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CONVIENT VECTOR SPACES EMBED INTO THE CAHIERS TOPOS
by Anders KOCK

RÉSUMÉ. Nous construisons un plongement plein de la catégorie des applications lisses entre espaces vectoriels convenables (Frölicher - Kriegl) dans l'un des topos connu comme un modèle de la Géométrie Différentielle Synthétique. L'étape essentielle consiste à étendre les foncteurs "points proches" de Weil du cas de dimension finie au cas convenable.

We construct a full embedding with good preservation properties of the Frölicher-Kriegl category $F$ (cf. [2, 3, 7, 9]) of "convenient" vector spaces, with all smooth maps, into the fully well-adapted model $C$ for synthetic differential geometry considered by Dubuc in [1], the so-called Cahiers topos (cf. also [4]). Each convenient vector space will, after the embedding, satisfy the vector form of the Axiom $1^W$ (Kock-Lawvere axiom, cf. [4]) for each Weil algebra $W$, and so the rich calculus of smooth maps in $F$ can be dealt with synthetically in $C$.

The idea of the construction is this: to construct a site of definition for the Cahiers topos, one utilizes that for each Weil algebra $W$, the endofunctor $-aW$ on the category of finite dimensional vector spaces with linear maps extends to an endofunctor on the category $f$ of finite-dimensional vector spaces and smooth maps, a construction which goes back to Weil [10]; the site is then the "semidirect product" $f \times W$ of $f$ and $W$ ($W$ being the category of Weil algebras). We then prove that $-aW$ can also be defined as an endofunctor on the category $F$ of convenient vector spaces and smooth maps. The semidirect product $F \ltimes W$ contains $f \times W$ as well as $F$, and the desired embedding $J : F \rightarrow C$ is then simply by "representing from the outside", i.e., utilizing the hom functor of $F \ltimes W$.

1. SOME CALCULUS IN CONVENIENT VECTOR SPACES.

We recall some facts about these, from [2, 3, 7, 8], cf. also [9] and [5].

A convenient vector space is a vector space over $\mathbb{R}$ equipped with a linear subspace $X'$ of the full algebraic dual $X^*$, such that $X'$ separates points, and with the following two completeness properties:

1. The bornology induced on $X$ by $X'$ is a complete bornology;
2. any linear $X \to \mathbb{R}$ which is bounded with respect to this bornology belongs to $X'$.

In the following $X, Y, Z, \text{etc.}$ always denote convenient vector spaces, $X = (X, X')$ etc. The vector space $\mathbb{R}^n$ carries a unique convenient structure, namely the full linear dual.

We recall that a map $c: \mathbb{R}^n \to X$ is called smooth (or a smooth plot on $X$) if for any $\varphi \in X'$, $\varphi \circ c: \mathbb{R}^n \to \mathbb{R}$ is smooth ($= C^\infty$). And a map $f: X \to Y$ is called smooth, if $f \circ c$ is smooth for any smooth plot $c$ on $X$.

The smooth linear maps $X \to \mathbb{R}$ turn out to be exactly the elements of $X'$.

A main motivation for the notion of convenient vector space is that the vector space $C^\infty(X, Y)$ of smooth maps from $X$ to $Y$ itself carries a canonical convenient structure, making the category of convenient vector spaces and their smooth maps into a cartesian closed category.

A map $f: X \to Y$ is said to have order $\geq k$ if there exists a smooth $f^*: X \times \mathbb{R} \to Y$ with

$$f(\lambda, x) = \lambda^k \cdot f^*(x, \lambda) \quad \forall \ x \in X \quad \forall \ \lambda \in \mathbb{R}.$$ 

In [5] (Theorem 2.13), we prove that $f$ is of order $\geq k$ iff for any $x \in X$ and $\varphi \in Y'$, the map

$$\mathbb{R} \to \mathbb{R} \quad \text{given by} \quad \lambda \mapsto \varphi(f(\lambda \cdot x))$$

is of order $\geq k$.

A map $f: X \to Y$ is homogeneous of degree $i$ if

$$f(\lambda \cdot x) = \lambda^i \cdot f(x) \quad \forall \ x \in X \quad \forall \ \lambda \in \mathbb{R},$$

and polynomial of degree $< k$ if it can be written as a sum

$$f = \sum f_i \quad (l = 0, \ldots, k-1)$$

with $f_i$ homogeneous of degree $i$. Since $Y'$ separates points, a map $f: X \to Y$ is homogeneous (resp. polynomial) with given degree iff for all $\varphi \in Y'$, $\varphi \circ f$ has the corresponding property.

One has the following results:

**Theorem 1.1.** Any smooth $g: X \to Y$ can uniquely be written as a sum of a polynomial map of degree $< k$, and a map of order $\geq k$.

