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A. D. ELMENDORF

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ACTIONS, FUNCTORS AND THE BAR CONSTRUCTION

by A.D. ELMENDORF

RÉSUMÉ. Sous de légères hypothèses, les actions d'une catégorie interne G sur un objet E sont équivalentes aux foncteurs de G vers une catégorie interne C (E) ne dépendant que de E. Par exemple, G peut être une catégorie topologique ou elle peut être interne à une quelconque catégorie localement cartésienne fermée telle qu'un topos. La "bar construction" peut alors être définie en termes de foncteurs au lieu d'actions.

The categorical bar construction was introduced by May in [8], § 12, in the context of topology, and generalized to internal categories by Meyer in [9, 10, 11 and 12]. Both make essential use of actions of categories, defined below, which they describe in the language of Ω-graphs. In [9], Meyer attributes to this author [4] the observation that actions of a topological category G are equivalent to continuous functors from G to the category of topological spaces. However, this is only true when G has a discrete object space. In this paper we establish a correspondence between actions and functors which is sufficiently general to apply to all topological categories (strictly speaking, categories internal to the category of weak Hausdorff k-spaces) as well as the more general setting of [12]. The crucial ingredient is the existence of an internal hom-object in the comma category K/B (called ($K\downarrow$ B) in [7], II.6) where K is the ambient category and B is the object element for the acting category. This condition is satisfied when K is the category of k-spaces and B is weak Hausdorff [1, 2, 6], as well as in any locally cartesian closed category K such as a topos [5], Theorem 1.42.

I would like to extend my thanks to L. Gaunce Lewis for many stimulating discussions and for sending me a copy of [6], to S.B. Niefield for a most enlightening conversation, and to K.R. Edwards for helpful criticism.

We begin by establishing a framework in which we can define the notion of an **action** of an internal category (e.g. a topological category). Although not the most general possible, the following hypotheses are convenient and satisfied in all the examples of interest.

Conventions 1. Throughout, K will be a fixed category which has products and pullbacks, and $\mathcal{B}$ will be a class of objects in $\mathcal{K}$ for which :

a) if $A \in \mathcal{B}$ and $B \in \mathcal{B}$, then $A \times B \in \mathcal{B}$, and
b) $\mathcal{K}/B$ is cartesian closed [7], p. 95, for all $B \in \mathcal{B}$.

For example, by [2] or [6], we could take $\mathcal{K}$ = category of k-spaces, $\mathcal{B}$ = weak Hausdorff k-spaces. Or we could take $\mathcal{K}$ to be any locally cartesian closed category and $\mathcal{B}$ all objects in $\mathcal{K}$. When we write $\mathcal{K}/B$, we will always assume that $\mathcal{K}$ and $B \in \mathcal{B}$ satisfy these conventions. We will write the internal hom-functor of $\mathcal{K}/B$ as

$$F_B : (\mathcal{K}/B)^{op} \times (\mathcal{K}/B) \to \mathcal{K}/B.$$

Notations 2. Suppose $A \in \mathcal{B}$, $B \in \mathcal{B}$, and $g : A \to B$. There are functors

$$g_* : \mathcal{K}/A \to \mathcal{K}/B \qquad \text{and} \qquad g^* : \mathcal{K}/B \to \mathcal{K}/A$$

defined by

$$g_*(E, f) = (E, gf) \quad \text{and} \quad g^*(E, f) = (g^*E, g^*f),$$

with the last defined by the pullback diagram

$$\begin{array}{ccc} g^*E & \longrightarrow & E \\ {\scriptstyle g^*f}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[g]{} & B \end{array}$$

It is an easy exercise that g_* is left adjoint to g^*.

Notations 3. Let M_1 and M_2 be objects of $\mathcal{K}/B \times B$ with structure maps $(T_i, S_i) : M_i \to B \times B$. We define an object $M_1 \square M_2$ in $\mathcal{K}/B \times B$ by the pullback diagram

$$\begin{array}{ccc} M_1 \square M_2 & \xrightarrow{\hat{S}_1} & M_2 \\ {\scriptstyle \hat{T}_2}\downarrow & & \downarrow{\scriptstyle T_2} \\ M_1 & \xrightarrow[S_1]{} & B \end{array}$$

with structure map $(T_1 \circ \hat{T}_2, S_2 \circ \hat{S}_1)$. It is easy to see that this is functorial in both M_1 and M_2. If M is an object of $\mathcal{K}/B \times B$ and (E, f) an object of $\mathcal{K}/B$, we will consider the object $M \square E$ of $\mathcal{K}/B$ defined as a pullback as above and with structure map $T \circ \hat{f}$. Again, this is functorial in both variables.

