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A DIRECT DESCRIPTION OF UNIFORM COMPLETION IN LOCALES AND A CHARACTERIZATION OF LT-GROUPS

by Igor KŘÍŽ

RÉSUMÉ. Isbell a construit la complétion uniforme d'un locale (au sens de fermeture absolue) à l'aide d'hyperespaces. Ici on donne une autre construction, plus directe, par générateurs et relations.

Comme application, on obtient "l'enveloppe locale" d'un groupe topologique, qui permet de caractériser les groupes topologiques qui sont des locales. Dans le cas commutatif, une telle caractérisation a été obtenue dans un article par Isbell, Kříž, Pultr et Rosický.

The question of uniform completeness in locales (in the sense of absolute closedness) was discussed first in Isbell's paper [3]. Among other results, Isbell gave a description of the uniform completion. Since the classical Cauchy-filter approach does not yield the absolute closedness property in locales, it was replaced by another construction the idea of which was to employ hyperspaces. This is also one of the reasons why Isbell called completion in locales "hypercompletion".

The main purpose of this paper is to introduce another description of uniform completion in locales, which does not use hyperspaces and in fact is even more straightforward than the original topological construction. It simply consists of writing down generators and defining relations. In connection with this we will also prefer the natural terminology without the prefix "hyper".

As an application, we give a general construction of a "localic hull" of a topological group, which yields a characterisation of those topological groups which are localic. It should be noted that for the commutative case a characterisation of this type was obtained in [4].

1. PRELIMINARIES.

1.1. Frames and locales. Basic facts concerning locales can be found in [5] (as well as in other papers, e.g. [1, 3]).

A *frame* is a complete lattice A in which the distributive law

$$a \wedge \vee S = \vee\{a \wedge x \mid x \in S\}$$

holds for any $a \in A$, $S \subset A$. We will write $1(A)$ or 1 for $\vee A$ and $0(A)$ or 0 for $\vee \emptyset$. A *frame morphism* is a mapping $f : A \to B$ of frames A, B preserving joins and finite meets. The category of frames will be denoted by Frm. The lattice $\Omega(X)$ of all open sets of a topological space X ordered by inclusion is a frame and a continuous mapping $f : X \to Y$ induces a frame morphism

$$\Omega(f) = f^{-1} : \Omega(Y) \to \Omega(X).$$

Thus, Ω becomes a contravariant functor from the category Top of topological spaces to Frm.

To make Ω a covariant functor, one introduces the category Loc $=$ Frm^{op} of *locales,* and writes Ω : Top $\to$ Loc. The functor Ω has a right adjoint

$$T : \mathrm{Loc} \to \mathrm{Top} \qquad \text{given by} \qquad A \mapsto T(A),$$

where $T(A)$ is the set of all frame morphisms $A \to \mathbf{2} = \Omega(.)$ together with the natural topology (see e.g. [5]).

There is a natural parallelism between the concepts in Top and Loc (see [3, 5]). It is, however, often not of much use in technical parts of the proofs presented. For that reason it is of an advantage to write morphisms as in frames, although we think in locales. Let us give some examples, which will be useful later on :

A frame morphism $f : A \to B$ is called an *embedding* of B into A iff it is surjective. An embedding is called *closed* if we have

$$f(x) = f(y) \quad \to \quad x \vee \bigvee_{f(u)=0} u = y \vee \bigvee_{f(u)=0} u.$$

A morphism f is *dense* iff

$$f(a) = 0 \quad \to \quad a = 0 .$$

It is easily checked that a dense closed emnbedding is an isomorphism.

Now let A be a frame. For $a, b \in A$ write $a \triangleleft b$ iff

$$(\exists\, c \in A)(b \vee c = 1 \;\;\&\;\; a \wedge c = 0).$$

The frame A is called *regular,* iff

$$(\forall\, a \in A) \;\; a = \vee\{\, b \mid b \triangleleft a\} .$$

1.1.1. *Proposition.* *The dense frame morphisms of regular frames are* *exactly the monomorphisms in the full subcategory of* Frm *generated by the regular frames.*

Proof. See [4].