In particular, $g$ is of order $\geq 1$ iff $g(0) = 0$.

In the light of the above mentioned equivalence of the two def-
The polynomial map in the theorem should be viewed as an approximating Taylor polynomial.

**Theorem 1.2.** Any smooth $i$-homogeneous map $h : X \to Y$ is of form

$$h(x) = H(x, \ldots, x)$$

for some unique symmetric $i$-linear map $H : X^i \to Y$.

This is Corollary 1.4 in [5].

**Theorem 1.3.** Let $f : \mathbb{R}^n \to X$ be smooth. Let $k \geq 0$ be an integer. There exist smooth functions $g_\alpha : \mathbb{R}^n \to X$ and elements $x_\alpha \in X$ such that, for all $t \in \mathbb{R}$,

$$f(t) = \sum_{|\alpha| < k} \frac{\partial^{|\alpha|} f(0)}{\partial \alpha} \cdot x_\alpha + \sum_{|\alpha| = k} \frac{\partial^{|\alpha|} f(0)}{\partial \alpha} \cdot g_\alpha(t)^{(k)}$$

(with standard conventions about multi-indices $\alpha$). The $x_\alpha$'s are uniquely determined.

Except for the uniqueness assertion, this follows immediately from [5], Theorem 2.12. The uniqueness of the $x_\alpha$'s follows easily from the corresponding result for the case $X = \mathbb{R}$ using that $X'$ separates points.

The $x_\alpha$'s in Theorem 1.3 are of course the "Taylor coefficients"

$$x_\alpha = \frac{1}{|\alpha|!} \cdot \frac{\partial^{|\alpha|} f(0)}{\partial \alpha} ;$$

however, they do not appear explicitly in the present article.

For any smooth $f : X \to Y$ and $x \in X$, the map

$$x_1 \mapsto f(x + x_1) - f(x)$$

can, by Theorems 1.1 and 1.2, be written as a sum of a smooth linear map $df_x$ and a map of order $\leq 2$. The map

$$X \times X \to Y$$

given by

$$(x, x_1) \mapsto df_x(x_1)$$

is smooth, and linear in the second variable, cf. e.g. [3]. Thus, it defines a map

$$Df : X \to L(X, Y)$$

where $L(X, Y)$ is the vector space of smooth linear maps $X \to Y$. There is a canonical structure of convenient vector space on $L(X, Y)$ making all the evaluation maps $L(X, Y) \to Y$ smooth and such that $Df$ is smooth.
2. JET CALCULUS AND WEIL PROLONGATIONS.

Let $I \subseteq C^\infty(\mathbb{R}^n)$ be an ideal. For any convenient vector space $X$, we let $I(X)$ be the set of those smooth $f : \mathbb{R}^n \to X$ such that for all $\varphi \in X'$, $\varphi \circ f \in I$. We say that $f_1 \equiv f_2 \mod I$ if $f_1 - f_2 \in I(X)$. This is an equivalence relation. An equivalence class is called a mod-1 jet into $X$. This notion will be proved to have good properties if $I$ is large enough: Let $M \subseteq C^\infty(\mathbb{R}^n)$ denote the (maximal) ideal of functions $i.e., functions of order $\geq 1$. Then $M^r$ is the ideal of functions of order $\geq r$. It is of finite codimension. We shall say that an ideal $I \subseteq C^\infty(\mathbb{R}^n)$ is a Weil ideal if, for some $r$, $M^r \subseteq I \subseteq M$. The residue ring $C^\infty(I)/I$ is then a Weil algebra (cf. e.g. [4] or [1] for the notion), and any Weil algebra comes about in this way. We shall use the letter $W$ to denote any Weil algebra, but with a given presentation by a Weil ideal $I$, and use "mod-1-jet" and "$W$-jet" synonymously.

We denote by $\text{Draw}$ or $W \times X$ the set of all $W$-jets into $X$. Since $M^r \subseteq I$, we may choose a finite set of polynomials $\sum_{i=1}^m h_i(t)\cdot x_i$ of degree $< r$ which form a basis in $C^\infty(\mathbb{R}^n)$ mod $I$. It then follows from Theorem 1.3 that any $W$-jet into $X$ has a representative of the form

$$h_1, \ldots, h_m \in \mathbb{R}[t_1, \ldots, t_n]$$

for unique $x_i \in X$, and thus $\text{Draw} \cong X^m$. This also justifies the notation, since $W \cong \mathbb{R}^m$. Likewise, if $f : X \to Y$ is linear, $f \circ W : \text{Draw} \to Y_{\text{Draw}}$ may of course be defined. Our aim is to define $f \circ W$ for any smooth $f : X \to Y$.

**Proposition 2.1.** If $f_1 \equiv f_2 \mod I$ (where $f_i : \mathbb{R}^n \to X$), then we have $g \circ f_1 \equiv g \circ f_2 \mod I$, for any smooth $g : X \to Y$.