The motivation here is to think of M_1 and M_2 as consisting of "arrows" between "elements" of B, with source given by S_i and target by T_i . Then $M_1 \square M_2$ consists of composable pairs of arrows. The intuition for $M \square E$ is to think of E as a family of objects (the "fibers") parametrized by the "elements" of B ; $M \square E$ then consists of ordered pairs of an arrow and an element of the source of the arrow.

The following basic observation is due to May in the topological setting.

Lemma 4. *The product* $M_1 \square M_2$ *turns* $K/B\times B$ *into a monoidal category ([7], Chap. VII ; the unit is* $\Delta : B \to B\times B$ *). A monoid in this category is precisely an internal category in* K *with object element* B *([5], Chapter 2).* ◊

In particular, this "box product" is associative (up to natural isomorphism) and it is easy to see that there is a natural isomorphism

$$(M_1 \square M_2)\square E \simeq M_1 \square (M_2 \square E).$$

Definition 5. Let M be a monoid in $K/B\times B$, so M is the morphism element of an internal category G with object element B. Let $I : B \to M$ and $\mu : M\square M \to M$ be the monoid structure maps, and let E be an object of K/B. An **action** of G on E consists of a map $\xi : M\square E \to E$ for which the following two diagrams commute :

a) **Unit :**

$$\begin{tikzcd} E \simeq B\square E \arrow[r, "I\square 1"] \arrow[dr, "="'] & M\square E \arrow[d, "\xi"] \\ & E \end{tikzcd}$$

b) **Associativity :**

$$\begin{tikzcd} M\square M\square E \arrow[r, "1\square\xi"] \arrow[d, "\mu\square 1"'] & M\square E \arrow[d, "\xi"] \\ M\square E \arrow[r, "\xi"] & E \end{tikzcd}$$

Our main theorem is the following :

Theorem 6. *For every object* E *of* K/B *there is an internal category* $C(E)$ *with object element* B *such that actions of* G *on* E *correspond naturally and bijectively to internal functors from* G *to* $C(E)$ *which are the identity on* B*.* (Such an internal functor is simply a morphism of monoids in $K/B\times B$.)

In the topological case, $C(E)$ is the category whose objects are the fibers in E over the points of B, with morphisms all continuous maps between fibers, topologized precisely so that Theorem 6 holds.

The actions considered here are left actions, but Theorem 6 addresses right actions as well. This involves the use of $E\square M$, which is defined just as one would expect from Notations 3. It is easy to see that right actions by G are the same thing as left actions by G^{op}, the internal category obtained from G by reversing source and target maps.

Corollary 7. *Right actions of G on* E *correspond naturally and bijectively to internal functors from* G^{op} *to* C(E) *which are the identity on* B. ◊

We may also consider bi-actions. This involves internal categories G and H with object elements B and A, morphism elements M and N respectively. A G-H **bi-action** on an object E of $K/B\times A$ consists of maps $\xi : M\square E \to E$ and $\eta : E\square N \to E$ such that ξ is a left action, η is a right action, and

$$\begin{tikzcd}
M\square E\square N \arrow[r, "\xi\square 1"] \arrow[d, "1\square\eta"'] & E\square N \arrow[d, "\eta"] \\
M\square E \arrow[r, "\xi"'] & E
\end{tikzcd}$$

commutes. It is a simple exercise to see that G-H bi-actions are the same thing as $(G\times H^{op})$-actions. (The structure map is the diagonal in the displayed diagram.)

Corollary 8. *There is a natural bijection between G-H bi-actions on* E *(where* E *is now an object of* $K/B\times A$*) and internal functors from* $G\times H^{op}$ *to* C(E) *which are the identity on* B×A. ◊

The special case $G = H$ is of particular interest in the bar construction, as we will see below.

Our first step in the proof of Theorem 6 must be to produce the category $C(E)$, which in the case of a topos is due to Bénabou ; see [5], Ex. 2.38, and for locally internal categories [5], p. 340. Since we are also considering weak Hausdorff k-spaces, which do *not* form a topos, our situation is more general than these, although the constructions are much the same.