Let Sp denote the subcategory of Frm generated by the objects $\Omega(X)$ where X are topological spaces. It can be shown (see [5]) that Sp is characterized by

$$\mathrm{Sp} = \{\alpha \in \mathrm{Frm} \mid \Omega T(\alpha) = \alpha\}.$$

1.2. Uniform locales. (For the details, see [9, 10]) A *cover* of a frame A is a set $S \subset A$ with the property that $\bigvee S = 1$. The system of all covers of A will be denoted by $C(A)$. Let us call a $U \in C(A)$ a *refinement* of a $V \in C(A)$ (and write $U \prec V$) iff

$$(\forall\, a \in U)(\exists\, b \in V)(a \leqq b).$$

If $U, V \in C(A)$ then we have a cover

$$U \wedge V = \{a \wedge b \mid a \in U \;\&\; b \in V\}.$$

Write for $x \in A$, $U \in C(A)$,

$$U^* = \{\bigvee S \mid S \subset U \;\&\; (a, b \in S \rightarrow a \wedge b \neq 0)\},$$

$$Ux = \bigvee\{a \mid a \in U \;\&\; a \wedge x \neq 0\}, \qquad U.U = \{Ux \mid x \in U\}.$$

Obviously $U.U \prec U^{**}$. Let $\mathcal{U} \subset C(A)$, $x, y \in A$. We write $x \overset{\mathcal{U}}{\lhd} y$, iff there exists a $U \in \mathcal{U}$ such that $Ux \leqq y$. Put

$$A_{\mathcal{U}} = \{x \in A \mid x = \bigvee\{y \mid y \overset{\mathcal{U}}{\lhd} x\}\}$$

The pair $(A, \mathcal{U})$, $\mathcal{U} \subset C(A)$ is called a *uniform frame* iff we have

$$(U \in \mathcal{U} \;\&\; U \prec V) \rightarrow V \in \mathcal{U}, \tag{i}$$

$$(U \in \mathcal{U} \;\&\; V \in \mathcal{U}) \rightarrow U \wedge V \in \mathcal{U}, \tag{ii}$$

$$(\forall\, U \in \mathcal{U})(\exists\, V \in \mathcal{U})\,(V^* \prec U), \tag{iii}$$

$$A_{\mathcal{U}} = A. \tag{iv}$$

The system $\mathcal{U}$ is called a *uniformity* on A. A *uniform basis* is a system $\mathcal{U} \subset C(A)$ satisfying (iii) and (iv). Similarly as in Top, we have the least uniformity generated by a given uniform basis (see [8]).

Let $(A, \mathcal{U})$, $(B, \mathcal{V})$ be uniform frames. A frame morphism $f : A \rightarrow B$

is called *uniformly continuous* with respect to U, V iff we have

$$(f_{\#})_{\#}(U) \subset V,$$

where "#" indicates the image function. In that case we write

$$f : (A, U) \to (B, V).$$

We call f a *uniform embedding* iff it is surjective and, moreover,

$$(f_{\#})_{\#}(U) = V.$$

We will often speak about *uniform locales*, related to uniform frames in the same way as locales relate to frames.

1.2.1. Lemma. *Each uniform locale is regular.*

Proof. See [8]. We could really claim "completely regular", but it would be never used in the sequel. ◊

1.3. In general, the products in Top and Loc need not coincide. Thus, a question arises as to what is the relation between groups in Top and Loc. It was shown in [4] and [7] that none of the concepts contained the other. We will be particularly interested in the fact that a topological group need not be a group in Loc.

Denote by "$\oplus$" and "+" the sum in Frm, resp. in Sp. (Realize that

$$\Omega(X \times Y) = \Omega(X) + \Omega(Y).)$$

In the sequel, an L-group means a cogroup object in Frm, while a T-group designates a cogroup object in Sp. (Note that if X is a topological group, then $\Omega(X)$ is a T-group and if A is a T-group or an L-group, then T(A) is a topological group.)

Thus, an L-group A consists of a multiplication $\mu : A \to A \oplus A$, an inverse $\iota : A \to A$ and a unit point $\varepsilon : A \to \mathbf{2}$ which satisfy the usual identitites

$$(\mu \oplus 1_A) \circ \mu = (1_A \oplus \mu) \circ \mu, \quad (\varepsilon \oplus 1_A) \circ \mu = (1_A \oplus \varepsilon) \circ \mu = 1_A,$$

$$\nabla \circ (\iota \oplus 1_A) \circ \mu = \nabla \circ (1_A \oplus \iota) \circ \mu = \sigma \circ \varepsilon,$$

where $\nabla : A \oplus A \to A$ is the codiagonal and σ is the initial morphism. A T-group consists of similar mappings

$$\mu : A \to A + A, \quad \iota : A \to A, \quad \varepsilon : A \to \mathbf{2}$$

satisfying analogous identities. We define homomorphisms of L-groups (T-groups) in the obvious way [4]. Note again that a homomorphism of T-groups is always a pre-image function of a homomorphism of topological groups. An *embedding* of L-groups will be an L-group homomorphism which is an embedding.