**Proof.** We have $f_1(0) = f_2(0)$ (say) since $f_1 \equiv f_2 \mod M$. Since $g \circ (f_i(x) - x_0) = \tilde{g} \circ f_i$ for $\tilde{g}(x) := g(x + x_0)$, it suffices to prove the result in the case $f_1(0) = f_2(0) = 0$.

So $f_1$ and $f_2$ may both be assumed to have order $\geq 1$. 

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To prove \( g \circ f_1 \equiv g \circ f_2 \mod I \) means by definition to prove

\[
\varphi \circ g \circ f_1 - \varphi \circ g \circ f_2 \in I,
\]

for any smooth linear \( \varphi : Y \to \mathbb{R} \), so let such \( \varphi \) be given. Change notation and write \( g \) for \( \varphi \circ g \). Then \( g : X \to \mathbb{R} \) may by Theorem 1.1 be written as a sum

\[
\sum_{q=0}^{r-1} h_q \circ G
\]

with \( h_q : X \to \mathbb{R} \) smooth homogeneous of degree \( q \), and \( G \) of order \( \geq r \). It suffices to prove that

\[
(2.1) \quad h_q \circ f_1 \equiv h_q \circ f_2 \mod I \quad \forall q = 0, \ldots, r-1
\]

and that

\[
(2.2) \quad G \circ f_1 \equiv G \circ f_2 \mod I.
\]

For (2.2), this is trivial; in fact each \( G \circ f_i \) \((i = 1, 2)\) has itself order \( \geq r \) since

\[
\text{order}(f_i) \geq 1 \quad \text{and} \quad \text{order}(G) \geq r.
\]

So

\[
G \circ f_i \in M^{\infty} \subset I, \quad i = 1, 2.
\]

For (2.1), we write, by Theorem 1.2 \( h_q \) in the form

\[
h_q(x) = H(x, \ldots, x),
\]

where \( H : X^q \to \mathbb{R} \) is smooth \( q \)-linear. For simplicity, let \( q = 2 \). Then

\[
H(f_1(t), f_1(t)) - H(f_2(t), f_2(t)) =
\]

\[
= H(f_1(t), f_1(t)) - H(f_2(t), f_1(t)) + H(f_2(t), f_1(t)) - H(f_2(t), f_2(t))
\]

\[
= H(f_1(t) - f_2(t), f_1(t)) - H(f_2(t), f_1(t)) - f_2(t)),
\]

and the result follows from

**Lemma.** Let \( H : X^q \to \mathbb{R} \) be \( q \)-linear smooth, and let \( I \) be an ideal in \( C^{\infty}(\mathbb{R}^q) \). If \( k : \mathbb{R}^n \to X \) belongs to \( I(X) \) then, for any smooth \( \ell_i : \mathbb{R}^n \to X \) \((i = 2, \ldots, q)\),

\[
(2.3) \quad H(k(t), \ell_2(t), \ldots, \ell_q(t)) \in I.
\]

**Proof.** Again, let \( q = 2 \) and write

\[
\ell_2(t) = |\alpha| < \alpha \cdot x_\alpha + L(t)
\]
with $L(t)$ or order $\geq r$. Then the function of $t$ displayed in (2.3) can be written

$$\sum_{a} t^a H(k(t), x_0) + H(k(t), L(t)).$$

The last term here clearly is a function of order $\geq r$, since $L$ is, and so is in $I$. But also each $H(k(t), x_0) \in I$ since they are of form $\phi \circ k$, $\phi \in X'$ (namely with $\phi = H(-, x_0)$), so is in $I$ since $k \in I(X)$. The Lemma, and thus the proposition, is proved.

For $g : X \to Y$ smooth there is thus an evident way of defining $g \circ \omega W : X \circ \omega W \to Y \circ \omega W$ so as to make $\omega W$ a functor, namely composing with $g$. If $j \in X \circ \omega W$ is a $\omega$-jet represented by $f : \mathbb{R}^n \to X$, we let $(g \circ \omega W)(j)$ be the $\omega$-jet represented by $g \circ f : \mathbb{R}^n \to Y$. If $g$ is smooth linear, $g \circ \omega W$ will then be the usual map with this notation.

Our next task is to make $\omega W$ into a functor which 'so takes values in $F$. Since $X \circ \omega W = X^m$, $X \circ \omega W$ inherits a structure of convenient vector space from that of $X^m$. The isomorphism $X \circ \omega W = X^m$ depends on a choice of basis mod $I$, but any other choice will define an invertible $m \times m$ matrix, which then defines also a smooth linear isomorphism $X^m \to X^m$, so the convenient vector space structure on $X \circ \omega W$ is well defined.

