes a mimicking of (3.2) on the level of polynomials, namely the linear isomorphism

$$R[s, t] \simeq R[t] \otimes R[s];$$

$\alpha_X$ and $\beta_X$ are surjective. The top isomorphism comes about purely algebraically by applying $- \otimes X$ to isomorphisms, well-known from algebra,

$$R[s, t]/I_1, I_2 \simeq R[s]/I_1 \otimes R[t]/I_2.$$

The maps $\alpha_X$ and $\beta_X$ are evidently natural in $X$ with respect to smooth maps; for the maps $\alpha_X \otimes X$ and $\beta_X \otimes X$ such naturality does not make sense, since $R[s, t] \otimes X$ is not functorial in $X$ with respect to smooth maps. However, this does not matter; the smooth natural isomorphism of the two bottom corners in (3.3) now follows from a piece of diagram chasing, namely the following Lemma whose proof we leave to the reader.

**Lemma.** Let $C$, $D$ and $E$ be functors $A \to B$, and assume for each $X \in A$ a commutative triangle

$$\begin{align*}
\begin{array}{ccc}
D(X) & \xrightarrow{\gamma_X} & E(X)
\alpha_X & \otimes & \beta_X \\
C(X) & \xrightarrow{\beta_X} & E(X)
\end{array}
\end{align*}$$

If all $\alpha_X$ are epic, and $\alpha$ and $\beta$ are natural in $X$, then so is $\gamma$. 


We have thus proved the first statement in the following theorem (the second assertion being trivial):

**Theorem 3.1.** The isomorphism (3.1) is natural with respect to smooth maps. Also $X \# R = X$, naturally with respect to smooth maps.

We end this section by remarking that the construction $X\circ W$ is also functorial in $W$. A homomorphism $F$ of Weil algebras

$$W_i = \mathcal{C}^\infty(\mathbb{R}^m)/I \xrightarrow{F} \mathcal{C}^\infty(\mathbb{R}^n)/J = W_j$$

can be represented by a smooth map

$$F : \mathbb{R}^m \to \mathbb{R}^n$$

with $F(0) = 0$,

and with $\varphi \circ F \in J$ whenever $\varphi \in I$. Then, for $f : \mathbb{R}^m \to \mathbb{R}$ representing an element $\{f\}$ of $W_1$, $f \circ F$ represents $F(\{f\}) \in W_2$. And if $f : \mathbb{R}^n \to X$ represents an element of $X\circ W_1$, $f \circ F$ represents $(X\circ F)(\{f\})$.

All said, $\circ$ defines a bifunctor

$$(X \# W) \to F$$

where $W$ is the category of Weil algebras. In fact, by Theorem 3.1, the monoidal category $(W, \circ, R)$ acts on $F$ in an associative unitary way (up to coherent isomorphisms). - Note that $\circ$ is the coproduct in $W$, $R$ the initial object. (Actually, $R$ is also terminal object in $W$.)

4. SEMIDIRECT PRODUCT OF CATEGORIES.

Let $W$ be any category with finite coproducts, denoted $\circ$, and with initial object denoted $R$, and let $G$ be a category on which $W$ acts (from the right, say), i.e., there is given a functor $\circ : G \times W \to G$, and there are given natural isomorphisms (for $X \in G$, $W \in W$):

$$(X \circ W_1) \circ W_2 \simeq X \circ (W_1 \circ W_2), \quad X \simeq X \circ R$$

which fit coherently with the associativity - and unit - isomorphisms of the monoidal category $(W, \circ, R)$.