1.3.1. Proposition. *L-groups are regular frames.*

Proof. Indeed, they are unifomizable (see [4, 7]). ◊

1.3.2. Proposition. *Any embedding of L-groups is closed.*

Proof. See [4]. ◊

Let us now specify what we mean by saying that a T-group (A, μ) is an L-group. First of all we should claim the existence of a lifting

$$\begin{tikzcd} & A \oplus A \arrow[d, "\tau"] \\ A \arrow[ur, dashed, "\mu_1"] \arrow[r, "\mu"'] & A+A \end{tikzcd} \tag{1.3.1}$$

where τ is the standard restriction mapping. Since τ is evidently a dense embedding (see [4]), it follows directly from 1.1.1, 1.3.1 that whenever μ_1 exists, it is uniquely determined and makes (A, μ_1) an L-group. T-groups which require this property will be called LT-groups.

2. SUBLOCALES AND FACTOR FRAMES.

In this section we give a general description of factorization in frames. The main idea of our approach belongs to Johnstone [6], who really proved the full strength of the theorem, but restricted the statement by unnecessary assumptions. In this paper we present the theorem in the full generality, although the case we really need is much closer to that of Johnstone [6].

2.1. The right adjoint of a frame morphism $f : A \to B$ will be denoted by $f_* : B \to A$ (recall that $f_*(x) = \bigvee_{f(y) \leq x} y$). Putting for a surjective f:

$$j = f_* \circ f : A \to A,$$

we obtain a natural 1-1 correspondence between the embeddings and the *nuclei* (see [11]). (Recall that a *nucleus* on A is a mapping $j : A \to A$ satisfying

$$j(a) \geqq a \tag{2.1.1}$$
$$j(j(a)) = j(a) \tag{2.1.2}$$
$$j(a \wedge b) = j(a) \wedge j(b) \tag{2.1.3}$$

for any two $a, b \in A$.) If j is a nucleus on A, then one can put

$$A_j = \{ a \in A \mid j(a) = a \}$$

and obtain a frame embedding $j : A \to A_j$, whose right adjoint is the inclusion.

2.2. A *join-basis* of a frame A is any subset $A' \subset A$ which satisfies

$$(\forall c \in A)(\exists S \subset A')(c = \vee S).$$

We call a subset $R \subset A \times A$ a *precongruence relation* if for any $a, b \in A$ with $a R b$ the set

$$\{ s \in A \mid (a \wedge s) R (b \wedge s) \}$$

is a join-basis of A.

2.2.1. Observation. Let $S \subset A \times A$ and let A' be a join-basis of A, which is closed under finite meets. Then

$$R(S, A') = \{ (a \wedge c, b \wedge c) \mid (a, b) \in S \ \&\ c \in A' \}$$

is a precongruence relation on A.

Take an $R \subset A \times A$. An element $a \in A$ will be called R- *coherent* if for any two $a, b \in A$ with $a R b$ we have

$$a \leqq u \quad \text{iff} \quad b \leqq u.$$

Denote by $\Rightarrow$ the operation of implication in A given by

$$c \leqq a \Rightarrow b \quad \text{iff} \quad c \wedge a \leqq b.$$

2.2.2. Lemma. *Let R be a precongruence on A and let $b \in A$ be R-coherent. Then the element $a \Rightarrow b$ is R-coherent for any $a \in A$.*

Proof. Let $x \ R \ y$. Since R is a precongruence relation, we can choose a set $Q \subset A$ such that

$$A = \vee Q \quad \text{and} \quad (\forall t \in Q)(x \wedge t \ R \ y \wedge t).$$

Now

$$x \leq a \Rightarrow b \text{ iff } x \wedge a \leq b \text{ iff } (\forall t \in Q)(x \wedge t \leq b)$$
$$\text{iff } (\forall t \in Q)(y \wedge t \leq b) \text{ iff } y \wedge a \leq b \text{ iff } y \leq a \Rightarrow b. \qquad \Diamond$$

2.2.3. *Theorem.* *Let* R *be a precongruence on* A*. Then the set* A(R) *of all the* R*-coherent elements together with the induced ordering is a frame and there exists a nucleus* $j : A \to A$ *such that* $A_j = A(R)$ *. Moreover,* $j : A \to A(R)$ *is universal among the join-preserving mappings* f *from* A *to complete lattices* B *satisfying*

$$a \, R \, b \quad \to \quad f(a) = f(b).$$

(More exactly, for any such f there exists a unique join-preserving $f_j : A(R) \to B$ such that $f = f_j \circ j$.)

