CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

MARCO GRANDIS

On distributive homological algebra, I. RE-categories

Cahiers de topologie et géométrie différentielle catégoriques, tome 25, nº 3 (1984), p. 259-301

<http://www.numdam.org/item?id=CTGDC_1984__25_3_259_0>

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

Vol. XXV-3 (1984)

ON DISTRIBUTIVE HOMOLOGICAL ALGEBRA, I. RE-CATEGORIES

by Marco GRANDIS

Résumé. Cet article est le premier d'une série de trois articles consacrée aux "théories homologiques distributives" (comme le complexe filtré ou le double complexe) et leurs modèles canoniques. Ici nous introduisons les RE-catégories, i.e. des catégories ordonnées involutives généralisant les catégories de relations sur les catégories exactes, et nous étudions leur 2-catégorie **RE** : celle-ci est strictement complète, un fait qui simplifiera notre approche.

0. Introduction.

0.1. *General outline.* Exact categories, in the sense of Puppe-Mitchell [20, 18], and their categories of relations are a flexible frame for studying basic homological facts. This series of three works is devoted to studying theories with values in exact categories and their canonical models, mostly in the distributive case. A review of results appeared in [9].

In Part I exact categories, more precisely their categories of relations, are generalized by RE-categories, i.e. involutive ordered categories satisfying certain conditions. RE-categories form a (strictly) complete 2-category **RE**, where (strict) universal problems can be solved.

Part II will introduce RE-theories and prove the existence of their canonical models, via the completeness of **RE** ; this also proves the existence of *bicanonical* models for theories with values in exact categories. A theory is *distributive* if its classifying exact category $\underline{E}_0$ is so (i.e. it has distributive lattice of subobjects) ; in this case $\underline{E}_0$ is an exact subcategory of I, the distributive exact category of sets and partial bijections, and the *canonical model has a set representation.*

Part III will supply the canonical models for some (distributive) theories of interest in homological algebra : e.g., the bifiltered object, the (discrete or real) filtered complex, the double complex, the filtered differential object. Their classifying categories can be "drawn" in the (discrete or real) plane, yielding quick "graphic" proofs of various results on spectral sequences as well as a tool of investigation which, in the author's opinion, justifies the task of proving (once for all) that the exhibited models are indeed canonical. For the differential filtered object one recovers the Zeeman diagram [23]; actually the need of precise foundations for that representation was the starting point of this research.

A notion of *distributive homological algebra,* concerning distributive theories in exact categories (or in RE-categories) arises from the above approach ; this notion is already present in Zeeman [23], when he points out that the bifiltered complex is not "distributive", and traces back to this fact the difficulty of that theory. Distributive theories not only stand out for having a set representation, but also, from a purely algebraic viewpoint, for the composability of canonical isomorphisms between subquotients, which characterizes distributive exact categories [7]. Last we notice that a non-trivial abelian category is never distributive : the frame of abelian categories seems to be too narrow for studying basic properties of homological systems.

0.2. As concerning Part I, the problem we are interested in can be guessed from the following trivial example.

The 2-category **EX** of exact categories, exact functors and natural transformations has no initial object ; however any *null* exact category $\underline{E}_0$ [1]) is *biinitial* [22] : for any exact category $\underline{E}$ there is an exact functor $\underline{E}_0 \to \underline{E}$, determined up to a unique isomorphism of functors [2]).

More generally, any small graph Δ determines a (strict) universal problem for diagrams $\Delta \to \underline{E}$ ($\underline{E}$ in **EX**) which has *no* solution, while it can be shown that the corresponding *biuniversal* problem has a solution (the bifree exact category generated by Δ), which is determined up to equivalence. It may be remarked that Freyd's Theorem on the initial object cannot be applied, because **EX** is not complete [17].

0.3. As well known, this situation is by no means confined to **EX**, but generally occurs for 2-categories formed by categories possessing certain limits or colimits and by the functors which preserve them (obviously, up to isomorphism). For example, see the canonical models for theories with values in toposes [16].