Lemma 9. *Let* π_1 *and* π_2 *be the projections from* B×B *to* B. *Then*

$$M\square E \simeq (\pi_1)_*(M \times_{B\times B} \pi_2{}^*E).$$

Proof. This follows from the diagram

$$\begin{tikzcd}
M\square E \arrow[r] \arrow[d] & \pi_2{}^*E \arrow[r] \arrow[d] & E \arrow[d] \\
M \arrow[r, "{(T,\, S)}"] \arrow[dr, "T"'] & B\times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \\
& B &
\end{tikzcd}$$

in which the squares are pullbacks. ◊

Corollary 10. *There is a natural isomorphism*

$$K/B(M\square E, E') \simeq K/B\times B\,(M, F_{B\times B}(\pi_2{*}E, \pi_1{*}E')).$$

Proof. We have

$$K/B(M\square E, E') \;=\; K/B\,((\pi_1)_*(M \times_{B\times B} \pi_2{*}E), E')$$

$$\simeq\; K/B\times B(M \times_{B\times B} \pi_2{*}E, \pi_1{*}E') \simeq K/B\times B(M, F_{B\times B}(\pi_2{*}E, \pi_1{*}E')). \qquad \Diamond$$

In the topological case, the space $F_{B\times B}(\pi_2{*}E, \pi_1{*}E')$ is precisely the space E.E' of Booth, Heath and Piccinini [3].

Notations 11. We will write M(E) for the object $F_{B\times B}(\pi_2{*}E, \pi_1{*}E)$ of $K/B\times B$.

Theorem 12. *There is an internal category* C(E) *with object element* B *and morphism element* M(E).

Proof. The unit map $I : B \to M(E)$ corresponds to id_E under

$$K/B(E, E) \simeq K/B(B\square E, E) \simeq K/B\times B\,(B, M(E)).$$

We define the **evaluation map** $\epsilon : M(E)\square E \to E$ to be the map corresponding to $id_{M(E)}$ under

$$K/B\times B\,(M(E), M(E)) \simeq K/B(M(E)\square E, E)$$

and the product $\mu : M(E) \square M(E) \to M(E)$ to be the map corresponding to $\epsilon \circ (1\square\epsilon)$ under

$$K/B(M(E)\square M(E)\square E, E) \simeq K/B\times B\,(M(E)\square M(E), M(E)).$$

We must now verify the commutativity of the **unit diagram**

$$\begin{tikzcd} M(E) \simeq B\square M(E) \arrow[r, "I\square 1"] \arrow[dr, "="'] & M(E)\square M(E) \arrow[d, "\mu"] & M(E)\square B \simeq M(E) \arrow[l, "1\square I"'] \arrow[dl, "="] \\ & M(E) & \end{tikzcd}$$

and the **associativity diagram**

$$\begin{tikzcd} M(E)\square M(E)\square M(E) \arrow[r, "1\square\mu"] \arrow[d, "\mu\square 1"'] & M(E)\square M(E) \arrow[d, "\mu"] \\ M(E)\square M(E) \arrow[r, "\mu"] & M(E) \end{tikzcd}$$

We first observe that ϵ is the counit at E of the adjunction

$$K/B(-\square E, -) \simeq K/B\times B(-, F_{B\times B}(\pi_2{}^*E, \pi_1{}^*-)),$$

so by standard properties of counits ([7], p. 80, Thm. 1), if

$$\alpha \in K/B(M\square E, E)$$

corresponds to $\hat{\alpha} \in K/B\times B\,(M, M(E))$, then $\alpha = \epsilon \circ (\hat{\alpha}\square 1)$. In particular, $\epsilon \circ (I\square 1) = id_E$. The left hand triangle in the unit diagram now commutes, being adjoint to the outer pentagon in

$$\begin{tikzcd}
B\square M(E)\square E \arrow[rr, "I\square 1\square 1"] \arrow[d, "1\square\epsilon"'] & & M(E)\square M(E)\square E \arrow[d, "1\square\epsilon"] \\
B\square E \arrow[rr, "I\square 1"] \arrow[drr, "\cong"'] & & M(E)\square E \arrow[d, "\epsilon"] \\
& & E
\end{tikzcd}$$

The right hand triangle is adjoint to

$$\begin{tikzcd}
M(E)\square E \arrow[rr, "1\square I\square 1"] \arrow[dr, "="'] & & M(E)\square M(E)\square E \arrow[dl, "1\square\epsilon"] \\
& M(E)\square E \arrow[d, "\epsilon"] & \\
& E &
\end{tikzcd}$$

and therefore commutes.

