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Minimal atlases of manifolds


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MINIMAL ATLASES OF MANIFOLDS *
by Alberto CAVICCHIOLI and Luigi GRASSELLI

RÉSUMÉ. On montre que chaque "ball-intersection atlas" minimal d'une n-variété M connexe et linéaire par morceaux a exactement n boules si la frontière de M est non vide. Ceci améliore divers résultats connus relatifs aux recouvrements par boules minimaux des variétés.

1. INTRODUCTION.

Given a connected compact n-manifold M, a natural invariant of M is the minimal number of balls which are needed to cover M.

Following [SN] the Ljusternik-Schnirelmann category (resp. the strong Ljusternik-Schnirelmann category), written cat M (resp. C(M)), is the minimal number of open contractible subsets (resp. of balls) of M which suffice to cover M. Obviously

\[ C(M) \geq \text{cat } M. \]

W. Singhof proved that C(M) = cat M if cat M is not too small compared with the dimension of M.

If M is a closed connected combinatorial n-manifold (n > 0) which is geometrically \([n/r]\) - connected, \(r \geq 2\), then M can be covered by \(r\) combinatorial balls \([\mathbb{Z}_2]\). If M is \(r\)-connected and \(r \leq n - 3\), then \([n/(r+1)]+1\) balls suffice to cover M as was later proved by E.C. Zeeman for PL-manifolds \([\mathbb{Z}_1]\) and by E. Luft in the topological case \([L]\).

Classical results for particular classes of spaces are:

1° A closed piecewise-linear 3-manifold covered by 3 open 3-balls is a 3-sphere-with-handles \([\mathbb{H}M]\).

2° If M is a locally trivial n-dimensional sphere bundle over a sphere, having a cross-section, then M admits coverings by 3 open \(n\)-balls \([M1]\).

Theorems which improve some quoted statements are obtained in [M2, PD, S1, S2] by making use of residual sets, a concept introduced in [DH].

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Relations between the Poincaré conjecture and ball coverings arguments are studied in [OS, Z2].

In order to cover a manifold with balls whose intersections are nice, R. Osborne and J. Stern proved this theorem: If $M$ is a closed $k$-connected topological $n$-manifold and $q = \min\{k, n-3\}$, then $M$ can be covered by $p$ open balls if $p(q+1) > n$. Further, these balls may be chosen so that the intersection of any collection of them is $(q-1)$-connected.

The boundary case is also considered in [OS, KT].

In the present paper, we prove that each minimal "ball-intersection atlas" of a connected piecewise-linear $n$-manifold $M$ has exactly $n$ balls if $\partial M$ is non-void. This improves some results of [OS] and [KT] in the piecewise-linear category.

2. NOTATIONS.

Let $\Delta_n$ be the set $\{0, 1, ..., n\}$ and $N_n = \Delta_n - \{0\}$. The symbol $\#A$ means the cardinality of the set $A$.

All (compact) spaces and maps considered belong to the piecewise-linear (PL) category in the sense of [H] or [Z1]. The prefix PL will always be omitted.

The ball-complexes $B_1, B_2$ are said to be abstractly isomorphic if there exists a bijection $f : B_1 \to B_2$ preserving the face-incidence relation.

An $n$-pseudocomplex $K$ is an $n$-dimensional principal ball-complex in which every $r$-ball, considered with all their faces, is abstractly isomorphic with the complex underlying an $r$-simplex ([HW], p. 49). $K$ is said to be a pseudodissection of the polyhedron $|K|$. By $S_r(K)$ and $K^r$, we respectively denote the set of all the $r$-balls of $K$ and the $s$-skeleton of $K$. We shall also call $r$-simplex (resp. vertex) each $r$-ball (resp. 0-ball) of $K$.