0.4. However in our case, that is for **EX**, one can observe that the required "limits", essentially kernels and cokernels, are subobjects and quotients and thus can be simulated by *endorelations* : just consider, instead of the subobject $m : M \to A$ (resp. the quotient $p : A \to P$) the projection

$$m\tilde{m} : A \to A \qquad \text{(resp. } \tilde{p}p : A \to A).$$

These endorelations do not present any problem of choice or quotientation, since

$$m : M \rightarrowtail A \qquad \text{and} \qquad n : N \rightarrowtail A$$

([1]) All the objects are zero-objects, that is initial and terminal ; in other words, $\underline{E}_0$ is equivalent to $\underline{1}$.

([2]) For "weak" limits in 2-categories we use the terminology of Street [22] on *bilimits*.

are equivalent monics iff $m\tilde{m} = n\tilde{n}$; they allow to define "~-kernels" (6.2) which are strictly preserved by (the symmetrization of) any exact functor.

Thus we are led to *simulate exactness by involutive ordered categories possessing suitable, uniquely determined, projections*; these are to be strictly preserved by good functors.

0.5. In §1, 2, we consider the 2-category of RO-categories, i.e., categories provided with a regular involution and a consistent order. It should be noticed that RO-transformations are *lax*-natural. Any RO-category $\underline{A}$ can be fully embedded in a factorizing one, Fct($\underline{A}$), whose objects are the projections of the former (§ 3).

In § 4, 5 the 2-category **RE** is introduced : a *RE-category* is a RO-category where any projection has a *numerator* and a *denominator* (still projections) and every object has suitable *null* projections ; RE-*functors* strictly preserve all that.

The connections between **EX** and **RE** are studied in § 6 : any category of relations Rel($\underline{E}$) on an exact category $\underline{E}$ is a RE-category ; conversely, any RE-category $\underline{A}$ is fully embedded in the factorizing RE-category Fct($\underline{A}$), which is isomorphic to the category of relations on its proper morphisms Prp Fct $\underline{A}$ (a *componentwise* exact category) : RE-categories can be characterized as the full subcategories of categories of relations on componentwise exact categories (6.6), or also as the Prj-full involutive subcategories of categories of relations on exact categories (6.8).

In § 7, 8 we introduce the transfer functor

$$\mathrm{Rst}_{\underline{A}} : \underline{A} \longrightarrow \mathit{Mlr}$$

of a RE-category $\underline{A}$ into the RE-category of modular lattices and modular relations [10]. We also study *transfer, distributive, boolean, idempotent* RE-categories ; in Parts II and III distributive and idempotent RE-theories will be the main object of investigation.

Last, §9 proves that **RE** is a complete 2-category.

0.6. *Conventions.* We generally use Mac Lane's [15] terminology for categories and Kelly-Street's [14, 13, 22] for 2-categories. However, a subobject will be a "chosen monic" rather than a class of equivalent monics ; the set of subobjects (resp. quotients) of the object A in the category $\underline{C}$ will be written $\mathrm{Sub}_{\underline{C}}(A)$ (resp. $\mathrm{Quo}_{\underline{C}}(A)$).

We choose, once for all, a reference universe U. A *U-category* $\underline{C}$ has objects and morphisms belonging to U, while we do not require it to have small Hom-sets ; $\underline{C}$ is *small* whenever the set of morphisms (and therefore also the set of objects) belongs to U. We write U-**CAT**, or **CAT** for short, the 2-category of U-categories, their functors and their natural transformations.

An *exact category* $\underline{E}$ (always in the sense of Puppe-Mitchell [20, 18]) is a category with *zero-object* 0 (initial and terminal), in which every morphism factorizes via a conormal epi and a normal monic. In particular every morphism $u : A' \to A''$ has kernels and cokernels, which will be written

$$(1) \qquad \ker u : \mathrm{Ker}\, u \rightarrowtail A' , \qquad \mathrm{cok}\, u : A'' \twoheadrightarrow \mathrm{Cok}\, u$$

and assumed to be, respectively, a subobject and a quotient. A functor between exact categories is *exact* whenever it preserves (up to equivalence) kernels and cokernels.

For exact categories and their categories of relations we refer to [20, 18, 3, 4, 2, 5, 7, 8, 10] ; the last reference contains a short review of results. Here an exact category will always be assumed to be a *well-powered U-category* (hence also well-copowered).