Proof. Define $j : A \to A$ by

$$j(a) = \bigwedge \{ u \geq a \mid u \text{ is R-coherent} \} .$$

Since a (finite or infinite) meet of R-coherent elements is evidently R-coherent again, we obtain

$$a \leq j(a) = j(j(a)).$$

Let us show that j preserves finite meets. We obviously have

$$j(a \wedge b) \leq j(a) \wedge j(b)$$

since the right hand element is R-coherent. Conversely, it holds

$$b \leq (a \Rightarrow j(a \wedge b)),$$
$$a \leq ((a \Rightarrow j(a \wedge b)) \Rightarrow j(a \wedge b))$$

while the right hand elements are R-coherent by Lemma 2.2.2. Thus we have

$$j(a) \wedge j(b) \leq (a \Rightarrow j(a \wedge b)) \wedge ((a \Rightarrow j(a \wedge b)) \Rightarrow j(a \wedge b)) \leq j(a \wedge b)$$

as required. - It remains to prove the universality of j . Let $f : A \to B$ preserve joins and let $a \, R \, b \to f(a) = f(b)$. Put for $a \in A$

$$s(a) = \bigvee \{ x \in A \mid f(x) \leq f(a) \}.$$

Since $s(a)$ is R-coherent and $s(a) \geq a$, we conclude that $s(a) \geq j(a)$, and consequently

$$f(a) \leq f(j(a)) \leq f(s(a)) .$$

On the other hand, since f preserves joins, we have $f(s(a)) \leq f(a)$. $\Diamond$

2.2.4. Corollary. *Preserve the notation of 2.2.3 and assume that* $R = R(S, A')$ *for a join-basis* A' *and a relation* $S \subset A \times A$. *Then for any frame morphism* $f : A \to B$ *which satisfies*

$$a \, S \, b \;\to\; f(a) = f(b)$$

there exists a unique frame morphism $g : A(R) \to B$ *satisfying* $f = g \circ j$.

Proof. Since f preserves finite meets, we have

$$a \, R \, b \;\to\; f(a) = f(b). \qquad \Diamond$$

2.3. Let A be a meet-semilattice. Denote by $\downarrow A$ the set of all downward-closed ("decreasing") subsets of A. Then $\downarrow A$ is a completely distributive lattice (in particular, a locale) and we have a mapping

$$\downarrow : A \to \downarrow A \quad \text{given by} \quad \downarrow a = \{x \in A \mid x \leqq a\} .$$

Moreover, one can prove

2.3.1. Proposition. *The mapping* $\downarrow$ *gives rise to a reflection from the category* MSL *of meet-semilattices to* Frm (see [11]). $\Diamond$

Of course, the word "reflection" applies only to the properties of the mapping $\downarrow$ and does not indicate that Frm should be a full subcategory of MSL. Considering 2.3.1, one sees that Proposition 1.1 from [6] is a special case of 2.2.3 for a certain type of precongruence relations on $\downarrow A$.

3. COMPLETION OF UNIFORM LOCALES.

In this section we give a description of uniform completion in terms of generators and defining relations. Throughout this paper, the completeness means the natural categorical phenomenon and not a generalization of the topological Cauchy-filter construction. It coincides with Isbell's concept of "hyper-completeness" (see [3]).

3.1. Let $(A, \mathcal{U})$ be a uniform locale. Denote by $S \subset \downarrow A \times \downarrow A$ the system of all pairs

$$(\downarrow a, k(a)), \qquad (\downarrow 1, c(U)), \qquad (\downarrow 0, \emptyset)$$

with $a \in A$, $U \in \mathcal{U}$, where k and c are given by

$$k(a) = \bigcup_{b \,\triangleleft_{\mathcal{U}}\, a} \downarrow b \qquad \text{and} \qquad c(U) = \bigcup_{a \in U} \downarrow a .$$

Put

$$\downarrow A' = \{\downarrow a \mid a \in A\}, \qquad R = R(S, \downarrow A').$$

(Realize that $\downarrow A'$ is meet-closed.) Recalling the notation of 2.2.3, write

$$\bar{A} = \downarrow A(R), \qquad j : \downarrow A \to \bar{A}.$$

Now both $\downarrow A$ and $\bar{A}$ are locales. Since $\bar{A} \subset \downarrow A$, we should be careful when indicating their localic operations. There is no trouble with the meets, which are preserved by the nucleus and hence coincide. On the other hand, there is a natural way to distinguish joins : we reserve the symbol " $\vee$ " for $\bar{A}$, while in $\downarrow A$ we simply use the set-theoretical symbol " $\cup$ ".

It follows immediately from the definition of S that $\downarrow A' \subset \bar{A}$. Hence, we have a dense embedding $p : \bar{A} \to A$ given by

$$p(u) = \bigvee \{ a \mid \downarrow a \leq u \}.$$

Put

$$\bar{U}_0 = \{\{\downarrow a \mid a \in U\} \mid U \in U\}.$$

We see easily that $\bar{U}_0$ is a system of covers, for

$$\bigvee \{ \downarrow a \mid a \in U \} = j(c(U)) = 1.$$

In fact we obtain a uniform basis. Denote by $\bar{U}$ the corresponding uniformity (see [8]).

The pair $(\bar{A}, \bar{U})$ together with the mapping $p : \bar{A} \to A$ (which is easily checked to be a uniform embedding) will be called the *completion* of a uniform locale (A, U).