**Proposition 2.2.** For $g : X \to Y$ smooth, the map $g \circ \omega W : X \circ \omega W \to Y \circ \omega W$ is smooth.

**Proof.** We first do the special case where $I = M^c \subset C^\infty(\mathbb{R}^n)$. As basis mod $I$, we may choose all monomials in $t_1, \ldots, t_n$ of degree $< r$. The statement is then just the fact that, for $g$ fixed, the $r$ degree partial derivatives $\partial^r f / \partial t^r$ depend in a smooth (in fact polynomial) way on the partial derivatives $\partial f / \partial t^r$ ("higher order chain rule"). Since I could not find a reference**, not even an exact statement, of this "evident" fact, I shall be more explicit. Write $g$ in the form

$$g = \sum_{q=0}^{r-1} h_q + G$$

with $h_q : X \to Y$ smooth homogeneous of degree $q$ and $G$ of order $\geq r$. It suffices to prove for each $h_q$ separately, and for $G$. Now, since a jet is represented by a function $f : \mathbb{R} \to X$ or order $\geq 1$, $G \circ f$ has order $\geq r$, so its partial derivatives of order $< r$ vanish, so depend smoothly on those of $f$. Now consider $h_q$. Write $h_q(x) = H(x_1, \ldots, x)$ where $H : X^q \to Y$ is smooth symmetric $q$-linear (Theorem 1.2). Since the partial derivatives of any $k : \mathbb{R}^n \to Z$ can be obtained from the $D^k$'s, by evaluation at the canonical basis vectors in $\mathbb{R}^n$, the result

**ADDED IN PROOF.** I thank the referee for providing the following two references: A. Bastiani, Applications différentiables et variétés différentiables de dimension infinie, J. Analyse Math. Jérusalem XIII (1964), 2-113; and P. Ver Eecke, Fondements du Calcul Différentiel, P.U.F., Paris 1984.

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can be obtained from the following Lemma (when writing $R^n$ for $X$, $X$ for $Y$ and $Y$ for $Z$).

**Lemma 2.3.** Let $H : Y^q \to Z$ be symmetric smooth $q$-linear. Then there is a fixed formula

$$D^P(H(f, \ldots, f)) = \Sigma H(D^{k_1}f, \ldots, D^{k_s}f)$$

valid for all smooth $f : X \to Y$.

**Proof** and more precise statement. Let

$$k(x) := H(f(x), \ldots, f(x)).$$

Then $D^{pk}(x ; x_1, \ldots, x_p)$ equals the following finite sum (2.4), whose index set is the set of partitionings of $p = \{1, 2, \ldots, p\}$ into $\leq q$ disjoint subsets $\pi(1), \ldots, \pi(s(\pi))$.

$$(2.4) \quad \frac{q}{n} \frac{1}{\pi(1)} f(x; x_{\pi(1)}), \ldots, \frac{q}{n} \frac{1}{\pi(s(\pi))} f(x; x_{\pi(s(\pi))}), f(x), \ldots, f(x)$$

$(q - s(\pi)) f(x)$'s; here

$$[q]_F \quad \text{denotes} \quad q(q-1) \ldots (q-r+1),$$

and if $B \subset p$ is a subset, with $b$ elements $i_1, \ldots, i_b$, then we have put

$$D_f^B(x ; x_B) := D^bf(x ; x_{i_1}, \ldots, x_{i_b}).$$

This formula is easily verified by induction, and the Lemma is proved.

Now let $I \subset M^\infty$ be a general Weil ideal. Choosing a basis $h_1, \ldots, h_n$ mod $I$ amounts to an $R$-linear splitting $\sigma$ of the projection

$$C^\infty(R^n)/M^\infty \to C^\infty(R^n)/I = W.$$ 

It induces a smooth linear splitting $\times_{W^\sigma}$ of

$$X^{m'} = X \times (C^\infty(R^n)/M^\infty) \xrightarrow{\pi_X} X_{W^\sigma} \cong X^m.$$ 

By the well-definedness result (Proposition 2.1), for $g : X \to Y$ smooth, $g \notin W$ equals the composite

$$X_{W^\sigma} \xrightarrow{\times_{W^\sigma}} X_{(C^\infty(R^n)/M^\infty)} \xrightarrow{g \times_{W^\sigma}} Y_{(C^\infty(R^n)/M^\infty)} \xrightarrow{\pi_Y} Y_{W^\sigma},$$

where the middle map is smooth by the special case already proved. Thus, the composite is smooth.

This proves the Proposition. Thus each Weil algebra $W$ defines an endofunctor $- \circ W : F \to F$. 

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3. TRANSITIVITY OF PROLONGATIONS.

