Oscar P. Bruno

Logical opens of exponential objects

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RÉSUMÉ. Soit $X = \tilde{A}$ et $Y = \tilde{B}$, avec $A = C^\infty(R^n)/J$, $B = C^\infty(R^n)/I$, deux objets représentables dans le topos de Dubuc. L'ensemble des sections globales de l'exponentielle $Y^X$ est identifié à l'ensemble $Z(I, A^n) \subset A^n$ de zéros de l'idéal $I$ dans $A^n$ et est ainsi muni d'une topologie $C^\infty$-compact-ouvert. Dans cet article, on étudie les ouverts de Penon de l'objet $Y^X$. On montre qu'ils coïncident avec les ouverts $C^\infty$-CO de $Z(I, A^n)$ dans le cas où $J$ a des extensions déterminées par des lignes (Définition 0.3) ou bien si $I = \{0\}$. On donne un exemple d'un ouvert de Penon qui n'est pas $C^\infty$-CO en prenant l'idéal $J$ de fonctions à germe nul.

INTRODUCTION.

Let $X = \frac{C^\infty(R^n)}{J}$, $Y = \frac{C^\infty(R^n)}{I}$ be two representable objects in the Dubuc topos $D$ (see Section 0) where $J$ has line determined extensions (0.3). The main result in this paper (Theorem 1.11) says that the global section functor $\Gamma$ establishes a bijection between Penon sub-objects of $Y^X$ and open subsets of $\Gamma(Y^X)$ in the $C^\infty$-CO topology. We show also that when $I = \{0\}$, we can assume $J$ arbitrary (1.12). However, the restriction on $J$ (of having line determined extensions) is seen to be unavoidable in general.

We precede the article with a Section 0 where we recall all these notions and fix the notations.

SECTION 0.

Let $D$ denote the Dubuc topos (see [3, 4]). We recall that $D$ is the topos defined by the following site :

i) The category $\mathcal{B}$, dual to that of finitely generated $C^\infty$-rings $C^\infty(R^n)/I$ presented by an ideal of local nature (see [4] and Remark below).

ii) The open cover topology (see [3] and Remark below).

0.1. Remark. i) Let $U$ be an open subset of $R^n$ (in most of the cases $U = R^n$) and $I \subset C^\infty(U)$ an ideal. Then $I$ is of local nature (or of local character, or germ determined) iff for every $f \in C^\infty(U)$, $f \in I$ iff there exists an open covering $\{U_\alpha\}$ of $U$ such that

$$f \mid U_\alpha \in I \mid U_\alpha = \text{ideal generated in } C^\infty(U_\alpha) \text{ by } \{h \mid U_\alpha : h \in I\}.$$
We remark that if \( I \subseteq C^\infty(\mathbb{R}^n) \) is an ideal of local character and \( U \) is an open subset of \( \mathbb{R}^n \), \( I|_U \) may not be of local character. If \( I \subseteq C^\infty(U) \) is any ideal, there exists a smallest local nature ideal \( \hat{I} \) which contains \( I \). In fact, \( f \in \hat{I} \) iff there exists an open covering \( \{ U_\alpha \} \) of \( U \) such that \( f|_{U_\alpha} \in I|_{U_\alpha} \). \( \hat{I} \) is called the local nature closure of \( I \).

ii) We recall that the generating covers of the open cover topology are families of the form

\[
j_{U_\alpha} : \overline{C^\infty(U_\alpha)} \rightarrow \overline{C^\infty(\mathbb{R}^n)}
\]

where \( \{ U_\alpha \} \) is an open covering of \( \mathbb{R}^n \) and \( j_{U_\alpha} \) are the maps corresponding to the restriction morphisms. The coverings of an arbitrary

\[
T = \overline{C^\infty(\mathbb{R}^n)}/I \in B
\]

are obtained by pulling-back these covers (see [3]). It can be seen then that they are families of the form

\[
\overline{C^\infty(U_\alpha)/I} U_\alpha \rightarrow \overline{C^\infty(\mathbb{R}^n)}/I
\]

where \( U_\alpha \) is a covering of the set of zeroes of \( I \), \( Z(I) \).

0.2. Remark. Let \( X = \overline{C^\infty(R^P)/J} \), \( Y = \overline{C^\infty(\mathbb{R}^n)}/I \in B \). We recall that the (cartesian) product of \( X \) and \( Y \) in \( B \) is

\[
X \times Y = \overline{C^\infty(R^{n+p})}/(J(\bar{x}, \bar{t})) \hat{=} J(\bar{t}, \bar{x})
\]

where this notation should be understood as follows: since we consider the elements of \( C^\infty(R^P) \) (resp. \( C^\infty(\mathbb{R}^n) \), \( C^\infty(R^{n+p}) \)) functions of the variables

\[
\bar{x} = (x_1, ..., x_P) \quad \text{(resp. } \bar{t} = (t_1, ..., t_P) \quad (\bar{x}, \bar{t}) = (x_1, ..., x_P, t_1, ..., t_P))\]

the ideal \( J \) is an ideal in the variable \( \bar{x} : J = J(\bar{x}) \). Now \( J(\bar{x}, \bar{t}) \) is the ideal generated in \( C^\infty(R^{n+p}) \) by the functions of \( J(\bar{x}) \). On the other hand, the symbol \( \hat{=} \) means the local nature sum, i.e., to sum and take local nature closure.