3.2. Let $f : (A, U) \to (B, U')$ be a uniformly continuous frame morphism. Define $g : A \to \bar{B}$ by putting

$$g(a) = \downarrow f(a)$$

(recall that elements of the form $\downarrow x$ are R-coherent). Using 2.3.1, we obtain a frame morphism $f_0 : \downarrow A \to \bar{B}$, satisfying

$$f_0(\downarrow a) = g(a) = \downarrow f(a) .$$

Since the mapping $f_0 \circ k : A \to \bar{B}$ preserves also finite meets, we obtain a unique frame morphism $f_1 : \downarrow A \to \bar{B}$ satisfying

$$f_1(\downarrow a) = f_0 \circ k(a) .$$

3.2.1. Lemma. *We have*

$$a \mathrel{S} b \;\to\; f_1(a) = f_1(b) .$$

Proof. Use 2.2.4. First, note that

$$f_1(\downarrow 0) = \downarrow 0 = f_1(\emptyset)\ .$$

Now, compute

$$f_1(\downarrow a) = f_o k(a) = f_o\Big(\bigcup_{b \triangleleft a} \downarrow b\Big) = f_o\Big(\bigcup_{c \triangleleft a} \bigcup_{b \triangleleft c} \downarrow b\Big) = \bigvee_{c \triangleleft a} f_o\Big(\bigcup_{b \triangleleft c} \downarrow b\Big) =$$
$$= \bigvee_{c \triangleleft a} f_o k(c) = \bigvee_{c \triangleleft a} f_1(c) = f_1\Big(\bigcup_{c \triangleleft a} c\Big) = f_1(k(a))\ .$$

Take a $U \in \mathcal{U}$. Choose a $W \in \mathcal{U}$ to satisfy $W.W \prec U$ (see 1.2) and conclude the proof by another calculation :

$$f_1(\downarrow 1) = \downarrow 1 = j(c(f_{\#}(W))) = j\Big(\bigcup_{y \in f_{\#}(W)} \downarrow y\Big) = \bigvee_{x \in W} \downarrow f(x) =$$

$$= \bigvee_{x \in W} f_o(\downarrow x) \leq \bigvee_{y \in U} \bigvee_{x \triangleleft y} f_o(\downarrow x) = \bigvee_{y \in U} f_1(\downarrow y) = f_1(c(U)). \qquad \lozenge$$

3.2.2. Theorem. *For a uniformly continuous morphism*

$$f : (A, \mathcal{U}) \to (B, \mathcal{U}')$$

there exists a unique frame morphism $\bar{f} : (\bar{A}, \bar{\mathcal{U}}) \to (\bar{B}, \bar{\mathcal{U}}')$ completing the diagram

$$\begin{array}{ccc} \bar{A} & \overset{\bar{f}}{\dashrightarrow} & \bar{B} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array}$$

Moreover, this morphism is uniformly continuous.

Proof. Realizing that the frames A, B, $\bar{A}$, $\bar{B}$ are uniform and hence regular, we deduce the uniqueness part of the theorem from 1.1.1 and from the density of p . Taking into account that

$$jk(a) = j(\downarrow a)$$

we obtain the existence part from 3.2.1 and 2.2.4. It remains to prove the uniform continuity of $\bar{f}$. Let

$$U_1 = \{\downarrow x \mid x \in U\}, \qquad U \in \mathcal{U}.$$

Choose again a $W \in \mathcal{U}$ to satisfy $W.W \prec U$. We compute

$$\bar{f}_{\#}(U_1) = \{\bar{f}(\downarrow x) \mid x \in U\} = \{f_1(\downarrow x) \mid x \in U\} =$$

$$= \{f_o(k(x)) \mid x \in U\} \succ \{f_o(\downarrow y) \mid y \in W\} =$$

$$= \{ \downarrow f(y) \mid y \in W\} = \{ \downarrow z \mid z \in f_{\#}(W) \} \in \overline{U}_o . \qquad \Diamond$$

We will call a uniform locale (A, U) *complete* if the completion $p : \overline{A} \to A$ is an isomorphism in Frm.

3.3. Theorem. *A frame (A, U) is complete iff each uniform embedding $f : (B, U_1) \to (A, U)$ is closed.*

Proof. If we assume that each uniform embedding $f : B \to A$ is closed, so is in particular the dense $p : \overline{A} \to A$. Thus, p is iso.
We prove the converse. Assume that A is complete. It suffices to prove that any dense

$$f : (B, U_1) \to (A, U)$$

is an isomorphism (in other cases we simply consider the restriction of f to the closure of A in B, see e.g. [3, 4, 6]). Thus, let f be dense. Put $j_B = f_* \circ f$.

