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ČECH METHODS AND THE ADJOINT FUNCTOR THEOREM

by Renato BETTI

RÉSUMÉ. Dans ce travail, on montre que des résultats classiques relatifs à la théorie de la forme et aux extensions de Čech peuvent être généralisés et appliqués dans le cadre des catégories basées sur une bicatégorie. Dans ce contexte, la condition de Čech donne un théorème sur les foncteurs adjoints.

INTRODUCTION.

It is known that to any topological space X a Čech system

$$\nu_X : \mathrm{Cov}\ X \to \mathrm{HPol}$$

can be associated. Here HPol denotes the homotopy category of polyhedra, Cov X is the (small, cofiltered) category of numerable coverings of X and refinements, and ν_X is the functor *nerve*.

Čech systems allow one to extend simplicial homology. The n-th homology group is defined by

$$H_n(X) = \varprojlim H_n(\nu_X U).$$

As Dold remarks ([9], p. 366), the Čech process for extending functors applies to many situations. Lee and Raymond [14] study the Čech methods of extending set-valued functors from categories of triangulable spaces and homotopy classes of maps to functors defined on more general topological spaces. Calder and Siegel [7, 8], and Frei [10] study Čech extensions at the level of continuous maps.

Čech systems are relevant in shape theory. With the terminology of Mardešić and Segal [15], the system ν_X is an HPol-expansion of the topological space X, i.e., for any polyhedron P there is a natural isomorphism

$$\varinjlim \mathrm{HPol}(\nu_X U, P) \simeq \mathrm{HTop}(X, P).$$

The shape category relative to the embedding HPol → HTop can be calculated by

$$\mathrm{Sh}(X, Y) \simeq \text{Pro-HPol}(\nu_X, \nu_Y)$$

and the notion of expansion thus allows one to generalize the Čech procedure to abstract situations.

The crucial point of the various Čech methods rests on the fact that there is a weakly cofinal functor from Cov X into the comma category $(X \downarrow \mathrm{HPol})$ of "polyhedra under X" (see Dold [9], p. 356, and Mardešić and Segal [15], p. 328).

The above property is assumed here as the starting point in defining the Čech condition. As a consequence, many facts relative to Čech extensions and to shape theory can be obtained as results of general category theory.

Moreover, in this setting, a double generalization is allowed. First, the Čech methods are considered with respect to an arbitrary family of small diagrams (as in Tholen [19, 20]). Second, everything works equally well for categories enriched in a bicategory. This extra generality is free of charge, but the main point is that Čech methods provide an adjoint functor Theorem in the enriched case (Čech condition = solution set).

The main application of the adjoint functor theorem we have in mind regards locally internal categories over a topos E. In [4] it is shown that these categories can be considered as enriched in the bicategory Span E and that many notions of locally internal category theory become standard notions of enriched category theory. The adjoint functor Theorem applies to this situation.

We begin by describing completions with respect to suitable families of indexing modules. Then the Čech condition is introduced and discussed. Special attention is paid to the Sets case. Finally, we deduce the adjoint functor Theorem as a generalization of known results in shape theory.

1. THE COMPLETION PROCESS.

We shall assume that B is a bicategory such that :

(i) it is locally small-complete and cocomplete, and local colimits are preserved by compositions on both sides ;

(ii) it admits right Kan extensions and right liftings.

Condition i in particular means that for any pair u, v of objects in B the hom-category $B(u, v)$ is small-complete and cocomplete. When B is the bicategory Span E , however, only finite limits and colimits exist in any hom-category. In this case we shall consider only limits indexed over one-object categories enriched in Span E (*internal-limits* in the terminology of [4], reminiscent of the fact that such categories are exactly categories internal to E).

Condition ii means that compositions on both sides with arrows of B have right adjoints. Namely, for any $f : u \to v$ the right adjoint to the functor

$$f.- : B(v, w) \to B(u, w)$$

is the (right) Kan extension, and it is denoted by $\hom^u(f, -)$. The right adjoint to the functor

$$-.f : B(w, u) \to B(w, v)$$

is the (right) lifting, and it is denoted by $\hom_u(f, -)$.