Recall that if \( H \rightarrow F \) is a subobject of \( F \) in a topos, then \( H \) is said to be Penon open iff the following formula holds internally:

\[
\forall h \in H \forall q \in F \ (\exists \ h = q) V q \in H
\]

(see [6, 1]).

Let \( \text{T} \) be the topos of sheaves over the site of Hausdorff topological spaces with open coverings, \( \text{Zar} \) be the topos of sheaves over the site given by the category dual to that of finitely presented \( k \)-algebras with coverings \( \Sigma a_j \rightarrow B \) where \( \Sigma a_j = 1 \) and \( k \) is an algebraically closed field; and let \( D \) be the Dubuc topos already presented. It has been proved by J. Penon (see [5]) that if either \( E = \text{T} \) or \( E = \text{Zar} \)
or $E = D$ and $F$ is representable, then a subobject of $F$ is Penon open iff it is representable and represented by: in the first case an open subset of $F$, in the second a Zariski open, and in the third, if $F = C^\infty(R^n)/I$ by a subobject of the form $C^\infty(U)/I|U$ where $U$ is an open subset of $R^n$. We study here Penon opens of $Y^X$ where

$$Y = C^\infty(R^n)/I, \quad X = C^\infty(R^p)/J \in D$$

are representables ($I$ and $J$ of local character). In some cases we will need to assume that the ideal $J$ has line determined extensions:

0.3. Definition (see [2]). An ideal $J \subseteq C^\infty(R^p)$ is said to have line determined extensions iff it satisfies the following condition: for every $n \in \mathbb{N}$ and $f \in C^\infty(R^{p+n})$, $f \in J(x, t)$ iff for every fixed $\bar{a} \in R^1$, $f(\bar{x}, \bar{a}) \in J$.

We recall from [2] that a large class of finitely generated ideals (including those generated by a finite number of analytic functions) have line determined extensions and there are some examples of non-finitely generated ideals which also have line determined extensions. As a matter of fact, these ideals are characterized as universally closed, i.e., $C^\infty$-CO closed ideals such that the extension $J(x, t)$ to $C^\infty(R^{p+n})$ for all $n$ is $C^\infty$-CO closed.

0.4. Definition. The $C^\infty$-CO topology in $C^\infty(R^l)$ is the topology for which a sequence $f_k$ of elements of $C^\infty(R^l)$ converges to $f \in C^\infty(R^l)$ iff $f_k$ and all its derivatives converge uniformly on compacts to $f$ and its respective derivatives.

A result which is closely related to the notion of ideal with line determined extensions is the following:

0.5. Theorem (Calderon-Reyes-Qué, see [7]). Let $C$, $D$ be closed subsets of $R^p$ and $R^n$ respectively, and let

$$J \subseteq C^\infty(R^p), \quad I \subseteq C^\infty(R^n) \quad \text{and} \quad S \subseteq C^\infty(R^{p+n})$$

be the ideals of all flat functions on $C$, $D$ and $C \times D$ respectively. (Recall that a function $f \in C^\infty(R^k)$ is said to be flat on a closed subset $K$ of $R^k$ iff $f$ and all its derivatives vanish on $K$.) Then

$$S = J(\bar{x}, \bar{t}) + I(\bar{t}, \bar{x}).$$

Finally we recall a well known lemma. By the way we remark that it is this lemma which implies that the congruence associated in the standard way to any ideal $I \subseteq C^\infty(R^n)$ is a $C^\infty$-ring congruence (see [4]).
0.6. Lemma. a) For every \( n+p \)-variables \( C^\infty \)-function \( h : \mathbb{R}^{n+p} \to \mathbb{R} \) and for every integer \( m \geq 0 \) there exist \( C^\infty \)-functions

\[
\begin{align*}
&f_k \text{ of } n \text{ variables } \{ k = (k_1, \ldots, k_p) : \Sigma k_i \leq m \}, \\
&\lambda_k \text{ of } n+p \text{ variables } \{ k = (k_1, \ldots, k_p) : \Sigma k_i = m+1 \}
\end{align*}
\]

such that the equality

\[
h(f, \lambda) = \sum_k f_k(f)\lambda^k + \sum_k \lambda_k(f, \lambda)\lambda^k
\]

holds for every

\[
(f, \lambda) = (t_1, \ldots, t_n, x_1, \ldots, x_p) \in \mathbb{R}^{n+p}
\]

where \( \lambda^k = x_1^{k_1} \cdots x_p^{k_p} \). Of course we have

\[
f_k(f) = \frac{1}{k!} \frac{\partial^k}{\partial f^k} h(f, 0).
\]

b) We will use this Lemma in the following particular case: If \( h \in C^\infty(\mathbb{R}^n) \) then there exist functions \( k_i \in C^\infty(\mathbb{R}^{2n}) \) such that, for every \( \tilde{y}_1, \tilde{y} \in \mathbb{R} \), we have

\[
h(\tilde{y}_1) - h(\tilde{y}) = \sum_{i=1}^n (y_1^i - y^i)k_i(\tilde{y}_1, \tilde{y}).
\]

SECTION 1.