3.3.1. Observation. We have

$$a \overset{U_1}{\lhd} b \quad \to \quad j_B(a) \overset{U_1}{\lhd} b .$$

Proof. Take a $U \in U_1$ with $Ua \leq b$. Put

$$c = \bigvee \{ x \in U \mid x \wedge a = 0\} .$$

We have

$$b \vee c = 1, \quad c \wedge a = 0,$$

hence

$$f(j_B(a) \wedge c) = f(j_B(a)) \wedge f(c) = f(a) \wedge f(c) = f(a \wedge c) = 0$$

and hence finally $j_B(a) \wedge c = 0$. $\Diamond$

3.3.2. Lemma. *For $a \in A$ we have*

$$\bigvee_{x \in k(a)} f_*(x) = f_*(a) .$$

Proof.

$$f_*(a) \geq \bigvee_{x \in k(a)} f_*(x) = \bigvee_{x \overset{U}{\lhd} a} f_*(x) \geq \qquad (f \text{ is onto and } f(Ub) \geq f_{\#}(U) f(b))$$

$$\geq \bigvee_{y \overset{U_1}{\lhd} f_*(a)} f_* f(y) = \bigvee_{y \overset{U_1}{\lhd} f_*(a)} j_B(y) \geq \bigvee_{y \overset{U_1}{\lhd} f_*(a)} y = f_*(a) . \qquad \Diamond$$

3.3.3. Lemma. *For $U \in U$ we have* $\bigvee_{y \in c(U)} f_*(x) = \downarrow 1.$

Proof. Let $U = f_{\#}(U_1)$. We have $f_{\#}(c(U_1)) = c(U)$ and thence

$$\bigvee_{x\in c(U)} f_*(x) = \bigvee_{x\in f_{\#}(c(U_1))} f_*(x) = \bigvee_{y\in c(U_1)} f_*f(y) = \bigvee_{y\in c(U_1)} j_B(y) \geqq 1 \,.$$

Take a $b \in B$. We have a commutative diagram

$$\begin{array}{ccc} \downarrow b & \overset{\bar{f}}{\longmapsto} & f(\downarrow b) \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle p_A} \\ b & \overset{f}{\longmapsto} & f(b) \end{array}$$

Since p_A is iso, we deduce

$$\downarrow f(b) = \bar{f}(\downarrow b) = f_1(\downarrow b) = f_0 k(b) = j(\bigcup_{x \triangleleft b}^{U_1} \downarrow f(x)). \tag{3.3.1}$$

Put

$$v = \bigcup \{ \downarrow z \mid f_*(z) \leqq b \} \,.$$

Considering 3.3.2, 3.3.3 and the inequality $f_*(0) \leqq b$ (which follows from the density of f), we see that v is S-coherent and hence R-coherent, since f_* preserves finite meets. On the other hand we have

$$\bigcup_{x \triangleleft b}^{U_1} \downarrow f(x) \leqq v$$

by Lemma 3.3.1. Thus, it follows from (3.3.1) that $\downarrow f(b) \leqq v$ which implies $f(b) \in v$ and in consequence

$$j_B(b) = f_*f(b) \leqq b \,.$$

Thus, j_B is the identity and hence f is iso. ◊

3.4. Theorem. *The completion mapping p gives rise to a reflection from the category of uniform locales to its full subcategory of all locales which are absolutely closed under uniform embeddings.*

Proof. It follows directly from 3.2.2, 3.3. ◊

4. A CHARACTERIZATION OF LT-GROUPS.

4.1. Let $(A, \mu, \iota, \varepsilon)$ be a T-group and let

$$'+' = T(\mu) : T(A)\times T(A) \to T(A)$$

be the corresponding (not necessarily commutative) group operation. For any $m \in A$ with $\varepsilon(m) = 1$ put

$$U_R^m = \{m+\xi \mid \xi \in T(A)\}, \qquad U_L^m = \{\xi+m \mid \xi \in T(A)\}.$$

Then we have certain uniform bases

$$\mathcal{V}_R = \{ U_R^m \mid m \in A, \varepsilon(m) = 1\}, \quad \mathcal{V}_L = \{U_L^m \mid m \in A, \varepsilon(m) = 1\}$$

on A. We denote the corresponding uniformities by $\mathcal{U}_R, \mathcal{U}_L$. By standard arguments, we obtain

$$\begin{aligned} \mathcal{U}_R &= \{ U \mid (\exists V \in \mathcal{V}_R)\, V \prec U\}, \\ \mathcal{U}_L &= \{ U \mid (\exists V \in \mathcal{V}_L)\, V \prec U\}. \end{aligned} \tag{4.1.1}$$

4.1.1. Lemma. *For $a, b \in A$ we have*

$$a \overset{\mathcal{U}_R}{\lhd} b \equiv (\exists m)(\varepsilon(m) = 1 \;\&\; m+a \leqq b),$$

$$a \overset{\mathcal{U}_L}{\lhd} b \equiv (\exists m)(\varepsilon(m) = 1 \;\&\; a+m \leqq b).$$