In order to fix the terminology, we recall the main notions relative to categories enriched in a bicategory, to modules and limits indexed by modules (see Kelly [13] for the notions of enriched category theory. Enrichments in a base bicategory can be found in [1]. Other descriptions are in Street [17], in [4] and in [5]).

A B-category X consists of objects $x, y, \ldots$. For any object x there is an underlying object e_x in the base bicategory B. Homs are provided by arrows $X(x, y) : e_x \to e_y$ and compositions and identities by suitable 2-cells

$$\begin{tikzcd} e_x \arrow[r, "X(x, y)"] \arrow[dr, "X(x, z)"'] & e_y \arrow[d, "X(y, z)"] \arrow[dl, phantom, "\swarrow"] \\ & e_z \end{tikzcd} \qquad \begin{tikzcd} e_x \arrow[r, bend left, "1"] \arrow[r, bend right, "X(x, x)"'] \arrow[r, phantom, "\Downarrow"] & e_x \end{tikzcd}$$

subject to associativity and unity laws.

One-object B-categories are denoted simply by the name of their only underlying object. In the case when B is a symmetric monoidal category V (regarded as a one-object bicategory) the above definition gives the usual notion of a category enriched in V.

A *B-functor* $F : X \to Y$ is a function on the objects which preserves the underlying objects; its effect on the homs is given by 2-cells

$$X(x, y) \to Y(Fx, Fy)$$

compatible with identities and compositions.

A *module* $\varphi : X \nrightarrow Y$ of B-categories assigns a component

$$\varphi(x, y) : e_x \to e_y$$

to every pair of objects, endowed with an action of X on the left and of Y on the right, in the sense that there are given 2-cells satisfying the usual axioms of associativity, unity and mixed associativity :

$$\begin{tikzcd} e_x \arrow[r, "\varphi(x,\, y)"] \arrow[dr, "\varphi(x,\, y')"'] & e_y \arrow[d, "Y(y,\, y')"] \arrow[dl, phantom, "\swarrow"] \\ & e_{y'} \end{tikzcd} , \qquad \begin{tikzcd} e_x \arrow[r, "\varphi(x,\, y)"] \arrow[dr, phantom, "\searrow"] & e_y \\ e_{x'} \arrow[u, "X(x',\, x)"] \arrow[ur, "\varphi(x',\, y)"'] & \end{tikzcd}$$

A *morphism* $\alpha \to \beta$ of modules

$$X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Y$$

is given by a family of 2-cells $\alpha(x, y) \to \beta(x, y)$ which is compatible with the actions.

Under our assumptions on the base bicategory B, all the B-categories with a small set of objects, and modules, constitute a bicategoy B-mod, where the composition of the modules $\varphi : X \nrightarrow Y$ and $\psi : Y \nrightarrow Z$ is defined as follows : $(\psi \circ \varphi)(x, z)$ is the coequalizer $\varinjlim \psi(y, z).\varphi(x, y)$ of the two actions

$$\sum_{y', y''} \psi(y'', z).\, Y(y', y'').\, \varphi(x, y') \rightrightarrows \sum_{y} \psi(y, z).\, \varphi(x, y).$$

It is easy to check that the hom is a module $X \nrightarrow X$ which is the identity under composition.

A functor $F : X \to Y$ gives rise to two modules $F_* : X \nrightarrow Y$ and $F^* : Y \nrightarrow X$ defined by

$$F_*(x, y) = Y(Fx, y) \quad \text{and} \quad F^*(y, x) = Y(y, Fx).$$

A calculation shows that F_* is left adjoint to F^* in the bicategory B-mod.

Given a functor $G : I \to X$ and a module $\psi : v \nrightarrow I$, the *limit of* G *indexed by* ψ is an object $\{\psi, G\}$ (if it exists) which represents the right lifting $\hom_I(\psi, G^*)$ of G^* through ψ:

$$\begin{tikzcd} I & X \arrow[l, "G^*"', "|" marking] \arrow[dl, "\hom_I(\psi,\, G^*) = X(-,\, \{\psi,\, G\})", "|" marking] \\ v \arrow[u, "|" marking] \arrow[ur, phantom, "\Uparrow"] & \end{tikzcd}$$

Definition. A family **J** of modules $u \nrightarrow I$ whose codomains are small categories is said to be *admissible* if :

(i) representables $I(i, -) : e_i \nrightarrow I$ are in **J** ;

(ii) whenever $\varphi : I \nrightarrow K$ is such that $\varphi(i, -) : e_i \nrightarrow K$ is in **J** for every i, the composite $\varphi \circ \psi$ with any ψ in **J** is again in **J**.