For any vector space $X$ and Weil algebras $W_1, W_2$ we have of course

\[(3.1) \quad X \# (W_1 \# W_2) \cong (X \# W_1) \# W_2\]

naturally in $X$ with respect to linear maps. Our aim in this section is to prove that for convenient vector spaces $X$, this isomorphism is natural in $X$ with respect to smooth maps.

Recall that we may consider as a subring

\[\mathbb{R}[t_1, ..., t_n] \subset C^\infty(\mathbb{R}^n)\]

Let $I \subset C^\infty(\mathbb{R}^n)$ be a Weil ideal representing the Weil algebra $W$. In the following commutative diagram with exact rows, $I'$ is defined as intersection (pullback):

\[
\begin{array}{cccccc}
0 & \to & I & \to & C^\infty(\mathbb{R}^n) & \to & C^\infty(\mathbb{R}^n)/W & \to & 0 \\
& & \downarrow & & \downarrow \alpha & & \downarrow & \\
0 & \to & I' & \to & \mathbb{R}[t_1, ..., t_n] & \to & \mathbb{R}[t_1, ..., t_n]/I' & \to & 0
\end{array}
\]

Since there is a basis mod $I$ consisting of polynomials, it follows that

\[C^\infty(\mathbb{R}^n) = \mathbb{R}[t_1, ..., t_n] + I\]

thus from the Noether isomorphism

\[P/P\cap I = (P+I)/I\]

it follows that $\alpha$ is an isomorphism. More generally, if $X$ is a convenient vector space, the subspace of $C^\infty(\mathbb{R}^n, X)$ consisting of smooth polynomial functions may be identified with $X \# \mathbb{R}[t_1, ..., t_n]$ (Theorem 1.3). So if we denote by $I(X)$ the subspace of functions $\mathbb{R}^n \to X$ which are $\equiv 0 \mod I$, and $I'(X)$ the polynomial functions among them, we have a commutative diagram with exact rows and with the left hand square a pullback:

\[
\begin{array}{cccccc}
0 & \to & I(X) & \to & C^\infty(\mathbb{R}^n, X) & \to & C^\infty(\mathbb{R}^n, X)/I(X) = X \# W & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & I'(X) = X \# I' & \to & X \# \mathbb{R}[t_1, ..., t_n] & \to & X \# W & \to & 0
\end{array}
\]

Henceforth, we shall write $I$ instead of $I(X)$ when the context (diagram) will inform us about $X$.

For the proof of naturality of (3.1) with respect to smooth maps,
we shall make essential use of the cartesian closedness of the category $F$ of convenient vector spaces with smooth maps: for $X$, $Y$ convenient vector spaces, the vector space $C^\infty(X, Y)$ of smooth maps $X \to Y$ carries a natural structure of convenient vector space making it the exponential object $Y^X$ in $F$. In particular

$$C^\infty(R^{n+m}, X) \cong C^\infty(R^m, C^\infty(R^n, X)), \tag{3.2}$$

natural in $X \in F$, and this will be the essence in the proof. Let $W_1$, $W_2$ be Weil algebras with presentation $C^\infty(R^n)/I_1$ and $C^\infty(R^m)/I_2$, respectively. Then $W_1 \otimes W_2$ has presentation $C^\infty(R^{n+m})/(I_1, I_2)$, where $(I_1, I_2)$ is the ideal generated by functions $h(s), g(s, t)$ with $h \in I_1$ and functions $h(s, t), g(t)$ with $g \in I_2$ (where $s = (s_1, \ldots, s_n)$ etc.). Consider the following commutative diagram (in which the two bottom corners represent the two sides of (3.1)):

$$\begin{array}{ccc}
R[s, t]/(I_1, I_2) & \xrightarrow{\alpha_X} & R[s, t]/I_2 \otimes R[s]/I_1 \\
\downarrow \cong & & \downarrow \\
C^\infty(R^{n+m}, X)/(I_1, I_2) & \xrightarrow{\alpha_X} & C^\infty(R^{n+m}, X) \xrightarrow{\beta_X} C^\infty(R^m, C^\infty(R^n, X))/I_2
\end{array} \tag{3.3}$$

Here $\alpha_X$ and $\alpha_X$ are evident, whereas $\beta_X$ utilizes (3.2) and $\beta_X$ utilizes a mimicking of (3.2) on the level of polynomials, namely the linear isomorphism

$$R[s, t] \cong R[t] \otimes R[s].$$

$\alpha_X$ and $\beta_X$ are surjective. The top isomorphism comes about purely algebraically by applying $- \otimes X$ to isomorphisms, well-known from algebra,

$$R[s, t]/(I_1, I_2) \cong R[s]/I_1 \otimes R[t]/I_2.$$

The maps $\alpha_X$ and $\beta_X$ are evidently natural in $X$ with respect to smooth maps; for the maps $\alpha_X$ and $\beta_X$ such naturality does not make sense, since $R[s, t] \otimes X$ is not functorial in $X$ with respect to smooth maps. However, this does not matter; the smooth natural isomorphism of the two bottom corners in (3.3) now follows from a piece of diagram chasing, namely the following Lemma whose proof we leave to the reader.