Next, we use another instance of $\alpha = \epsilon \circ (\hat{\alpha}\square 1)$ with $\alpha = \epsilon \circ (1\square\epsilon)$ to conclude that

$$\epsilon \circ (1\square\epsilon) = \epsilon \circ (\mu\square 1).$$

This implies the commutativity of the following hexagon, which is adjoint to the associativity diagram :

$$\begin{tikzcd}
& & M(E)\square M(E)\square M(E)\square E \arrow[dll, "\mu\square 1\square 1"'] \arrow[dd, "1\square 1\square\epsilon"] \arrow[drr, "1\square\mu\square 1"] & & \\
M(E)\square M(E)\square E \arrow[dd, "1\square\epsilon"'] & & & & M(E)\square M(E)\square E \arrow[dd, "1\square\epsilon"] \\
& & M(E)\square M(E)\square E \arrow[dll, "\mu\square 1"'] \arrow[drr, "1\square\epsilon"] & & \\
M(E)\square E \arrow[drr, "\epsilon"'] & & & & M(E)\square E \arrow[dll, "\epsilon"] \\
& & E & &
\end{tikzcd}$$

Proof of Theorem 6. Let M be the morphism element of G, so M is a monoid in K/B×B. A functor from G to $\mathcal{C}$(E) which is the identity on B is precisely a map of monoids from M to M(E), i.e., an element α of K/B×B(M, M(E)) for which

c)
$$\begin{tikzcd} M \arrow[rr, "\alpha"] & & M(E) \\ & B \arrow[ul, "I"] \arrow[ur, "I"'] & \end{tikzcd}$$

and

d)
$$\begin{tikzcd} M\square M \arrow[r, "\alpha\square\alpha"] \arrow[d, "\mu"'] & M(E)\square M(E) \arrow[d, "\mu"] \\ M \arrow[r, "\alpha"'] & M(E) \end{tikzcd}$$

commute. We associate to α the corresponding element ξ of K/B(M□E, E), and must show that functors correspond to actions. The unit diagram (a) of definition 5 is adjoint to (c) above, so these two conditions are equivalent.

We next observe that

$$\begin{tikzcd} M\square E \arrow[r, "\xi"] \arrow[d, "\alpha\square 1"'] & E \\ M(E)\square E \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

commutes, being adjoint to the trivial diagram

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & M(E) \\ M(E) \arrow[ur, "="'] & \end{tikzcd}$$

Consequently,

$$\begin{tikzcd} M\square M\square E \arrow[r, "1\square\xi"] \arrow[d, "\alpha\square\alpha\square 1"'] & M\square E \arrow[r, "\xi"] \arrow[d, "\alpha\square 1"] & E \\ M(E)\square M(E)\square E \arrow[r, "1\square\epsilon"'] & M(E)\square E \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

commutes, since the square is simply the triangle with $\alpha\square$- applied to it. Therefore

$$\epsilon\circ(1\square\epsilon)\circ(\alpha\square\alpha\square 1) = \xi\circ(1\square\xi).$$

The equivalence of (b) and (d) now follows from the following expanded version of (b) :

$$\begin{tikzcd} M\square M\square E \arrow[rr, "1\square\xi"] \arrow[dr, "\alpha\square\alpha\square 1"] \arrow[dd, "\mu\square 1"'] & & M\square E \arrow[dd, "\xi"] \\ & M(E)\square M(E)\square E \arrow[dr, dashed, "\epsilon\circ(1\square\epsilon)"] & \\ M\square E \arrow[rr, "\xi"] & & E \end{tikzcd}$$

The lower triangle is adjoint to (d), and the upper is the formula just derived. ◊

The following theorem shows that there is no real loss of generality in restricting ourselves to functors which are the identity on objects.