An admissible family corresponds to what is called a "family of coverings" in [3]. Here we follow the terminology of Tholen [20].

Observe that any functor $F : I \to K$, regarded as a module, satisfies the assumption in ii above. Hence compositions of the type $F.\psi$ are again in **J**, whenever ψ is in **J**.

A category X is said to be **J**-*complete* if it admits all limits indexed by the modules of **J**. A functor is called **J**-*continuous* if it preserves the **J**-indexed limits.

Example. The classical construction of pro-categories by Grothendieck and Verdier is obtained by taking E = Sets and **J** = trivial modules into small cofiltered categories.

For categories indexed over a base topos E (see Johnstone and Joyal [10]) one should consider B = Span E and **J** = all modules into one-object categories which are cofiltered when regarded as internal categories.

This case is an instance of the following basic example.

The Span E **case.** Span E is the bicategory whose objects are those of E and whose arrows $u \to v$ are "spans" (h, k) of maps of E :

$$u \xleftarrow{\;h\;} w \xrightarrow{\;k\;} v.$$

The 2-cells $(h, k) \to (f, g)$ are defined to be those maps p such that

$$fp = h \qquad \text{and} \qquad gp = k.$$

Composition in Span E is given by pullback and $(1_u, 1_u)$ is the identity.

A map f of E becomes the arrow $(1, f)$ of Span E. Such arrows are characterized (up to isomorphism) by the property that they have a right adjoint. For this reason, in a general bicategory, an arrow f having a right adjoint f^o is called a *map*.

It is easy to check that categories internal to E become exactly one-object categories enriched in Span E. Moreover [4] shows that locally internal categories are enriched categories X which admit restrictions along maps. This fact means that, given a map $f : u \to v$ and an object x of X over v, the module $f^o.X(-, x) : X \,\mathrel{+\!\!\!\to}\, v$ is representable :

$$f^o\,.X(-, x) \simeq X(-, x_f).$$

Restrictions along maps are limits indexed by maps. The family **J** of all maps is admissible, hence the locally internal categories are exactly the complete ones with respect to this **J**.

Another example, in the case B = Span E , depends on the fact that, for a locally internal category regarded as a Span E -category, the property of being strong tensored is implied by **J**-completeness with **J** = all modules into one-object categories (see [4], where this property is referred to as *internal completeness).*

Suppose an admissible family **J** is given. To each category X we associate a new category P^oX which is the free **J**-completion.

Definition. The objects of P^oX over u are diagrams of the type :

$$u \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!+\;} I \xrightarrow{F} X$$

where φ is in **J**. The homs are defined by right liftings. Composition and identities are defined by the universal property of liftings :

$$P^oX((F, \varphi), (G, \psi)) = \hom_J(\psi, G^*.F.\varphi)$$

$$\begin{tikzcd} I \arrow[r, "F"] & X \arrow[d, phantom, "\Longrightarrow"] & J \arrow[l, "G"'] \\ u \arrow[u, "\varphi", "+"'] \arrow[rr, "+"] & & v \arrow[u, "\psi"', "+"] \end{tikzcd}$$

Remark. The composite module $G^*.F$ exists even if X is not small :

$$G^*.F(i, j) = X(Fi, Gj).$$

Moreover, given any module $\beta : u \longrightarrow\!\!\!\!\!\!\!+\; X$, the right lifting $\hom_X(G.\psi, \beta)$ exists and is isomorphic to $\hom_J(\psi, G^*.\beta)$. Hence

$$P^oX((F, \varphi), (G.\psi)) \simeq \hom_X(G.\psi, F.\varphi).$$

As a consequence we have that two objects (F, φ) and (G, ψ) are isomorphic in P^oX iff $F.\varphi \simeq G.\psi$.