We prove first some auxiliary results (1.1 to 1.5).

Let \( B = C^\infty(\mathbb{R}^n)/I \), \( A = C^\infty(\mathbb{R}^n)/J \) be any two \( C^\infty \)-rings in \( \mathcal{B}^{op} \) and

\[
X = \overline{A}, \quad y = \overline{B} \in B \subset D.
\]

Let \( \Gamma : D \to \text{Sets} \) be the global section functor \( \Gamma(F) = \text{Hom}(l, F) \).

We have

1.1. Proposition. \( \Gamma(Y^X) = Z(I, \mathcal{A}^0) \), where

\[
Z(I, \mathcal{A}^0) = \{ (f_1, \ldots, f_n) \in \mathcal{A}^0 : \forall h \in I, h(f_1, \ldots, f_n) = 0 \}
\]

(Notice that the last definition makes sense since smooth functions may be evaluated in \( C^\infty \)-rings.)

1.2. Definition. The \( C^\infty \)-CO topology on \( A \) is the quotient topology determined by the \( C^\infty \)-CO topology of \( C^\infty(\mathbb{R}^n) \) (see 0.4).

The \( C^\infty \)-CO topology of \( \mathcal{A}^0 \) is just the product topology, and we give the subspace topology to \( Z(I, \mathcal{A}^0) \).

Recall that the quotient map \( C^\infty(\mathbb{R}^n) \to A \) is open, thus it follows:

1.3. Lemma. If a sequence \( h_k \) of elements of \( A \) converges to \( h \in A \) in the \( C^\infty \)-CO topology, then there exist a sequence \( \{ f_k \} \subset C^\infty(\mathbb{R}^n) \)
and $f \in C^\infty(\mathbb{R}^p)$ such that

1) $[f_k] = h_k$ and $[f] = h$.

(The brackets mean "equivalence class of").

2) $f_k$ converges to $f$ in the $C^\infty$-CO topology of $C^\infty(\mathbb{R}^p)$.

1.4. Lemma. Let $X$, $Y$ be as above and $I : H \rightarrow Y^X$ be a subobject of $Y^X$. Then $H$ is Penon open iff it satisfies the following conditions:

a) For every representable sheaf $T = C^\infty(\mathbb{R}^k)/K \in B$, arrow $q : T \rightarrow Y^X$ and $s_o \in Z(K) \subset \mathbb{R}^k$, so : $1 \rightarrow T$, if $q \circ s_o$ factors through $H$, then there exists a neighborhood $V$ of $s_o$ in $\mathbb{R}^k$ such that $q \circ j_V$ factors through $H$ (where $j_V : C^\infty(V)/K \rightarrow C^\infty(\mathbb{R}^k)/K$ is the map corresponding to the restriction). In other words, if 
$q(s_o) \in H$ then there exists a neighborhood $V$ of $s_o$ in $\mathbb{R}^k$ such that $q \circ s_o \in H$.

b) If $T = C^\infty(\mathbb{R}^k)/K$ is any representable sheaf and $q, h$ are arrows $q : T \rightarrow Y^X$, $h : T \rightarrow H$, and there exists a sequence $s_r$ of elements of $Z(K)$ converging to $s_o \in Z(K)$ such that $q \circ s_r = h \circ s_r$, then $q \circ s_o$ factors through $H$. (Notice that this condition is vacuous if the ideal $J$ is $C^\infty$-CO closed since in this case we have $q \circ s_o = h \circ s_o$.)

Proof. Kripke-Joyal semantics (see [1]) tells us that $H$ is Penon open iff for every $T \in B$ and for every $q : T \rightarrow Y^X$, $h : T \rightarrow H$ there exists a covering of $T$ such that $(q \circ f_1, h \circ f_2)$ verifies the formula $\neg(h = q)$ and $q \circ f_2$ factors through $H$. We must prove that this K-J statement is equivalent to the statement of the Lemma.

Statement of the Lemma implies K-J statement: Assume $H$ verifies the statement of the Lemma. Because of the sheaf axiom on $H$ it suffices to show that for every $s_o \in Z(K)$ either

i) there exists an open neighborhood $V$ of $s_o$ in $\mathbb{R}^k$ such that $q \circ j_V$ factors through $H$, or

ii) There exists an open neighborhood $V$ of $s_o$ in $\mathbb{R}^k$ such that for $s \in V \cap Z(K)$ we have $h \circ s \neq q \circ s$.