Proof. Straightforward. $\lozenge$

4.2. Denote the completions of the uniform locales $(A, \mathcal{U}_R)$, $(A, \mathcal{U}_L)$ by A_R resp. A_L and put $A_{RL} = A_R \cap A_L$ (recall that both A_R, A_L are subsets of $\downarrow A$). Clearly A_{RL} is a frame and the diagram of canonical embeddings

$$\begin{tikzcd} & A_R \arrow[dr] & \\ \downarrow A \arrow[ur, "j_R"] \arrow[dr, "j_L"'] & & A_{RL} \\ & A_L \arrow[ur] & \end{tikzcd}$$

is a pushout. The composition mapping $\downarrow A \to A_{RL}$ will be denoted by j_{RL}.

4.2.1. Lemma. *The equalities*

$$\begin{aligned} \downarrow a &= \bigvee\{\downarrow b \mid (\exists m)\; m+b \leqq a \;\&\; \varepsilon(m) = 1\} \\ &= \bigvee\{\downarrow b \mid (\exists m)\; b+m \leqq a \;\&\; \varepsilon(m) = 1\}, \end{aligned} \tag{4.2.1}$$

$$\bigvee\{\downarrow b \mid b \in U_R^n\} = \bigvee\{\downarrow b \mid b \in U_L^n\} = 1, \quad \downarrow 0 = j_{RL}(\emptyset)$$

hold in A_{RL} for any $a, n \in A$. On the other hand, any frame morphism $f : A \to B$ satisfying

$$\begin{aligned} f(\downarrow a) &= \bigvee\{f(\downarrow b) \mid (\exists m)(m+b \leqq a \;\&\; \varepsilon(m) = 1)\} = \\ &= \bigvee\{f(\downarrow b) \mid (\exists m)(b+m \leqq a \;\&\; \varepsilon(m) = 1)\} \end{aligned} \tag{4.2.2}$$

$$\bigvee\{f(\downarrow b) \mid b \in U_R^m\} = \bigvee\{f(\downarrow b) \mid b \in U_L^m\} = 1\,, \quad f(\downarrow 0) = f(\emptyset),$$

for all $a, n \in A$, *can be (uniquely) factorized through* j_{RL}.

Proof. It is easily obtained from (4.1.1), 3.1 and 2.2.4. ◊

4.3. Let $p_R\colon A_R \to A$, $p_L\colon A_L \to A$ be the completion mappings and let $p_{RL}\colon A_{RL} \to A$ be their colimit. The mappings

$$(p_{RL} \oplus p_{RL})_* \circ \tau_* \circ \mu : A \to A_{RL} \oplus A_{RL}\,, \quad (p_{RL})_* \circ \iota : A \to A_{RL}$$

(where τ is taken from 1.3.1 and the asterisk indicates adjunction) are easily checked to preserve finite meets. Using 2.3.1, we obtain unique frame morphisms

$$\nu : \downarrow A \to A_{RL} \oplus A_{RL}\,, \quad \varkappa : \downarrow A \to A_{RL}$$

satisfying

$$\nu(\downarrow a) = (p_{RL} \oplus p_{RL})_* \tau_* \mu(a), \qquad \varkappa(\downarrow a) = (p_{RL})_* \iota(a).$$

4.3.1. Theorem. *The morphisms* ν, $\varkappa$ *can be factorized through* j_{RL}.

Proof. By Lemma 4.2.1 it suffices to prove the equalities (4.2.2). The relations

$$\nu(\downarrow 0) = \nu(\emptyset) = \downarrow 0 \oplus \downarrow 0, \qquad \varkappa(\downarrow 0) = \varkappa(\emptyset) = \downarrow 0$$

are trivial. Computing

$$\varkappa(\downarrow a) = \downarrow\iota(a) = \bigvee\{\downarrow b \mid (\exists m)\ b+m \leqq \iota(a) \ \&\ \varepsilon(m) = 1\} =$$

$$= \bigvee\{\downarrow \iota(b) \mid (\exists m)\ \iota(b) + \iota(m) \leqq \iota(a) \ \&\ \varepsilon(m) = 1\} =$$

$$= \bigvee\{\downarrow \iota(b) \mid (\exists m)\ m+b \leqq a \ \&\ \varepsilon(m) = 1\} =$$

$$= \bigvee\{\varkappa(\downarrow b) \mid (\exists m)\ m+b \leqq a \ \&\ \varepsilon(m) = 1\}$$

and

$$\bigvee\{\varkappa(\downarrow b) \mid b \in U_R^n\} = \bigvee\{\downarrow b \mid b \in U_L^{\iota(n)}\} = 1$$

we conclude the proof for the case of $\varkappa$ (the remaining identities are analogous). In the case of ν, we calculate, first,