A functor $F : I \to P^oX$ gives rise to a module $\hat{F} : I \longrightarrow\!\!\!\!\!\!\!+\; X$ defined by $\hat{F}(-, x) = F_i . \varphi_i$, where

$$(e_i \overset{\varphi_i}{\longrightarrow\!\!\!\!\!\!\!+\;} J_i \xrightarrow{F_i} X)$$

is the object Fi of P^oX.

When X is small, P^oX classifies modules into X.

Theorem. *P^oX is J-complete. Moreover any functor $G : X \to Z$ into a J-complete category Z can be extended (uniquely up to equivalence) to a J-continuous functor $G' : P^oX \to Z$.*

Proof. The proof follows the original one by Grothendieck and Verdier. It can also be obtained by suitably adapting to the enriched case the proof of Johnstone and Joyal [10], or that of Tholen [18]. We repeat it in its main lines, in order to fix terminology and to underline the role of admissible families and the generalization to the enriched case.

To assign $F : I \to P^oX$ means to assign the objects

$$Fi = (e_i \underset{\varphi_i}{\longrightarrow\!\!\!\!\!\!\!+\;} J_i \underset{F_i}{\longrightarrow} X)$$

of P^oX, and to specify the effect on arrows :

$$I(i, j) \to \hom_X(F_i . \varphi_i, F_j . \varphi_j).$$

By considering together all the objects of the categories J_i we get a new (small) category J. Namely, the objects of J over u are pairs (i, α) where α is an object over u in J_i. The homs are given by :

$$J((i, \alpha), (j, \beta)) = X(F_i(\alpha), F_j(\beta)).$$

By also considering together the functors F_i we can define a fully faithful functor $\overline{F} : J \to X$ whose effect on objects is $\overline{F}(i, \alpha) = F_i(\alpha)$.

So far we have defined the functor part $\overline{F}$ of the limit $\{\psi, F\}$. Now we define a module $\Lambda : I \nrightarrow J$ by $\Lambda(i, -) = U_i . \varphi_i$, where $U_i : J_i \to J$ is the functor which takes α into (i, α).

The functoriality of U_i depends on the functoriality of F_i. The module properties of Λ can be checked directly :

$$\begin{tikzcd} & J \arrow[dl, "\overline{F}"'] & & \\ X & & I \arrow[ul, "\Lambda"', "|" marking] \arrow[ll, "F", "|" marking] & u \arrow[l, "\psi"', "|" marking] \end{tikzcd}$$

To check that $(\overline{F}, \Lambda . \psi)$ is the limit $\{\psi, F\}$, we observe that $F_i \simeq \overline{F} . U_i$ hence (by the previous remark) the objects F_i and $(\overline{F}, \Lambda(i, -))$ are isomorphic in P^oX.

A calculation gives the result. The crucial point of this proof is to show that the module $\Lambda(i, -)$ is in **J** for each i (so $\Lambda . \psi$ is in **J**). By definition,

$$\Lambda(i, -) = U_i . \varphi_i \qquad \text{with} \qquad \varphi_i \in \mathbf{J}$$

Hence also $\Lambda(i, -) \in \mathbf{J}$.

There is a fully faithful $Y : X \to P^oX$ given by

$$Yx = (e_x \xrightarrow{1} \!\!\!\!\!\!\!+\;\; e_x \xrightarrow{x} X).$$

Let us now consider a functor $G : X \to Z$. If Z is **J**-complete, then $G'(H, \varphi) = \{\varphi, GH\}$ defines a **J**-continuous functor :

$$\begin{tikzcd} I \arrow[r, "H"] & X \arrow[r, "Y"] \arrow[d] & P^oX \arrow[dl, "G'"] \\ u \arrow[u, "\varphi", "|" marking] \arrow[r] & Z & \end{tikzcd}$$

The functor G' extends G and is uniquely defined (up to equivalence). q.e.d.

Remark. With the notation of the previous theorem we have $\hat{F} \simeq \overline{F} . \Lambda$.

Hence, when X is small, the limit $\{\psi, F\}$ can be computed as $\hat{F}.\psi$. Moreover the object (K, φ) of P^oX is isomorphic to the limit of the diagram

$$u \xrightarrow{\ \psi\ } I \xrightarrow{\ F\ } P^oX$$

iff $K.\varphi \simeq \hat{F}.\psi$.