So, take $s_o \in Z(K)$ and assume that point ii is not verified. It follows that there exists a sequence $s_r$ of points of $Z(K)$ converging to $s_o$ such that for every $r \in \mathbb{N}$, $q \circ s_r = h \circ s_r$. We remark that this does not
imply $q \circ \tilde{s}_o = i \circ h \circ \tilde{s}_o$, but in virtue of b it follows that $q \circ \tilde{s}_o$ factors through $H$ and so, by a, we have that $\tilde{s}_o$ verifies point i.

**K-J statement implies statement of the Lemma**: a) Take $q : T \rightarrow Y^X$ and consider the following commutative diagram

$$
\begin{array}{ccc}
1 & \rightarrow & T \\
\downarrow & & \downarrow q \\
H & \rightarrow & Y^X
\end{array}
$$

With this data we may consider the arrows

$$q : T \rightarrow Y^X \quad \text{and} \quad T \xrightarrow{a} 1 \xrightarrow{q_1} H.$$ 

By K-J statement, there exists a covering such that $(q \circ f_1, i \circ q_1 \circ a \circ f_1)$ verifies the formula $\exists \tilde{s}_o (q = h)$ and $q \circ f_2$ factors through $H$. Since $T_1, T_2$ is a covering $\tilde{s}_o : 1 \rightarrow T$ must factor either through $T_1$ or through $T_2$. But it cannot factor through $T_1$ since this would imply that

$$(q \circ \tilde{s}_o, i \circ q_1 \circ a \circ \tilde{s}_o) = (q \circ \tilde{s}_o, i \circ q_1)$$

verifies the formula $\exists \tilde{s}_o (q = h)$, which contradicts the commutativity of (1).

b) Immediate.

1.5. **Lemma** (see [5]). Let $F$ be an object in the topos $D$.

a) The correspondence $R \rightarrow \Gamma(R)$ from the set of subobjects of $F$ to the set of subsets of $\Gamma(F)$ has a right adjoint $E$, i.e., for every $S \subseteq \Gamma(F)$, there exists $E(S) \subseteq F$ such that for every $R \subseteq F$ we have

$$\Gamma(R) \subseteq S \iff R \subseteq E(S).$$

Moreover $\Gamma(E(S)) = S$. In fact, for $S \subseteq \Gamma(F)$, $E(S)$ is defined by the following rule: an arrow $f : T \rightarrow F$ ($T \in B$) factors through $E(S)$ iff $\Gamma(f) : \Gamma(T) \rightarrow \Gamma(F)$ factors through $S$.

b) If $H \rightarrow F$ is Penon open, then $E(\Gamma(H)) = H$.

**Proof.** a) It must be seen that the sub-presheaf defined is actually a sheaf. This is easily done.
b) It must be seen that an arrow \( f : T \to F \) factors through \( H \) iff it factors through \( E(\Gamma(H)) \). Now \( \Rightarrow \) is immediate. To see \( \Leftarrow \) assume \( f : T \to F \) factors through \( E(\Gamma(H)) \). This means that for every global section \( s_o : 1 \to T \), \( f \circ s_o \) factors through \( H \). Now use 1.4 a and the sheaf axiom on \( H \).

1.6. Proposition. Let \( W \) be a \( C^\infty \)-CO open subset of \( Z(I, \mathcal{A}^R) \). Then \( E(W) \subseteq Y^X \) is Pence open.

Proof. We use 1.4. Let us see that \( E(W) \) verifies 1.4.a. Take arrows as in the commutative diagram

\[
\begin{array}{ccc}
1 & \xrightarrow{s_o} & T \\
 & \leftarrow \, q \, \searrow \, & Y^X
\end{array}
\]

\[T = C(\mathcal{K}/K(\mathcal{S})) \text{.} \]

It follows that \( q \circ s_o \in W \). Now \( q \) is represented by an

\[ [\tilde{r}] = ([r_1], \ldots, [r_p]), \text{ and so } q \circ s_o \text{ is represented by } [\tilde{r}(s_o, \tilde{x})] \in W. \]

So, since \( W \) is open, it follows that \( [\tilde{r}(s, \tilde{x})] \in W \) for every fixed \( \tilde{s} \) in a certain neighborhood \( V' \) of \( s_o \) in \( Z(K) \). Then, calling \( V \subseteq \mathcal{K} \) an open set such that \( V' = V \cap Z(K) \) we have that \( q \circ j_V \) factors through \( E(W) \), where

\[ j_V : C^\infty(V)/K(V) \to C^\infty(\mathcal{K}) \]

is the arrow corresponding to the restriction morphism. Let us now see that \( E(W) \) verifies 1.4.b. To do this, take arrows \( q : T \to Y^X, h : T \to E(W) \) and a sequence \( s_r \) of elements of \( Z(K) \) converging to

\[ s_o \in Z(K) \] such that \( q \circ s_r = i \circ h \circ s_r \).