$$\nu(\downarrow a) = \bigvee\{\downarrow b \oplus \downarrow c \mid b+c \leqq a\} =$$

$$= \bigvee\{\downarrow b \oplus \bigvee\{\downarrow x \mid (\exists m)\ x+m \leqq c \ \&\ \varepsilon(m) = 1\} \mid b+c \leqq a\} =$$

$$= \bigvee\{\downarrow b \oplus \downarrow x \mid (\exists c)(\exists m)(x+m \leqq c \ \&\ b+c \leqq a)\} =$$

$$= \bigvee\{\downarrow b \oplus \downarrow x \mid (\exists m)\ b+x+m \leqq a\} \leqq$$

$$\leq \bigvee_{\substack{t+m \lnsim a\\ \varepsilon(m)=1}} \; \bigvee_{b+x \lnsim t} \downarrow b \oplus \downarrow x = \bigvee_{\substack{t+m \lnsim a\\ \varepsilon(m)=1}} \nu(\downarrow t) \leq \nu(\downarrow a)$$

To obtain the last identity, assume

$$\varepsilon(m) = 1, \qquad \varepsilon(n) = 1, \qquad n+n \leq m$$

and compute

$$\bigvee\{ \nu(\downarrow b) \mid b \in U_R^m \} = \bigvee\{ \downarrow x \oplus \downarrow y \mid (\exists\, b \in U_R^m)\; x+y \leq b\} \geq$$

$$\geq \bigvee\{ \downarrow x \oplus \downarrow y \mid (\exists \xi,\ \xi \in T(A))\, x = n+\xi \ \&\ y = -\xi + n + \xi \} =$$

$$= \bigvee\{ \downarrow x \oplus \downarrow y \mid (\exists \xi \in T(A))\ x = n+\xi \ \&\ y \in U^{-\xi+n+\xi} \} =$$

$$= \bigvee\{ \downarrow x \oplus 1 \mid x \in U_R^n \} = 1 \oplus 1 = 1.$$

The symmetry argument concludes the proof. ◊

4.3.2. Theorem. *We have an L-group* $(A_{RL}, \bar{\mu}, \bar{\iota}, \bar{\varepsilon})$ *making diagrams*

$$\begin{tikzcd}
A_{RL} \arrow[r, "\bar{\mu}"] \arrow[ddd, "p_{RL}"'] & A_{RL} \oplus A_{RL} \arrow[d, "p_{RL} \oplus p_{RL}"] \\
 & A \oplus A \arrow[d, "\tau"] \\
 & \\
A \arrow[r, "\mu"] & A + A
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{RL} \arrow[r, "\bar{\iota}"] \arrow[d, "p_{RL}"'] & A_{RL} \arrow[d, "p_{RL}"] \\
A \arrow[r, "\iota"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "p_{RL}"'] \arrow[dr, "\bar{\varepsilon}"] & \\
A \arrow[r, "\varepsilon"] & 2
\end{tikzcd}
\qquad (4.3.1)$$

commutative. Thus, any T-group can be embedded into an L-group as a dense subgroup of its spatial reflection.

Proof. We define $\bar{\mu}$, $\bar{\iota}$ as factors of ν, $\varkappa$ through j_{RL}. Recall that τ and p_{RL} are dense and hence monomorphisms in regular frames, and for that reason $(A_{RL}, \bar{\mu}, \bar{\iota}, \bar{\varepsilon})$ is an L-group. ◊

4.3.3. Theorem. *A T-group A is an LT-group iff the diagram*

$$\begin{tikzcd}
 & A_R \arrow[dr, "p_R"] & \\
\downarrow A \arrow[ur, "j_R"] \arrow[dr, "j_L"'] & & A \\
 & A_L \arrow[ur, "p_L"'] &
\end{tikzcd}$$

is a pushout.

Proof. The sufficiency follows directly from 4.3.2. Thus, let $A, \mu_1 : A \to A \oplus A$ be an LT-group. Then the following diagram is commutative, since τ is dense. This makes (A, μ_1) a dense subgroup of $(A_{RL}, \bar{\mu})$. We deduce that p_{RL} is iso by Theorem 1.3.2.

$$\begin{tikzcd}
A_{RL} \arrow[r, "\overline{\mu}_1"] \arrow[d, "\rho_{RL}"'] & A_{RL} \oplus A_{RL} \arrow[d, "\rho_{RL} \oplus \rho_{RL}"] \\
A \arrow[r, "\mu"] & A \oplus A
\end{tikzcd}$$
◊

4.3.4. Remark. Note that if the T-group A is complete with respect to U_R or U_L (i.e., we have $A_R = A$ or $A_L = A$) the condition of Theorem 4.3.3 is obviously satisfied and hence A is an LT-group.

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