Another consequence of the construction given in the proof of the theorem is that each object (F, ψ) of P^oX is a limit. Namely, $(F, \psi) \simeq \{\psi, YF\}$.

2. THE Sets CASE.

When B = Sets and $\mathbf{J}$ = all modules, then

$$P^oX \simeq (\mathrm{Sets}^X)^{op}.$$

For a general $\mathbf{J}$ we have the following situation.

Any module $\varphi: \mathbf{1} \nrightarrow I$ can be identified with a functor $\bar{\varphi} : I \to$ Sets. Consider the category $\mathrm{el}\,\bar{\varphi}$ of elements of $\bar{\varphi}$: its objects are pairs (i, α) with $\alpha \in \bar{\varphi} i$, its arrows $(i, \alpha) \to (j, \beta)$ are arrows $h : i \to j$ of I such that $\bar{\varphi} h(\alpha) = \beta$. Now, for any (F, φ) we have a canonical decomposition $F = \bar{F}.T^*_\varphi$

$$\begin{array}{ccc} I & \xrightarrow{\ F\ } & X \\ \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \bar{F}} \\ \mathbf{1} & \longrightarrow & \mathrm{el}\,\bar{\varphi} \end{array}$$

where $\bar{F}(i, \alpha) = Fi$ and $T_\varphi : \mathrm{el}\,\bar{\varphi} \to \mathbf{1}$ is the trivial functor.

Hence, if we suppose that $\mathbf{J}$ is closed under the formation of the categories of elements, we have a more classical description of P^oX :

$$\begin{aligned} P^oX((F, \varphi), (G, \psi)) &\simeq P^oX((\bar{F}, T^*_\varphi), (\bar{G}, T^*_\psi)) \\ &\simeq \varprojlim_{(j, \beta)} \hom(T^*_\psi(j, \beta), \bar{G}^*\bar{F}^*.T^*_\varphi(i, \alpha)) \\ &\simeq \varprojlim_{(j, \beta)} \hom(1, \varinjlim_{(i, \alpha)} (\bar{G}^*.\bar{F})((i, \alpha), (j, \beta)) \\ &\simeq \varprojlim_{j} \varinjlim_{i} X(Fi, Gj), \end{aligned}$$

where (j, β) varies in $\mathrm{el}\,\bar{\psi}$ and (i, α) in $\mathrm{el}\,\bar{\varphi}$.

3. THE ČECH CONDITION.

In [2] the shape category of an arrow is defined in any bicategory with enough colimits (following Bourn and Cordier [6]). The aim was to show that basic properties of shape categories rely on a module calculus which is also relevant in dealing with the inverse system approach. An attempt was made to consider the Čech condition in B-cat. Here we introduce the Čech condition relative to an admissible family of modules.

Definition. The functor $G : A \to X$ satisfies the Čech *condition* relative to J if there exist $\sigma : X \to P^oA$ and the natural isomorphisms $F_x.\varphi_x \simeq G^*.x$ where (F_x, φ_x) denotes $\sigma(x)$:

$$\begin{tikzcd} A \arrow[r, "G"] & X \\ J_x \arrow[u, "F_x"] \arrow[ur, phantom, "\simeq"] & e_x \arrow[u, "x"'] \arrow[l, "\varphi_x", "\shortmid" marking] \arrow[ul, "\shortmid" marking] \end{tikzcd}$$

In other words, G^* is of the form $\hat{\sigma}$ for a functor $\sigma : X \to P^oA$.

When A is small and J consists of all modules into small categories, then any G satisfies the Čech condition. The reason for this is that P^oA classifies modules $u \nrightarrow A$. In the Sets case this corresponds to the fact that, when A is small, also $(x \downarrow G)$ is a small category.

Remark. When B = Sets, the previous condition amounts to

$$\varinjlim A(F(i, \alpha), a) \simeq X(x, Ga)$$

for each object a, where the colimit is taken along el $\bar{\varphi}$. In other words

$$\varinjlim_i A(Fi, a) \simeq X(x, Ga).$$

In the classical case, this means that the embedding HPol $\to$ HTop satisfies the Čech condition relative to the class of small cofiltered categories. In a more abstract way it means that F is an A-expansion of x in the sense of Mardešić and Segal [15].