Given a simplex $s$ in an $n$-pseudocomplex $K$, the disjoined star $\text{std}(s, K)$ is defined to be the disjoint union of the $n$-simplexes of $K$ containing $s$, with re-identification of the $(n-1)$-faces containing $s$ and of their faces. The subcomplex

$$\text{lk}(s, K) = \{ \tau \in \text{std}(s, K) \mid \tau \cap s = \emptyset \}$$

is called the disjoined link of $s$ in $K$. If $K$ is a pseudodissection of a manifold, the star $\text{st}(s, K)$ and the link $\text{lk}(s, K)$ of a simplex $s$ in $K$ are not necessarily balls or spheres; however, $\text{std}(s, K)$ and $\text{lk}(s, K)$ are the balls or spheres obtained by a minimal set of severings on $\text{st}(s, K)$ and $\text{lk}(s, K)$ respectively. A vertex $v$ of an $n$-pseudocomplex $K$. 

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will be called a cone-vertex if it belongs to all n-simplexes of K (or, equivalently, if \( \text{st}(v, K) = K \)).

An r-simplex s of a closed n-pseudomanifold K (cf. [SP]) is said to be regular (resp. singular) if \( \text{lk}(s, K) \) is (resp. is not) a combinatorial \((n-r-1)\)-sphere.

An identification system of a principal n-pseudocomplex K is defined to be a set \( G \) of simplicial isomorphisms such that, for any pair
\[
S_{n-1}^\alpha, S_{n-1}^\beta \in S_{n-1}(K),
\]
there exists at most one map
\[
\varphi_{\alpha \beta} : S_{n-1}^\alpha \to S_{n-1}^\beta
\]
belonging to \( G \). Let \( \sim_G \) be the equivalence relation on
\[
S(K) = \bigcup_{r \in \Delta_n} S_r(K)
\]
defined as follows:
\[
s_h^G \sim_G s_k^G \text{ iff } s_h^G = s_k^G \text{ or there exists a sequence of isomorphisms in } G \text{ (or their inverses) taking one to the other.}
\]
The symbol \( \hat{K}_G \) will denote the quotient complex \( S(K)/\sim_G \).

3. MINIMAL BALL COVERINGS.

Let \( M \) be a closed connected n-manifold and \( B = \{ B_i \mid i \in \mathbb{I} \} \) be a finite set of closed \( r \)-balls such that \( M = \bigcup_{i \in \mathbb{I}} B_i \).

Definition 1. \( B \) is said to be a \( P_0 \)-ball covering if it satisfies the following property:

\((P_0)\) For every \( i, j \in \mathbb{I} \) (\( i \neq j \)),
\[
B_i \cap B_j = \partial B_i \cap \partial B_j
\]
has \((n-1)\)-manifolds as connected components.

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\]
has \((n-1)\)-balls as connected components.

\( B \) is said to be a \( P_2 \)-ball covering if it satisfies the following property

...
(Pₙ) For every J ⊆ I, ∑ J = k, for k ≤ n+1,
\[ \bigcap_{j \in J} B_j = \bigcap_{j \in J} (\overline{B_j}) \]
has \((n-k+1)\)-balls as connected components.

Obviously \(P_2 \Rightarrow P_1 \Rightarrow \ldots \Rightarrow P_\emptyset\).

**Definition 2.** Let \(M\) be an \(n\)-manifold with \(h > 0\) boundary components \(M_j\) \((j \in \mathbb{N}_h)\) and \(B = \{ B_i \mid i \in I \}\) be a finite set of closed \(n\)-balls such that \(M = \bigcup_{i \in I} B_i\). B is said to be a \(P_\alpha\)-ball covering \((\alpha \in \Delta_2)\) of \(M\) if \(B\) satisfies the property \(P_\alpha\) and
\[ B_j = \{ B_i \cap M_j \mid i \in I \} \]
is a \(P_\alpha\)-ball covering of the closed \((n-1)\)-manifold \(M_j\), for every \(j \in \mathbb{N}_h\).

Note that a \(P_\emptyset\)-ball (resp. \(P_1\)-ball) covering is a ball covering (resp. strong ball covering) in the sense of [IY, KT] (resp. [FG2]).