Let

\[ [\tilde{r}], [\tilde{g}] \in (C^\infty(\mathcal{K}))/J(\tilde{r}) \]

represent \( h \) and \( q \) respectively. The equality \( q \circ s_r = i \circ h \circ s_r \) means that

\[ [\tilde{r}(s_r, \tilde{x})] = [\tilde{g}(s_r, \tilde{x})] \] in \( (C^\infty(\mathcal{K}))/J \)

for every \( r \in \mathbb{N} \), or, in the other words,

\[ \tilde{r}(s_r, \tilde{x}) - \tilde{g}(s_r, \tilde{x}) \in \mathcal{J}. \]

Now \( \tilde{g}(s_o, \tilde{x}) + (r(s_r, \tilde{x}) - \tilde{g}(s_r, \tilde{x})) \) \( C^\infty \)-CO converges to \( \tilde{r}(s_o, \tilde{x}) \) as \( r \to \infty \).

Then

\[ [\tilde{g}(s_o, \tilde{x})] = [\tilde{g}(s_o, \tilde{x}) + (\tilde{r}(s_r, \tilde{x}) - \tilde{g}(s_r, \tilde{x}))] \]
converges to \([f(\xi, \lambda)]\). But we know that \([f(\xi, \lambda)] = h \circ \xi\) is in \(W\). So, \([g(\xi, \lambda)] \in W\) or, in other words, \(q \circ \xi\) factors through \(E(W)\).

In order to prove the converse of 1.6 (in the case that \(J\) has line determined extensions) we need two lemmas.

1.7. Lemma (Glueing Lemma). If a sequence \(f_\lambda\) of elements of \(C^\infty(R^p)\) and \(f \in C^\infty(R^p)\) are such that for every compact set \(K \subset R^p\) and every \(d \in N\) there exists \(\lambda, \alpha \in R\) such that

\[
|D^{\alpha}(f_\lambda - f)| < \lambda \cdot e^{-\lambda}
\]

in \(K\) for \(|\alpha| < d\) and \(\lambda \geq \lambda_0\) for certain \(\lambda_0 \in N\) then there exists \(F \in C^\infty(R^{p+1})\) such that

\[
\begin{cases}
F(\lambda, s) = f_\lambda(\lambda) & \text{if } 1/\lambda - 1/\lambda(\lambda + 1) < s < 1/\lambda + 1/\lambda(\lambda + 1) \\
F(\lambda, s) = f(\lambda) & \text{if } s \leq 0
\end{cases}
\]

and \(F(\lambda, s_0)\) belongs to the ideal generated by \(\{f_\lambda : \lambda \in N\} \cup \{f\}\) for every fixed \(\xi_0 \in R\).

Proof. We may assume \(f = 0\). Take \(\psi \in C^\infty(R)\) such that

\[
\text{supp}(\psi) \subset (-1, 1) \quad \text{and} \quad \psi([-1, 1]) = 1.
\]

Let us call

\[
\phi_\lambda(s) = \psi(2\lambda(s - 1/\lambda)).
\]

We have that \(\text{supp}(\phi_\lambda) \cap \text{supp}(\phi_k) = \emptyset\) if \(\lambda \neq k\).

It is easily seen that

\[
F(\lambda, s) = \begin{cases}
\phi_\lambda(\lambda) \phi_k(s) & \text{if } s \in \text{supp}(\phi_\lambda) \\
0 & \text{otherwise}
\end{cases}
\]

is \(C^\infty\) and has the required properties.

1.8. Lemma. a) Assume \(J\) has line determined extensions. Let \(\overline{h}_k\) be a sequence of elements of \(Z(I, \Omega^\infty \cap C^\infty)\) converging to \(h \in Z(I, A^\infty)\). Let \(N \subset C^\infty(R)\) be the ideal of all functions vanishing at \(1/\lambda\) and \(0 : 1 \in N\), and let \(S = C^\infty(R)/N\). (We call \(S\) the generic convergent sequence). Then there exists a subsequence \(\overline{h}_{\lambda_k}\) of \(\overline{h}_k\) and an arrow

\[
F : S \rightarrow Y^X
\]

such that \(F \circ 1/\lambda = \overline{h}_{\lambda_k}\) and \(F \circ 0 = h\),

where \(1/\lambda : 1 \rightarrow S\) and \(0 : 1 \rightarrow S\) are the arrows corresponding to evaluation at \(1/\lambda\) and \(0\) respectively.

b) Let \(J\) be any ideal \([\text{of local character}]\) and \(I = \{0\}\). Let \(\overline{h}_k\)
be a sequence of elements of \( Z(\{0\}, \mathcal{A}^n) = \mathbb{A}^n \) converging to \( h \in \mathbb{A}^n \). Then, there exists a subsequence \( h_{kg} \) of \( h_k \) and an arrow

\[
F : R \to C^\infty(R^p) \quad \text{such that} \quad F \circ 1/\lambda = h_{kg}, \quad \text{and} \quad F \circ 0 = h,
\]

where \( R = C^\infty(R) \) is the line.