We construct a new category $G \circ W$ as follows: the objects are pairs $(X, W)$ with $X \in G$, $W \in W$. An arrow $(X_1, W_1) \to (X_2, W_2)$ is a pair of arrows in $G$ and $W$,

$$(X_1 \overrightarrow{f} X_2 \# W_1, \quad W_2 \overrightarrow{\varphi} W_1),$$

and the composite of this pair with
is the pair (associativity isomorphisms omitted, by coherence):

\[
\begin{array}{c}
\xymatrix{ X_1 \ar[r]^{g} & X_2 \ar[r]^{\gamma} & W_2 }
\end{array}
\]

Identity arrow is

\[
\begin{array}{c}
\xymatrix{ X \ar[r]^{X_W} & X_W, \ar[r]^{id_W} & W, }
\end{array}
\]

There is a full embedding \( j : G \to G \times W \) given by \( X \mapsto (X, R) \) and

\[
\begin{array}{c}
\xymatrix{ (X_1, f) \ar[r] & (X_2, f) \ar[r] & X_2 \times R, \ar[r]^{id_R} & W. }
\end{array}
\]

**Proposition 4.1.** The inclusion \( j : G \to G \times W \) preserves all those inverse limits which are preserved by all \(-W\).

**Proof.** We prove the case of binary products only (which is all we need for what follows). We have in fact more generally

\[
(4.2) \quad (Z_1, W_1) \times (Z_2, W_2) = (Z_1 \times Z_2, W_1 \times W_2)
\]

due to the string of conversions

\[
\begin{array}{c}
\xymatrix{ (Y, W) \ar[r] & (Z_1 \times Z_2, W_1 \times W_2) \\
(Y \times (Z_1 \times Z_2) \times W = (Z_1 \times W) \times (Z_2 \times W), \ar[r] & W_1 \times W_2 \ar[r] & W \\
(Y \times Z_1 \times W_1 \times W_1 \ar[r] & (W_1 \times W_1, \ar[r] & W)_{i=1,2} \\
((Y, W) \times (Z_1, W_1))_{i=1,2} 
\end{array}
\]

**Proposition 4.2.** If \( G \) has exponential objects \( Y^X \) which are preserved by each \(-W\) in the sense \( Y^X \times W = (Y \times W)^X \) and if each \(-W\) preserves finite products, then \( j \) preserves exponential objects.

**Proof.** We have bijective correspondences

\[
\begin{array}{c}
\xymatrix{ (Z, W) \ar[r] & (Y^X, R) \\
Z \times Y^X \times W = (Y \times W)^X \\
(Z \times X, W) \ar[r] & (Y, R) \\
(Z, W) \times (X, R) \ar[r] & (Y, R) 
\end{array}
\]

where we for the last conversion utilized \((4.2)\), which we may by the second assumption made.
If the initial object $R$ of $W$ is also terminal, we have a canonical functor $\pi : G \times W \to G$, given on objects by $\pi(X, W) = X$ and with $\pi$ applied to the arrow (4.1) given as

$$X_1 \to X_2 \otimes W \to X_3 \otimes R \cong X_2.$$ 

Clearly $\pi \circ j = \text{id}_G$, and there is a natural map making $j(\pi(X, W))$ a retract of $(X, W)$. (In fact, if each $- \otimes W$ preserves finite products, it follows from (4.2) that

$$\text{(4.3)} \quad (Z, W) \cong (Z, R) \times (1, W),$$

and $(1, W)$ is an object in $G \times R$ which has a unique point (= map from the terminal object).)

5. THE EMBEDDING.

We consider now the category $F$, with the "action" $\otimes$ of $W$, the category of Weil algebras, as described in §2 and §3, and we form $F \otimes W$. The full subcategory $f \subset F$ of finite dimensional vector spaces is stable under the action, so that we get $f \otimes W$ as a full subcategory of $F \otimes W$.

We describe (essentially following [1]) a Grothendieck topology on $f \otimes W$ which will make it a site of definition for the Cahiers topos.

We declare the following families to be covering:

$$\text{(5.1)} \quad (X_i, W) \xrightarrow{a_i = (f_i, \text{id})} (X, W), \quad i \in I$$

if $a_i : X_i \to X$ form an open covering.