Let \(M\) be a connected \(n\)-manifold. For \(\alpha \in \Delta_2\), define :
\[ b_\alpha(M) = \min \{ \#B \mid B \text{ is a } P_\alpha\text{-ball covering of } M \} \]
Obviously,
\[ b_\emptyset(M) \leq b_1(M) \leq b_2(M) \]
The following results are known.

**Proposition 1.**
1° If \(M\) is a closed \(n\)-manifold, \(b_2(M) = n + 1\) [P1, FG1].
2° If \(M\) has non-empty connected boundary, \(b_3(M) \leq n\) [FG2].
3° If \(M\) has non-empty boundary, \(b_3(M) \leq n\) [KT].

The statements 2 and 3 of the above proposition can be obtained as easy consequences of the following :

**Proposition 2.** If \(M\) is a connected \(n\)-manifold with non-empty boundary, then \(b_2(M) = n\).

**Proof.** We first prove that \(b_2(M) \leq n\) by exhibiting a \(P_2\)-ball covering \(B^*\) of \(M\) with \(n\) balls. Let \(M_i\) \((i \in \mathbb{N}_n)\) be the boundary components of \(M\), \(M_i'\) a copy of \(M_i\) and \(\varphi_i : M_i \rightarrow M_i'\) the identification map. Let \(w_i\) \((i \in \mathbb{N})\) be a point such that the adjunction space
\[ Q = M_1 \cup_{\varphi_1} (w_1 \ast M_1') \cup_{\varphi_2} \ldots \cup_{\varphi_n} (w_n \ast M_n') \]
is a closed \(n\)-pseudomanifold.
Moreover, if $K$ is a simplicial triangulation of $Q$, the set of the singular simplexes of $K$ is $\{w_i \mid i \in \mathbb{N}_n\}$ and the disjoined star of each simplex of $K$ is strongly-connected.

We give an inductive algorithm for constructing a pseudodissection $K_p$ ($0 \leq p \leq n$) of $Q$ such that $S_0(K_p)$ has $p$ regular cone-vertices. Let $K_0 = K$.

Let now $A_j$ $(j \in \mathbb{N}_p)$ be a regular cone-vertex of $K_p$. There exist a finite sequence $\xi_1 = \{\sigma_{\alpha}^{n-p}\}_{\alpha=0}^{s}$ of all the $(n-p)$-simplexes of $K_p$ not containing $A_1, ..., A_p$ and a finite sequence $\xi_2 = \{w_i\}_{i=0}^{u}$ of $(n-1)$-simplexes of $K_p$ such that, for every $\beta \in \mathbb{N}_s$,

$$\sigma_{\beta}^{n-1} \in st(\sigma_{\gamma}^{n-p}, K_p) \cap st(\sigma_{\gamma}^{n-p}, K_p)$$

for some $\gamma < \beta$. For each $\sigma_{\alpha}^{n-p} \in \xi_1$, consider the disjoined star $std(\sigma_{\alpha}^{n-p}, K_p)$ and glue them pairwise together by identifying the two copies of every $(n-1)$-simplex of $\xi_1$. The pseudocomplex $B$ so obtained is a pseudodissection of an $n$-ball. Moreover, there exists an identification system $G$ on $B$ such that the quotient $B/G$ is isomorphic with $K_p$. Define $A_{p+1}$ as an interior point of $B$ and set $\Sigma = A_{p+1} \ast \delta B$. If $G'$ is the identification system induced by $G$ on $\Sigma$, set $K_{p+1} = \Sigma G'$. There exist a finite sequence $\xi_2 = \{\nu_\delta\}_{\delta=0}^{u}$ of all the vertices of $K_p$ different from the regular cone-vertex $A_j$ $(j \in \mathbb{N}_n)$ and a finite sequence $\xi_3 = \{w_i\}_{i=0}^{v}$ of $(n-1)$-simplexes of $K_p$ such that, for every $\delta \in \mathbb{N}_v$,

$$\nu_\delta \in st(\nu_\delta, K_p) \cap st(\nu_\delta, K_p)$$

for some $\mu < \delta$. Note that

$$\{ w_i \}_{i=1}^{v} \subset \{ \nu_\delta \}_{\delta=0}^{u}.$$

By the strong connectedness of $std(w_i, K_p)$, it is possible to obtain a triangulated $n$-ball $B_i$ $(i \in \mathbb{N}_h)$ such that:

1° all the vertices of $B_i$ belong to $\partial B_i$,
2° $w_i$ is a cone-vertex of $B_i$,
3° there exists an identification system $G_i$ on $B_i$ such that $B_i G_i$ is isomorphic with $std(w_i, K_p)$.

Let $\xi_3$ be the finite sequence obtained from $\xi_2$ by considering the disjoined stars of all the regular vertices of $\xi_2$ and all the $n$-balls $B_i$'s. By identifying the elements of $\xi_3$ along suitable $(n-1)$-simplexes of $\xi_2$, we can obtain exactly $h$ triangulated $n$-balls $D_1, ..., D_h$ such that

$$\{w_k \mid k \in \mathbb{N}_h\} \cap D_i = \{w_i\}.$$
There exist an identification system $G^*$ induced by $\xi$ and a triangulated $n$-ball $E$ obtained from $C_1, \ldots, C_h$ such that $|E_{G^*}| = \mathbb{Q}$, $A_1, \ldots, A_n$ are cone-vertices of $E_{G^*}$ and

$$S_0(E_{G^*}) = \left\{ A_j \mid j \in \mathbb{N}_h \right\} \cup \left\{ w_j \mid j \in \mathbb{N}_n \right\}.$$  
Set $T = E_{G^*}$. If $T'$ is the first barycentric subdivision of $T$, define

$$B = \left\{ B_i \mid i \in \mathbb{N}_{n+h} \right\},$$

where

$$B_j = \text{st}(A_j, T') \quad \text{if} \quad 1 \leq j \leq n,$$
$$B_j = \text{st}(w_j, T') \quad \text{if} \quad n+1 \leq j \leq n+h.$$

Note that, by construction,

$$B_i \cap B_j = \emptyset \quad \text{if} \quad i \neq j, \quad \text{and} \quad i, j \in \mathbb{N}_{n+h} - \mathbb{N}_h.$$

$B^* = \left\{ B_i \mid i \in \mathbb{N}_n \right\}$ is a $P_2$-ball covering of $M$.

Now we show that no such covering of smaller cardinality exists. Let $B = \left\{ B_i \mid i \in \mathbb{N}_k \right\}$ be a $P_2$-ball covering of $M$. For each $i \in \mathbb{N}_k$, $H_j(B_i \cap M)$ is the $j$-th homology group. The Mayer-Vietoris sequence gives:

$$\ldots \to H_j(B_i \cap M) \oplus H_j(B_i \cap M_s) \to \cdots \to H_j(B_i \oplus B_j \cap M) \to H_{j-1}(B_i \cap M_s) \to \cdots$$

Then

$$H_j((B_i \cup B_j) \cap M) = 0 \quad \text{if} \quad j \geq 2,$$

while, for $j = 1$, it is a free abelian group (possibly zero). By induction on $m \geq k$, the Mayer-Vietoris sequence gives:

$$0 = H_j((\bigcup_{i=1}^{m-1} B_i) \cap M) \oplus H_j(B_m \cap M) \to H_j((\bigcup_{i=1}^{m-1} B_i) \cap M_s) \to$$
$$\to H_{j-1}((\bigcup_{i=1}^{m-1} B_i) \cap M_s) = 0.$$

Then

$$H_j((\bigcup_{i=2}^m B_i) \cap M) = 0 \quad \text{if} \quad j \geq m,$$

while, for $j = m-1$, it is a free abelian group. If $k < n$, setting $m = k$, we have that

$$H_j(M \cap M_s) = H_j(M_s)$$

vanishes for $j \geq k$ and is a free abelian group for $j = k - 1$. In particular $H_{n-1}(M_s) = 0$ and $H_{n-2}(M_s)$ is a free abelian group.
This is a contradiction because either $H_{n-1}(M_\delta) = \mathbb{Z}$ or $H_{n-1}(M_\delta) = 0$ and $H_{n-2}(M_\delta)$ has torsion, $M_\delta$ being a closed $(n-1)$-manifold.