**Proof.** We prove only point a. Point b follows similarly although more directly. By 1.3, there exists a sequence \( \{f_k\} \subseteq C^\infty(R^p) \) and \( f \in C^\infty(R^p) \) such that \( f_k \) converges to \( f \) and \( \{f_k\} = h_k, \{f\} = h \). Let us take a subsequence \( f_{kg} \) of \( f_k \) such that

\[
\left| \partial^\alpha(f_{kg}^i - f^i) \right| < e^{-\lambda} \quad (1 \leq i \leq n)
\]

in \([-\lambda, \lambda]\) for every \( \alpha \) such that \( |\alpha| \leq \lambda \). Thus, by 1.7, there exists an \( F \in C^\infty(R^{p+1}) \) such that

\[
\left\{ \begin{array}{ll}
F(\bar{x}, s) = f_{kg}(\bar{x}) & \text{for } s \in \left( \frac{1}{\lambda} - \frac{1}{4\lambda(\lambda+1)}, \frac{1}{\lambda} + \frac{1}{4\lambda(\lambda+1)} \right) \\
F(\bar{x}, s) = f(\bar{x}) & \text{for } s \leq 0.
\end{array} \right.
\]

We will show that this \( F \) defines an arrow \( F : S \to Y^X \). As it happens, such an arrow is a zero of \( I \) in

\[
C^\infty(R^{p+1})/(N(s, \bar{x}) \in J(\bar{x}, s))^N
\]

(Recall that \( J \) means "local nature closure of the sum"). So, we must show that

\[
[F] \in (C^\infty(R^{p+1})/(N(s, \bar{x}) \in J(\bar{x}, s)))^N
\]

is a zero of \( I \). Take \( g \in I \). We have that

\[
g([F]) = [g(F)] \in C^\infty(R^{p+1})/(N(s, \bar{x}) \in J(\bar{x}, s))
\]

(this is the \( C^\infty \)-ring structure in a quotient of this type, see [4]). And

\[
g(F(\bar{x}, 0)) = g(f(\bar{x})) \in J
\]

and for every

\[
\lambda \in \mathbb{N} \quad \text{and} \quad s \in \left( \frac{1}{\lambda} - \frac{1}{4\lambda(\lambda+1)}, \frac{1}{\lambda} + \frac{1}{4\lambda(\lambda+1)} \right),
\]

\[
g(F(\bar{x}, s)) = g(f_{kg}(\bar{x})) \in J. \text{ Now from 0.6.b, it follows that } g(f_{kg}), g(f) \text{ satisfy the hypothesis of 1.7 (because } f_{kg}, f \text{ do). Call } G \in C^\infty(R^{p+1}) \text{ the function given by 1.7 :}
\]

\[
\left\{ \begin{array}{ll}
G(\bar{x}, s) = g(f_{kg}(\bar{x})) & \text{if } \frac{1}{\lambda} - \frac{1}{4\lambda(\lambda+1)} < s < \frac{1}{\lambda} + \frac{1}{4\lambda(\lambda+1)} \\
G(\bar{x}, s) = g(f(\bar{x})) & \text{if } s \leq 0
\end{array} \right.
\]

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and for every fixed $s_0 \in \mathbb{R}$, $G(x, s_0) \in J$. Since $J$ has line determined extensions, it follows that $G(x, s) \in J(x, s)$. On the other hand,

$$g(F) - G \in C^\infty(\mathbb{R}^{d+1})$$

is a function flat on $\mathbb{R}^d \times (\{1/k : k \in \mathbb{N}\} \cup \{0\})$. So, by 0.5,

$$(g(F) - G) \in N(s, x).$$

So

$$g(F) = G + (g(F) - G) \in N(s, x) \cap J(x, s).$$

It is immediate to verify that the arrow $F$ just defined verifies

$$F \circ 1/k = H_{k_0} \quad \text{and} \quad F \circ 0 = H.$$

1.9. Proposition. Assume $J$ has line determined extensions and let $U$ be a Penon open subobject of $Y^X$. Then $\Gamma(U)$ is a $C^\infty$-CO open subset of $Z(I, A^n)$.

Proof. Suppose $\Gamma(U)$ is not $C^\infty$-CO open in $Z(I, A^n)$. This means that there is a sequence $h_k$ of elements of $Z(I, A^n) \setminus \Gamma(U)$ $C^\infty$-CO converging to a certain $h \in \Gamma(U)$. By 1.8.a, there exist a subsequence $h_{k_0}$ of $h_k$ and an arrow

$$F : S = C^\infty(\mathbb{R})/N \rightarrow Y^X$$

such that

$$F \circ 1/k = h_{k_0} \quad \text{and} \quad F \circ 0 = h.$$

Now, since $U$ is Penon open, we have from 1.4 that there exists an open neighborhood $V$ of $0 \in \mathbb{R}$ such that $F \circ j_V$ factors through $U$. This is a contradiction.