**Lemma.** Let $C$, $D$ and $E$ be functors $A \to B$, and assume for each $X \in A$ a commutative triangle

$$\begin{array}{ccc}
D(X) & \xrightarrow{\gamma_X} & E(X) \\
\downarrow \alpha_X & & \downarrow \beta_X \\
C(X) & \xrightarrow{\gamma_X} & E(X)
\end{array}$$

If all $\alpha_X$ are epic, and $\alpha$ and $\beta$ are natural in $X$, then so is $\gamma$. 

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We have thus proved the first statement in the following theorem (the second assertion being trivial):

**Theorem 3.1.** The isomorphism (3.1) is natural with respect to smooth maps. Also $X \circ R = X$, naturally with respect to smooth maps.

We end this section by remarking that the construction $X \circ W$ is also functorial in $W$. A homomorphism $F$ of Weil algebras

$$W_1 = C^\infty(R^n)/I \xrightarrow{F} C^\infty(R^m)/J = W_2$$

can be represented by a smooth map

$$\bar{F} : R^m \to R^n$$

with $\bar{F}(0) = 0$,

and with $\varphi \circ \bar{F} \in J$ whenever $\varphi \in I$. Then, for $f : R^n \to R$ representing an element $\{f\}$ of $W_1$, $f \circ \bar{F}$ represents $F(\{f\}) \in W_2$. And if $f : R^m \to X$ represents an element of $X \circ W_1$, $f \circ \bar{F}$ represents $(X \circ W)(\{f\})$.

All said, $\circ$ defines a bifunctor

$$F \times W \to F$$

where $W$ is the category of Weil algebras. In fact, by Theorem 3.1, the monoidal category $(W, \circ, R)$ acts on $F$ in an associative unitary way (up to coherent isomorphisms). - Note that $\circ$ is the coproduct in $W$, $R$ the initial object. (Actually, $R$ is also terminal object in $W$.)

**4. SEMIDIRECT PRODUCT OF CATEGORIES.**

Let $W$ be any category with finite coproducts, denoted $\circ$, and with initial object denoted $R$, and let $G$ be a category on which $W$ acts (from the right, say), i.e., there is given a functor $\circ : G \times W \to G$, and there are given natural isomorphisms (for $X \in G$, $W_1 \in W$):

$$(X \circ W_1) \circ W_2 \simeq X \circ (W_1 \circ W_2), \quad X \simeq X \circ R$$

which fit coherently with the associativity - and unit - isomorphisms of the monoidal category $(W, \circ, R)$.

We construct a new category $G \ltimes W$ as follows: the objects are pairs $(X, W)$ with $X \in G$, $W \in W$. An arrow $(X_1, W_1) \to (X_2, W_2)$ is a pair of arrows in $G$ and $W$,

$$f : (X_1 \xrightarrow{X_2 \circ W_1}, W_2 \xrightarrow{W_2} W_1)$$

and the composite of this pair with
is the pair (associativity isomorphisms omitted, by coherence):

\[(X_2 \xrightarrow{g} X_1 \otimes W_1, \ x \xrightarrow{Y} W_2)\]

Identity arrow is

\[(X \cong X \otimes R \xrightarrow{id} X \otimes W, \ id_W).\]

There is a full embedding \(j : G \to G \times W\) given by \(X \mapsto (X, R)\) and

\[(X_1 \xrightarrow{f} X_2) \mapsto (X_1 \xrightarrow{f} X_2 \cong X_2 \otimes R, \ id_R).\]

**Proposition 4.1.** The inclusion \(j : G \to G \times W\) preserves all those inverse limits which are preserved by all \(- \otimes W\).

**Proof.** We prove the case of binary products only (which is all we need for what follows). We have in fact more generally

\[(Z_1, W_1) \times (Z_2, W_2) \cong (Z_1 \times Z_2, W_1 \otimes W_2)\]

due to the string of conversions

\[
\begin{align*}
(Y, W) &\longrightarrow (Z_1 \times Z_2, W_1 \otimes W_2) \\
Y \to (Z_1 \times Z_2) \otimes W = (Z_1 \otimes W) \times (Z_2 \otimes W), & \quad W_1 \otimes W_2 \to W \\
\frac{Y \to Z_1 \otimes W, \ W_i \to W}{((Y, W) \to (Z_{i}, W_i))_{i=1,2}}
\end{align*}
\]