Let $i$ and $j$ denote the following full inclusions

$$F \otimes W \xrightarrow{i} f \otimes W \xrightarrow{j} F$$

Any $Y \in F$ defines a functor $J(Y) : (f \otimes W)^{\text{op}} \to \text{Sets}$, namely

$$J(Y) = \hom_{f \otimes W}(j(-), f(Y)).$$

So $J(Y)$ is "representable from the outside". We may omit $i$ and $j$ from notation.

**Proposition 5.1.** $J(Y)$ is a sheaf.

**Proof.** Let $\{a_i\}$ be a covering, as in (5.1), in $f \otimes W$, and let

$$b_i : (X_i, W) \to Y$$

be a compatible family $(Y \in F)$. We should construct a map
The data of the $b_i$'s amount to $b_i : X \to Y\mathfrak{a}W$ and the compatibility condition for the $b_i$'s implies one for the $B_i$'s. The required map $c$ amounts to a map $c : X \to Y\mathfrak{a}W$. Also $\pi_1(a_i) : X_i \to X$ form an open covering. So the crux is to observe that any convenient vector space $Z$ (in our case $Z = Y\mathfrak{a}W$) represents (from the outside) a sheaf on the site $\mathfrak{f}$ (with open coverings as its topology). This follows from concreteness of the categories $\mathfrak{f}$ and $\mathfrak{F}$, and the fact that smoothness of a set theoretic map $X \to Y$ between convenient vector spaces may be tested by smooth plots on an open covering of $X$ and with finite dimensional domains.

We leave the full details to the reader. At this point, it would have been an advantage to consider the categories $\mathfrak{f}$ and $\mathfrak{F}$ consisting of open subsets of finite dimensional, resp. convenient vector spaces, with $W$ acting on them (which it does by the same construction as the one of §2.3) because the open coverings in $\mathfrak{f}$ and $\mathfrak{F}$ admit pullbacks which are furthermore preserved by $-\mathfrak{a}W$.

We can now state our main theorem; $C$ denotes the Cahiers topos (= sheaves on $\mathfrak{f}\times\mathfrak{W}$):

**Theorem 5.2.** The functor $J : \mathfrak{F} \to C$ is full and faithful. It preserves finite products, and it preserves exponentials $Y^X$ provided $X$ is finite dimensional.

**Remark.** By the remarks just before the statement of the theorem it follows that the embedding $J$ may be extended to the category $\mathfrak{F}$ of open subsets of convenient vector spaces, and their smooth maps, and thus possibly also to some category of "manifolds modelled on convenient vector spaces".

**Proof.** When $J$ is composed with the global-sections functor $\Gamma : C \to \text{Sets}$, we get the faithful underlying-set functor $|.| : \mathfrak{F} \to \text{Sets}$, so $J$ is faithful. To test fullness, let $f : J(X) \to J(Y)$ be a map in $C$. We get a set theoretic map $|f| : X \to Y$, which we have to test is smooth. But again, smoothness may be tested by checking with smooth plots $c : \mathbb{R}^n \to X$ (in fact $n - 1$ suffices), and since $\mathbb{R}^n \in f \subseteq \mathfrak{f}\times\mathfrak{W}$, smoothness of $|f|$ follows. To see $J(|f|) = f$, just apply the faithful $|.|$.

Next we argue that $J$ preserves finite products. It is clear from the construction that $-\mathfrak{a}W : \mathfrak{F} \to \mathfrak{F}$ preserves finite products for each $W \in \mathfrak{W}$. Hence, by Proposition 4.1, $J : \mathfrak{F} \to \mathfrak{F}\times\mathfrak{W}$ preserves finite products, and hence so does $J$, for standard categorical reasons (essentially, "Yoneda embedding preserves limits").