1.10. Proposition. Let $J$ be any local character ideal and $I = \{0\} \subset C^\infty(\mathbb{R})$. Let $U$ be a Penon open subobject of $Y^X = C^\infty(\mathbb{R})^X$. Then $\Gamma(U)$ is a $C^\infty$-CO open subset of

$$A^n = (C^\infty(\mathbb{R})/J)^n = Z(\{0\}, A^n).$$

Proof. Similar to the proof of 1.9 (use 1.8.b instead of 1.8.a).

From 1.5, 1.6 and 1.9 it follows:

1.11. Theorem. Let

$$X = C^\infty(\mathbb{R})/J = \overline{A}, \quad Y = C^\infty(\mathbb{R})/I = \overline{B}$$

and let us assume that $J$ has line determined extensions. Then the mapping $U \mapsto \Gamma(U)$ from the set of subobjects of $Y^X$ to the set of subobjects of $Z(I, A^n)$ determines a bijection between the set of Penon open subobjects of $Y^X$ and the set of $C^\infty$-CO open subsets of $Z(I, A^n)$.

Example. An easy instance of 1.11 is $D^0 (D = C^\infty(\mathbb{R})/(X^2))$. One may see
that its open subobjects "coincide" with usual open subsets of $\mathbb{R}$.

As it was said in the Introduction the hypothesis on $J$ of having line determined extensions is essential: it cannot be avoided in general (see Example 1.14 below). However Theorem 1.11 holds in some cases for ideals $J$ not having line determined extensions. This is the case for instance, if the ideal $I$ is $\{0\}$.

1.12. Theorem. Let

$$X = C^{\omega}(\mathbb{R})/J = \tilde{A}, \quad Y = C^{\omega}(\mathbb{R}^2) = \tilde{B},$$

where $J$ is any ideal of local character. Then the mapping $\tilde{\Gamma}(U)$ from the set of subobjects of $Y^X$ to the set of subsets of $\mathbb{R}^2$ determines a bijection between the set of Penon open subobjects of $Y^X$ and the set of $C^\omega$-CO open subsets of $\mathbb{R}^2$.

Proof. Follows from 1.5, 1.6 and 1.10.

1.13. Examples. i) Consider $\mathbb{R}^R$ in the Dubuc topos, where $R = C^{\omega}(\mathbb{R})$ is the line. In this case, Theorem 1.12 just says that $\tilde{\Gamma}$ establishes a bijection between the set of Penon open subobjects of $\mathbb{R}^R$ and the set of $C^\omega$-CO open subsets of $C^{\omega}(\mathbb{R})$. This was conjectured by M. Bunge at the workshop which took place in Aarhus in June 1983 and answered independently by I. Moerdijk and the author.

ii) Let $\Delta = C^{\omega}(\mathbb{R})/J$ where $J$ is the ideal of all $f \in C^{\omega}(\mathbb{R})$ such that $f$ vanishes in a neighborhood of $0 \in \mathbb{R}$. $R^\Delta$ is the internal ring of germs at $0$ of smooth one-variable functions. By 1.12, the Penon topology of $\mathbb{R}^\Delta$ "coincides" with the $C^\omega$CO topology on $C^{\omega}(\mathbb{R})/\Delta$ which is the set-theoretical ring of germs at $0$ of smooth one variable functions.

1.14. Example. Let $w \in C^{\omega}(\mathbb{R}^2)$ be a function vanishing in

$$C = \{(x, y) \in \mathbb{R}^2 : \ |x| \leq |y|\} \cup \{(x, y) : y = 0\}$$

and different from zero everywhere else. Let $I \subset C^{\omega}(\mathbb{R}^2)$ be the ideal generated by $w$, and $J \subset C^{\omega}(\mathbb{R})$ be the ideal of all smooth functions vanishing in a neighborhood of $0 \in \mathbb{R}$. Let

$$X = C^{\omega}(\mathbb{R})/J \quad \text{and} \quad Y = C^{\omega}(\mathbb{R}^2)/I.$$

We have that

$$\tilde{\Gamma}(Y^X) = \{(f_1, [f_2]) \in (C^{\omega}(\mathbb{R})/J)^2 : w(f_1(x), f_2(x)) \text{ vanishes in a neighborhood of } 0 \in \mathbb{R}\}.$$

Let $V \subset \tilde{\Gamma}(Y^X)$ be the set

$$V = \{(f_1, [0]) : \frac{df_1}{dx}(0) \neq 0\} \subset \tilde{\Gamma}(Y^X).$$
Our example is $E(V) \to Y$ (see 1.5): $E(V)$ is Penon open in $Y^X$ while it is easily seen that $\Gamma(E(V)) = V$ is not $C^\infty$-open in $\Gamma(Y^X)$. In order to see that $E(V)$ is Penon open we need the following lemma, whose proof we omit.