The Čech condition still allows a description of the shape category Sh_G of the functor G :

$$Sh_G(x, y) \simeq hom_A(G^*.y, G^*.x) \simeq P^oA(\sigma(x), \sigma(y)).$$

Moreover, if G is shape adequate (i.e.,

$$X(x, Ga) \simeq Sh_G(x, Ga)),$$

then it can be proved (as in [2]) that each object x, regarded in Sh_G, is the limit $\{F.\varphi, D.G\}$ of the composite

$$A \xrightarrow{G} X \xrightarrow{D} Sh_G$$

indexed by $F.\varphi$.

Čech extensions. Suppose that Z is a J-complete category. Given $\sigma : X \to P^oA$, any $F : A \to Z$ can be extended to X by

$$\check{F}_\sigma x = \{\varphi, FX\}$$

where (X, φ) denotes $\sigma(x)$.

The effect of $\check{F}_\sigma$ on the arrows is defined by the counit

$$\varphi \to (\{\varphi, FX\}, FX\text{-})$$

of the limit $\{\varphi, FX\}$.

In the classical case, this is the Čech extension

$$\check{F}_\sigma x = \varprojlim F.\nu x .$$

Theorem. *Given* $G : A \to X$ *and* $\sigma : X \to P^oA$, *then* $\hat{\sigma} \simeq \check{G}^*$ *iff, for any* $F : A \to Z$ *into a* J *-complete category* Z, *the functor* $\check{F}_\sigma$ *is isomorphic to the right Kan extension* $\mathrm{Ran}_G F$ *of* F *along* G.

Proof. In one direction the proof is an obvious generalization of the classical result that Čech extension = Kan extension for homotopy functors (Dold [9], Lee and Raymond [14]) :

$$\check{F}_\sigma x \simeq \{\varphi, FX\} \simeq \{G^*.x, F\varphi\} \simeq (\mathrm{Ran}_G F)(x).$$

Conversely, let us denote by B^{e_a} the category whose objects over u are arrows $u \to e_a$ and whose homs are right liftings :

$$\hom_{e_a}(x, y) = B^{e_a}(y, x).$$

It is easy to see that $B^{e_a} = P^o e_a$ relative to the admissible family of all modules. Hence it admits small limits indexed by any module. We use the categories B^{e_a} as codomains of representable functors

$$A(-, a) : A \to B^{e_a}.$$

It is easy to check that

$$\{\alpha, A(-, a)\} \simeq \alpha_a \qquad \text{for any} \quad \alpha : u \nrightarrow A.$$

Hence :

$$A(\check{-}, a)_\sigma(x) \simeq (\mathrm{Ran}_G A(-, a))(x)$$

entails

$$(X.\ \varphi)_a \simeq (G^*.x)_a \qquad \text{for each } a .$$

q.e.d

4. THE ADJOINT FUNCTOR THEOREM.

By generalizing a result of Stramaccia [16] relative to the classical case of directed sets, Tholen [20] proved, for B = Sets, that the Čech condition = Pro-adjointness. This is still the case in our general setting. Moreover it can be shown in a precise way that the Čech condition is a solution-set condition.

We denote by $\bar{G} : P^oA \to P^oX$ the extension of $G : A \to X$.

Theorem. *The functor* $G : A \to X$ *satisfies the Čech condition relative to* J *iff* $\bar{G}$ *has a left adjoint.*

Proof. The proof follows the original one by Stramaccia [16]. Suppose that $\bar{G}$ has a left adjoint Λ. Then, for each object a we have :

$$P^oA(\Lambda x,\ a) \simeq P^oX(x,\ \bar{G}a) \simeq X(x,\ Ga) \simeq (G^*.x)(a).$$

We have also :

$$P^oA(\Lambda x,\ a) \simeq \hom_{e_a}(1,\ a^*.K.\varphi) \simeq a^*.K.\varphi \simeq (K.\varphi)(a)$$

where : $\Lambda x = (K,\ \varphi)$. Hence $K.\varphi \simeq G^*.x$ and G satisfies the Čech condition relative to **J**.