Theorem 13. *Let G be an internal category with object element* $B \in \mathcal{B}$*, let* $q : B \to A$ *with* $A \in \mathcal{B}$*, and let* $E \in K/A$*. Then internal functors from G to C(E) which are q on objects correspond naturally and bijectively to internal functors from G to* $C(q^*E)$ *which are* id_B *on objects.*

Proof. It is easy to see that if M is a monoid in $K/A\times A$ (such as M(E)) then $(q\times q)^*(M)$ is a monoid in $K/B\times B$. Next, the diagram of pullbacks

$$\begin{tikzcd} M\square q^*E \arrow[r] \arrow[d] & q^*E \arrow[r] \arrow[d] & E \arrow[d] \\ M \arrow[r, "S"] & B \arrow[r, "q"] & A \end{tikzcd}$$

shows that

$$q_*(M\square q^*E) = (q\times q)_*M\square E$$

for any M. Consequently

$$K/B\times B(M, M(q^*E)) \simeq K/B(M\square q^*E, q^*E) \simeq K/A(q_*(M\square q^*E), E)$$
$$\simeq K/A((q\times q)_*M\square E, E) \simeq K/A\times A((q\times q)_*M, M(E)) \simeq K/B\times B(M, (q\times q)^*M(E))$$

so by the Yoneda lemma,

$$M(q^*E) \simeq (q\times q)^*M(E).$$

Letting M be the morphism object of G, functors from G to C(E) which are q on objects *are* maps of monoids from M to $(q\times q)^*M(E)$, and the theorem follows. ◊

We conclude with some observations about the bar construction. Given an object E of $K/B\times B$ and an internal category G with object element B and morphism element M, Corollary 8 tells us that G-G bi-actions on E are equivalent to internal functors $Z : G\times G^{op} \to C(E)$. Given such data, we may follow Meyer [12] (see also [9, 10, 11]) and define the **bar construction** $B_*(G, Z)$ as a simplicial object of K. The n-th object $B_n(G, Z)$ is the pullback

$$\begin{tikzcd} B_n(G, Z) \arrow[r] \arrow[d] & E \arrow[d, "{(S, T)}"] \\ \underset{(n \text{ terms})}{M\square \dots \square M} \arrow[r] & B\times B \end{tikzcd}$$

i.e.,
$$B_n(G, Z) = M^n \times_{BxB} \tau^*E,$$

where $\tau : BxB \to BxB$ is the interchange map. If we consider $\Delta : B \to BxB$, it is easy to see that

$$\Delta^*(M^n \square E) \simeq B_n(G, Z) \simeq \Delta^*(E\square M^n)$$

as objects of K (but not of K/B). The boundary maps are given by :

$$\partial_o = \Delta^*(\xi_R \square 1), \qquad \partial_n = \Delta^*(1\square \xi_L),$$

$$\partial_j = \Delta^*(1_E \square 1_M^{j-1} \square \mu \square 1_M^{n-j-1}), \qquad 0 < j < n,$$

where
$$\xi_R : E\square M \to M \quad \text{and} \quad \xi_L : M\square E \to M$$

are the action structure maps. Degeneracies are all of the form $\Delta^*(1\square I\square 1)$.

When $E = CxD$ for C and D objects of K/B with a right G-action on C and a left G-action on D, this reduces to

$$B_n(G, Z) \equiv B_n(C, G, D) = C\square M^n \square D,$$

with structure maps essentially as given by May in [8], §7. In this *split* case, by [12], Cor. 4.6, $B_*(G, Z)$ is the nerve of an internal category $[G, Z]$, so we may iterate by considering actions of $[G, Z]$ and so on. The condition Z must satisfy in the split case involves the following lemma.

Lemma 14. *Let C and D be objects of K/B, so CxD is an object of K/BxB. There is a natural inclusion map*

$$\iota : C(C) \times C(D) \to C(CxD).$$

Proof. The desired functor corresponds to the product $\epsilon_C \times \epsilon_D$ of evaluation maps under

$$K/BxB((M(C)\square C) \times (M(D)\square D), CxD) \simeq K/BxB((M(C)xM(D))\square(CxD), CxD)$$
$$\simeq K/BxBxBxB(M(C)xM(D), M(CxD)). \qquad \Diamond$$

The following corollary is now a straightforward consequence of Theorem 6, Corollaries 7 and 8, and Lemma 14.

Corollary 15. *If $Z : GxG^{op} \to C(E)$ is an internal functor which is id_{BxB} on objects, the associated action splits as $E = CxD$ with a right action on C and a left action on D iff there are internal functors $X : G \to C(D)$ and $Y : G^{op} \to C(C)$ such that $Z = \iota \circ (XxY)$.* $\Diamond$

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Department of Mathematics
Cornell University
ITHACA, N.Y. 14853. U.S.A.