1.15. Lemma. Let $V, C, X, Y$ be as above and

$$F = (F_1, F_2) \in C^\infty(C^k(R^{k+1}))^2$$

be such that for certain $\bar{s}_0 \in R$, $[F(\bar{s}_0, x)] \in V$, but there exists a sequence $\bar{s}_r$ of points of $R$, $\bar{s}_r \to \bar{s}_0$ as $r \to \infty$ such that $[F(\bar{s}_r, x)] \in \Gamma(Y^X)\forall x$. Then there exist a sequence $x_r$ of real numbers $x_r \to 0$ as $r \to \infty$ and $r_0 \in N$ such that for $r > r_0$ we have $F(\bar{s}_r, x_r) \notin C$.

Let us now see that $E(V)$ is Penon open. We use 1.4. Let us see first that $E(V)$ verifies 1.4.b. Take $T = C^\infty(R^S)/K$, a pair of arrows: $q : T \to Y^X$, $h : T \to E(V)$ and a sequence $\bar{s}_r$ of elements of $Z(K)$ converging to $\bar{s}_0 \in Z(K)$ such that $q \circ \bar{s}_r = i \circ h \circ \bar{s}_r$.

Let us assume that $q$ and $i \circ h$ are represented by

$$[\bar{q}], [\bar{g}] \in (C^\infty(C^k(R^{k+1})/(\mathbb{K}\bar{s}_0, x)) \otimes J(x, \bar{s}))^2$$

respectively. It follows that $q \circ \bar{s}_r$, $i \circ h \circ \bar{s}_r$ are represented by

$$[\bar{q}(\bar{s}_r, x)], [\bar{g}(\bar{s}_r, x)] \in \Gamma(Y^X) \subset (C^\infty(R)/J)^2.$$

We have

$$\bar{q}(\bar{s}_r, x) - \bar{g}(\bar{s}_r, x) \in J$$

for all $r \in N$, then

$$\bar{q}(\bar{s}_0, x) - \bar{g}(\bar{s}_0, x) \in \text{closure}(J^2).$$

Since closure($J$) is the ideal of all flat functions at $0 \in R$ and $[\bar{g}(\bar{s}_0, x)] \in V$, it follows that $\bar{q}(\bar{s}_0, x) \in V$, as the reader may check (use that $W(\bar{q}(\bar{s}_0, x))$ must vanish in a neighborhood of $0 \in R$).

Now, let us see that $E(V)$ verifies 1.4.a. Take $T = C^\infty(R^S)/K$, an arrow $q : T \to Y^X$ and $\bar{s}_0 : 1 \to T$, $\bar{s}_0 \in Z(K)$ such that $q \circ \bar{s}_0$ factors through $E(V)$, i.e., $q \circ \bar{s}_0 \in V$. We have that $q$ is represented by an element

$$[\bar{F}] \in Z(1, (C^\infty(C^k(R^{k+1})/(\mathbb{K}\bar{s}_0, x)) \otimes J(x, \bar{s}))^2)$$

and so, $q \circ \bar{s}_0$ is represented by

$$[\bar{F}(\bar{s}_0, x)] \in \Gamma(Y^X).$$

We must show that there exists an open neighborhood $W$ of $\bar{s}_0$ in $R$ such that $q \circ j_w$ factors through $E(V)$. Now, by 1.5, the condition

"$t \circ j_w$ factors through $E(V)$"
Assume that such W does not exist. This means that there exists a sequence $s_r$ of points of $Z(K) \subseteq \mathbb{R}^k$ converging to $s_0$ and such that

$$F(s_r, x) \in I(Y^X) \setminus V.$$

By 1.15, it follows that there exists a sequence $x_r$ of real numbers which tends to zero as $r \to \infty$ such that $F(s_r, x_r) \notin C$ (i.e., $w(F(s_r, x_r)) \neq 0$). Now, we know that $w(F) \in K(s, x) \updownarrow J(x, s)$ and so, since the functions of $J$ vanish in a neighborhood of 0, there exists a neighborhood $W$ of $(s_0, 0) \in \mathbb{R}^{k+1}$ such that in $W$, $w(F)$ is an element of $K(s, x)$. For some $\epsilon > 0$

$$(s_0^1 - \epsilon, s_0^1 + \epsilon) \times \ldots \times (s_0^k - \epsilon, s_0^k + \epsilon) \times (-\epsilon, \epsilon)$$

is contained in $W$, and so we should have $w(F(s_r, x)) = 0$ for $x \in (-\epsilon, \epsilon)$ and every $r \geq r_0$ for some $r_0 \in \mathbb{N}$. This is a contradiction.

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Departamento de Matematicas
F.C.E.N., Pab. I
Ciudad Universitaria
1428, BUENOS AIRES. ARGENTINA