**Proposition 4.2. If** \(G\) **has exponential objects** \(X^X\) **which are preserved by each** \(- \otimes W\) **in the sense** \(Y^X \otimes W \cong (Y \otimes W)^X\) **and if each** \(- \otimes W\) **preserves finite products, then** \(j\) **preserves exponential objects.**

**Proof.** We have bijective correspondences

\[
\begin{align*}
(Z, W) &\to (Y^X, R) \\
Z \to Y^X \otimes W = (Y \otimes W)^X \\
Z \times X &\to Y \otimes W \\
(Z \times X, W) &\to (Y, R) \\
(Z, W) \times (X, R) &\to (Y, R)
\end{align*}
\]

where we for the last conversion utilized (4.2), which we may by the second assumption made.
If the initial object $R$ of $W$ is also terminal, we have a canonical functor \( \pi : G \times W \to G \), given on objects by \( \pi(X, W) = X \) and with \( \pi \) applied to the arrow (4.1) given as

\[
X_1 \to X_2 \otimes W \xrightarrow{X_2 \otimes !} X_2 \otimes R = X_2.
\]

Clearly \( \pi \circ f = \text{id}_G \), and there is a natural map making \( f(\pi(X, W)) \) a retract of \((X, W)\). (In fact, if each \(-\otimes W\) preserves finite products, it follows from (4.2) that

\[(4.3) \quad (Z, W) \simeq (Z, R) \times (1, W),\]

and \((1, W)\) is an object in \( G \times R \) which has a unique point (= map from the terminal object).)

5. THE EMBEDDING.

We consider now the category \( F \), with the "action" \( \otimes \) of \( W \), the category of Weil algebras, as described in §2 and §3, and we form \( F \times W \). The full subcategory \( F \subset F \) of finite dimensional vector spaces is stable under the action, so that we get \( f \times W \) as a full subcategory of \( F \times W \).

We describe (essentially following \cite{1}) a Grothendieck topology on \( f \times W \) which will make it a site of definition for the Cahiers topos \( T_{-\otimes} \). We declare the following families to be covering:

\[
(X_i, W) \xrightarrow{a_i \otimes \text{id}} (X, W), \quad i \in I
\]

if \( \pi(a_i) : X_i \to X \) form an open covering.

Let \( i \) and \( j \) denote the following full inclusions.

\[
\begin{array}{ccc}
F \times W & \xrightarrow{i} & F_k W \\
\downarrow & & \downarrow \text{j} \\
F & & F
\end{array}
\]

Any \( Y \in F \) defines a functor \( J(Y) : (F_k W)^{\text{op}} \to \text{Sets} \), namely

\[
J(Y) = \text{hom}_{F_k W}(f(-), j(Y)).
\]

So \( J(Y) \) is "representable from the outside". We may omit \( i \) and \( j \) from notation.

**Proposition 5.1.** \( J(Y) \) is a sheaf.

**Proof.** Let \( \{a_i\} \) be a covering, as in (5.1), in \( f \times W \), and let

\[
b_i : (X_i, W) \to Y
\]

be a compatible family \( Y \in F \). We should construct a map
The data of the \( b_i \)'s amount to \( b_i : X \to Y \) and the compatibility condition for the \( b_i \)'s implies one for the \( E_i \)'s. The required map \( c \) amounts to a map \( c : X \to Y \). Also \( \pi(a_i) : X_i \to X \) form an open covering. So the crux is to observe that any convenient vector space \( Z \) (in our case \( Z = Y \)) represents (from the outside) a sheaf on the site \( f \) (with open coverings as its topology). This follows from concreteness of the categories \( f \) and \( F \), and the fact that smoothness of a set theoretic map \( X \to Y \) between convenient vector spaces may be tested by smooth plots on an open covering of \( X \) and with finite dimensional domains.

We leave the full details to the reader. At this point, it would have been an advantage to consider the categories \( f \) and \( F \) consisting of open subsets of finite dimensional, resp. convenient vector spaces, with \( W \) acting on them (which it does by the same construction as the one of §2.3) because the open coverings in \( f \) and \( F \) admit pullbacks which are furthermore preserved by \( -\otimes W \).

We can now state our main theorem; \( C \) denotes the Cahiers topos (= sheaves on \( f \times W \)):

**Theorem 5.2.** The functor \( J : F \to C \) is full and faithful. It preserves finite products, and it preserves exponentials \( Y^X \) provided \( X \) is finite dimensional.

**Remark.** By the remarks just before the statement of the theorem it follows that the embedding \( J \) may be extended to the category \( F \) of open subsets of convenient vector spaces, and their smooth maps, and thus possibly also to some category of "manifolds modelled on convenient vector spaces".