Remark. For the proof of $b_2(M) \geq n$ it is sufficient that each $B_j$ is a $P_2$-ball covering of $M_j$ without assuming the property $P_2$ for $B$ in the interior of $M$.

Note that Proposition 2 improves the statement of the Theorem 4.1 in [OS] in the case $q = 0$.

4. MINIMAL ATLASES.

A BI-atlas (ballintersection atlas) of a closed connected $n$-manifold $M$ in the sense of [P2] is a finite covering

$$A = \{ V_\alpha \mid \alpha \in A \}$$

de M such that :

a) each $V_\alpha$ is an open $n$-ball,

b) the intersection of any number of $V_\alpha$'s has open balls as connected components.

In order to define a concept of BI-atlas for manifolds with boundary, we need the following

Definition 3. Let $M$ be a connected $n$-manifold. An open subset $P$ of $M$ is said to be an open $n$-quasi-ball if $P$ is homeomorphic with the union of an open $n$-ball $B$ with a finite number (possibly null) of open disjoint $(n-1)$-balls on $\partial B$.

Definition 4. A finite covering $A = \{ V_\alpha \mid \alpha \in A \}$ of a connected $n$-manifold $M$ with $h$ ($h > 0$) boundary components $M_i$ ($i \in N_h$) is said to be a BI-atlas if the following conditions hold :

a') each $V_\alpha$ is an open $n$-quasi-ball,

b') the intersection of any number of $V_\alpha$'s has open quasi-balls as connected components,

c') $A_i = \{ V_\alpha \cap M_i \mid \alpha \in A \}$

is a BI-atlas of the closed $(n-1)$-manifold $M_i$ ($i \in N_h$).

Let us define

$$a(M) = \min \{ \# A \mid A \text{ is a BI-atlas of } M_j \}.$$ 

A BI-atlas $A$ of $M$ such that $\# A = a(M)$ is said to be a minimal atlas of $M$. 

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In ([P2], Proposition 5.1), M. Pezzana proved that $a(M) = n + 1$ for every closed connected $n$-manifold $M$.

**Proposition 3.** If $M$ is a connected $n$-manifold with $h$ ($h > 0$) boundary components $M_i$ ($i \in \mathbb{N}_h$), $a(M) = n$.

**Proof.** Let $Q$ be the closed $n$-pseudomanifold constructed as in Proposition 2 starting from $M$. If $T = E_Q$ is the pseudodissection of $Q$ obtained in Proposition 2, the interior of the space $|\text{std}(A_i, T)|$, underlying the disjoined star of each cone-vertex $A_i \in S_T(T)$ ($i \in \mathbb{N}_h$), is an open $n$-ball of $Q$. If $T'$ is the first barycentric subdivision of $T$, set

$$B_i = |\text{st}(A_i, T')|.$$  

The polyhedron $M' = \bigcup_{i=1}^{n} B$ is homeomorphic with $M$.

Since $M' \subset Q$, the collection

$$A = \{|\text{std}(A_i, T)| \cap M' | \quad i \in \mathbb{N}_h\}$$

is a BI-atlas of $M'$ such that $|A| = n$. In fact, each connected component of $|\text{std}(A_i, T)| \cap \partial M'$ is an open collar of the $(n - 1)$-ball

$$|\text{std}(b_{1I}, T')| \cap \partial M',$$

$b_{1I}$ being the barycenter of the edge $<A_i, w_r>$ for some singular vertex $w_r \in S_T(T)$. This proves that $a(M) \leq n$.

Conditions $b'$ and $c'$ of Definition 4 give $a(M) \geq n$, according to a Mayer-Vietoris argument as in Proposition 2. \(\diamond\)
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