Conversely, suppose that G satisfies the Čech condition and $\sigma : X \to P^oA$ is such that $\hat{\sigma} \simeq G^*$. Take an object (F, φ) in P^oX :

$$u \xrightarrow{\ \varphi\ } \!\!\!\!\!\!+\ I \xrightarrow{\ F\ } X.$$

For each object i in I, consider the object

$$\Lambda'(i) = (K_i,\ \psi_i) = \sigma(Fi).$$

The assignment $i \mapsto \Lambda'(i)$ amounts to a functor $\Lambda' : I \to P^oA$ because G satisfies the Čech condition, and the limit $\{\varphi, \Lambda'\}$ provides an object $(K,\ \psi) = \Lambda(F,\ \varphi)$ in P^oA.

This functor $\Lambda : P^oX \to P^oA$ is the required left adjoint to $\bar{G}$. Indeed, by a direct calculation we have that $\Lambda' \simeq G^*.F$ because G satisfies the Čech condition and moreover, $K.\psi \simeq \Lambda'.\varphi$ because $(K,\ \psi)$ is the limit $\{\varphi\ ,\ \Lambda'\}$. Hence $K.\psi \simeq G^*.F.\psi$. So for any object

$$v \xrightarrow{\ \tau\ } \!\!\!\!\!\!+\ J \xrightarrow{\ H\ } A$$

in P^oA we have :

$$P^oX((F,\ \varphi),\ (GH, \tau)) \simeq \hom_J(\tau,\ H^*.G^*.F.\varphi) \simeq \hom_J(\tau,\ H^*.K.\psi)$$

$$\simeq P^oA((K,\ \psi),\ (H, \tau)). \qquad \text{q.e.d.}$$

We are now in a position to prove the Adjoint Functor Theorem. It generalizes a result of Giuli [11], relative to reflective subcategories. For suitable classes of diagrams it is also contained in Tholen [20].

Theorem. *Let* A *be* J*-complete. Then* $G : A \to X$ *has a left adjoint iff :*

(i) it satisfies the Čech condition relative to J *, and*
(ii) it is J *-continuous.*

Proof. If G has a left adjoint, it preserves any limit. Moreover, it satisfies the Čech condition relative to the admissible family consisting of representables (hence it satisfies the Čech condition relative to any $\mathbf{J}$).

Conversely, we show that there exists the right Kan extension $\mathrm{Ran}_G 1$ of 1 along G, and it is preserved by G. Consider $\sigma(x) = (F, \varphi)$ for a given object x. By a calculation involving the universal properties of liftings, we have that $\hom_A(G^*.x, 1)$ exists and is isomorphic to $\hom_I(\varphi, F^*)$. Now we know that :

$$\hom_I(\varphi, F^*) \simeq A(-, \{\varphi, F\}).$$

Hence

$$\{\varphi, F\} \simeq \{G^*.x, 1\} \simeq (\mathrm{Ran}_G 1)(x).$$

It is now enough to observe that $\mathrm{Ran}\ 1$ is preserved by G because it is a limit. q.e.d.

Remark. The theorem provides a variety of "adjoint functor theorems", depending on various completeness assumptions on A.

When B = Sets and $\mathbf{J}$ = all modules into small categories, it gives the classical Freyd Theorem.

When B = Span E and $\mathbf{J}$ = modules into one-object categories, it extends the Freyd Theorem to locally internal categories and internal-completeness.

Example. When $\mathbf{J}$ is the admissible family of adjoint pairs of modules, the theorem becomes simpler. In this case $\mathbf{J}$-complete categories are the Cauchy-complete ones (see [3], or Street [18]).

Under the assumption that A is Cauchy-complete the proof is reduced to an easy calculation. $F.\varphi$ is left adjoint to $\psi.F^*$, hence $F.\varphi$ is representable. So

$$F.\varphi \simeq A(a, -) \quad \text{and} \quad X(x, G-) \simeq A(a, -) :$$

$$\begin{tikzcd} A \arrow[r, "G"] & X \\ I \arrow[u, "F"] \arrow[r, shift right, "\psi"'] & e_x \arrow[u, "x"'] \arrow[l, shift right, "\varphi"'] \end{tikzcd}$$

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