**Proof.** When \( J \) is composed with the global-sections functor \( \Gamma : C \to \text{Sets} \), we get the faithful underlying-set functor \( \Gamma : F \to \text{Sets} \), so \( J \) is faithful. To test fullness, let \( f : J(X) \to J(Y) \) be a map in \( C \). We get a set theoretic map \( \Gamma(f) : X \to Y \), which we have to test is smooth. But again, smoothness may be tested by checking with smooth plots \( c : R^n \to X \) (in fact \( n - 1 \) suffices), and since

\[
\Gamma(f) \in f \subset f \times W,
\]

smoothness of \( \Gamma(f) \) follows. To see \( J(\Gamma(f)) = f \), just apply the faithful \( \Gamma \).

Next we argue that \( J \) preserves finite products. It is clear from the construction that \( -\otimes W : F \to F \) preserves finite products for each \( W \in W \). Hence, by Proposition 4.1, \( J : F \to F \times W \) preserves finite products, and hence so does \( J \), for standard categorical reasons (essentially, "Yoneda embedding preserves limits").

Finally, to argue for exponentials, we note that the functors

\[
-\otimes W : F \to F
\]

satisfy
In fact, if $W$ is $m$-dimensional as a vector space, both sides are isomorphic, by smooth linear isomorphisms, to

$$\langle Y^X \rangle^m \simeq \langle Y^W \rangle^X.$$  

This isomorphism is in fact natural with respect to smooth maps, because if $h_1, \ldots, h_m \in C^\infty(\mathbb{R}^n)$ is a basis mod I, an element of $Y^X \otimes W$ has a unique representative of form

$$t \mapsto \sum_{j=1}^m h_j(t) \cdot \xi_j (\xi_j \in Y^X),$$

and under the isomorphism, this element goes to

$$x \mapsto [t \mapsto \sum_{j=1}^m h_j(t) \cdot \xi_j (\xi_j \in Y)],$$

the square bracket here representing an element of $Y \otimes W$. The passage thus described is clearly natural. So $- \otimes W$ satisfies the conditions of Proposition 4.2, so that $j : F \to F \otimes W$ preserves exponentiation. The rest of the argument is now purely categorical; let $A \in F \otimes W$, and let $A$ be the object of $C$ which it represents. For $X \in F$ and $Y \in F$, we then have

$$\text{hom}_C(A, J(Y)) = \text{hom}_{F \otimes W}(A, j(Y)) = \text{hom}_{F \otimes W}(A, j(Y)) = \text{hom}_{F \otimes W}(A \otimes j(X), j(Y)) = \text{hom}_C(A \otimes j(X), j(Y)),$$

the last equality provided $A \otimes j(X) \in F \otimes W$, which will be the case since $X \in F$. The theorem is proved.

6. RETROSPECT.

Having Theorem 5.2, as well as the full power of synthetic reasoning in $C$, many of the constructions and comparisons that we worked hard to get, become very transparent. For a Weil algebra $W$, let $W$ denote the ("infinitesimal") object in $C$ which it represents. Then $F \otimes W$ becomes the full subcategory of $C$ of objects of form $J(X) \otimes W$ ($X \in F$, $W \in W$), this being identified with $(X, W) \in F \otimes W$. A $W$-jet into $X$ becomes simply a map $W \to J(X)$, explaining the functoriality of the jet notion. Also, $X \otimes W$ goes by $J$ to $J(X) \otimes W$, explaining the properties of the functor $- \otimes W$, e.g. the transitivity

$$(X \otimes W_1) \otimes W_2 \simeq X \otimes (W_1 \otimes W_2)$$

is simply the categorical law $(A^B)^C \simeq A^{B \times C}$.
Let us finally remark that each \( J(X) \) evidently will be an \( R \)-module object \( (R = J(R)) \), and that it will satisfy the "vector form of Axiom 1\(^W \)" (cf. [4]), in the sense that, if \( n \) is the linear dimension of \( W \), we have an isomorphism \( J(X)^n \rightarrow J(X)^W \) constructed out of a linear basis \( h_1, \ldots, h_m \) for \( R[t_1, \ldots, t_n] \mod I \) (where \( W = R[t]/I \)) as the map with synthetic description

\[
(6.1) \quad (x_1, \ldots, x_m) \mapsto [(t_1, \ldots, t_n)] \mapsto \sum h_i(t) x_i
\]

(\( \overline{W} \) being identified with a sub"set" of \( \mathbb{R}^n \), namely the "zero-set of \( I \) "). This follows essentially from the fact that in \( F \) we have an isomorphism \( X^m = \mathbb{R}^m \) given by the same formula (6.1).

From the validity of Axiom 1\(^W \) for \( J(X) \) it follows, in turn, that \( J(X) \) is infinitesimally linear in the strong (Bergeron-) sense, cf. [6]; the argument is as in [6], Proposition 1.2, with \( R \) replaced by \( J(X) \).

REFERENCES.