Finally, to argue for exponentials, we note that the functors $-\mathfrak{a}W : \mathfrak{F} \to \mathfrak{F}$ satisfy
In fact, if \( W \) is \( m \)-dimensional as a vector space, both sides are isomorphic, by smooth linear isomorphisms, to

\[
(Y^X)_m \simeq (Y^m)^X.
\]

This isomorphism is in fact natural with respect to smooth maps, because if \( h_1, \ldots, h_m \in C^\infty(\mathbb{R}^n) \) is a basis mod \( I \), an element of \( Y^X \otimes W \) has a unique representative of form

\[
\xi \mapsto \sum h_j(t), \xi_j \in Y^X,
\]

and under the isomorphism, this element goes to

\[
x \mapsto \left[ \xi \mapsto \sum h_j(t), \xi_j(x) \right],
\]

the square bracket here representing an element of \( Y \otimes W \). The passage thus described is clearly natural. So \( - \otimes W \) satisfies the conditions of Proposition 4.2, so that \( j : F \to FK\mathcal{W} \) preserves exponentiation. The rest of the argument is now purely categorical; let \( A \in \mathcal{F} \times \mathcal{W} \), and let \( \overline{A} \) be the object of \( \mathcal{C} \) which it represents. For \( X \in \mathcal{F} \) and \( Y \in \mathcal{F} \), we then have

\[
\text{hom}_\mathcal{C}(\overline{A}, J(Y^X)) = \text{hom}_{FK\mathcal{W}}(A, j(Y^X)) = \text{hom}_{FK\mathcal{W}}(A, j(Y)^j(X)) = \text{hom}_{FK\mathcal{W}}(Axj(X), J(Y)),
\]

the last equality provided \( Axj(X) \in \mathcal{F} \mathcal{K}\mathcal{W} \), which will be the case since \( X \in \mathcal{F} \). The theorem is proved.

6. RETROSPECT.

Having Theorem 5.2, as well as the full power of synthetic reasoning in \( \mathcal{C} \), many of the constructions and comparisons that we worked hard to get, become very transparent. For a Weil algebra \( W \), let \( \mathcal{C} \) denote the ("infinitesimal") object in \( \mathcal{C} \) which it represents. Then \( \mathcal{F} \times \mathcal{W} \) becomes the full subcategory of \( \mathcal{C} \) of objects of form \( J(X) \times \mathcal{W} \) \( (X \in \mathcal{F}, W \in \mathcal{W}) \), this being identified with \( (X, W) \in \mathcal{F} \mathcal{K}\mathcal{W} \). A \( W \)-jet into \( X \) becomes simply a map \( W \to J(X) \), explaining the functoriality of the jet notion. Also, \( X \otimes W \) goes by \( J \) to \( J(X)W \), explaining the properties of the functor \( - \otimes W \), e.g. the transitivity

\[
(X \otimes W_1) \otimes W_2 \simeq X \otimes (W_1 \otimes W_2)
\]

is simply the categorical law \( (A^B)^C \simeq A^{B \times C} \).
Let us finally remark that each $J(X)$ evidently will be an $R$-module object ($R = J(R)$), and that it will satisfy the "vector form of Axiom 1\textsuperscript{W}" (cf. [4]), in the sense that, if $m$ is the linear dimension of $W$, we have an isomorphism $J(X)^m \cong J(X)^W$ constructed out of a linear basis $h_1, ..., h_m$ for $R[t_1, ..., t_n] \mod I$ (where $W = R[t]/I$) as the map with synthetic description

$$(6.1) \quad (x_1, ..., x_m) \mapsto [(t_1, ..., t_n)] \mapsto \Sigma h_i(t) x_i$$

($\overline{W}$ being identified with a sub"set" of $R^n$, namely the "zero-set of $I$ ").

This follows essentially from the fact that in $F$ we have an isomorphism $X^m \cong \mathfrak{a}W$ given by the same formula (6.1).

From the validity of Axiom 1\textsuperscript{W} for $J(X)$ it follows, in turn, that $J(X)$ is infinitesimally linear in the strong (Bergeron-) sense, cf. [6]; the argument is as in [6], Proposition 1.2, with $R$ replaced by $J(X)$.

REFERENCES.