If $F : \mathbf{A} \to \mathbf{B}$ is a 2-functor between 2-categories, and B an object in its codomain, an *i-universal arrow*

$$(A_0, \ b_0 : B \longrightarrow F(A_0))$$

from B to F will be given by an object A_0 of **A** and a morphism b_0 of **B** such that :

a) for every morphism $b : B \to F(A)$ in **B** there is some morphism $a : A_0 \to A$ in **A** verifying $b = F(a)b_0$,

b) for every cell

$$\beta : b_1 \to b_2 : B \to F(A)$$

in **B** there is some cell

$$\alpha : a_1 \to a_2 : A_0 \longrightarrow A$$

in **A** verifying $\beta = F(\alpha)\, b_0$; α is determined by a_1 and a_2 .

Notice that the condition a is formally superfluous, but useful for checkings. The morphism $a : A_0 \to A$ in a is determined up to a unique isomorphic cell α of **A** such that $1_b = F(\alpha)\, b_0$; the object A_0 is determined up to equivalence in **A**.

Every 2-universal arrow is i-universal, and every i-universal one is biuniversal ; i-universal arrows compose in the usual way.

1. RO-categories.

1.1. An *involutive* category $\underline{A} = (\underline{A}, \sim)$ is a category provided with an involution $\sim : \underline{A} \to \underline{A}$, i.e. a contravariant endofunctor, identical on objects and involutory, whose result on the morphism $a : A' \to A''$ will be written $\tilde{a} : A'' \to A'$. An involutive category is selfdual.

Assume now that the involution of $\underline{A}$ is *regular,* that is

$$a = a\tilde{a}a \quad \text{for each morphism } a.$$

An endomorphism $e : A \to A$ is called a *projection* if it is symmetrical and idempotent ($e = \tilde{e} = ee$) ; an equivalent condition is

$$e = \tilde{e}e , \quad \text{or also} \quad e = e\tilde{e} .$$

The projections of A form a set $\mathrm{Prj}_{\underline{A}}(A)$ canonically ordered by :

(1) $\quad e \propto f \quad$ if $\quad e = ef$ (iff $e = fe$, iff $e = fef$).

It is well known that the composition of e, $f \in \mathrm{Prj}(A)$ is always idempotent :

$$ef.ef = eff.eef = ef(ef)^{\sim}ef = ef ;$$

it is a projection iff e and f commute.

1.2. Definition. A *RO-category* $(\underline{A}, \sim, \leqq)$ will be a category $\underline{A}$ provided with a regular involution $\sim$ and with an order relation $\leqq$ on parallel morphisms, consistent with composition and involution. We always assume $\underline{A}$ to be a *U-category with small projection-sets* ; notice that these have two order relations, $\leqq$ and $\propto$ (1.1.1), which are generally different.

For any exact category $\underline{E}$, the category $\mathrm{Rel}(\underline{E})$ of relations on $\underline{E}$ has such a structure (see Calenko [3, 4] ; for generalizations [2, 5] ; the smallness of projection-sets follows from [5], Ch. 3, III-IV).

A RO-category $(\underline{A}, \sim, \leqq)$ has an obvious 2-category structure, with 2-cells given by $\leqq$, and three opposite RO-categories : the *first opposite* $(\underline{A}^*, \sim^*, \leqq)$, *the second opposite* (or *order opposite)* $(\underline{A}, \sim, \geqq)$, and the *biopposite* $(\underline{A}^*, \sim^*, \geqq)$; one easily sees that

$$(\underline{A}, \sim, \leqq) \quad \text{and} \quad (\underline{A}^*, \sim^*, \leqq)$$

are isomorphic, and so the other two.

In the following $(\underline{A}, \sim, \leqq)$, or $\underline{A}$ for short, is a RO-category.

1.3. For any morphism $a : A' \to A''$ there are two *transfer mappings of projections* ([6], § 2.17-18):

(1) $$a_P : \mathrm{Prj}(A') \to \mathrm{Prj}(A''), \qquad a_P(e) = ae\tilde{a},$$

(2) $$a^P : \mathrm{Prj}(A'') \to \mathrm{Prj}(A'), \qquad a^P(f) = \tilde{a}fa \ (= \tilde{a}_P(f))$$

yielding two projections associated with a:

(3) $$\underline{c}(a) = a^P(1_{A''}) = \tilde{a}a \in \mathrm{Prj}(A'),$$

(4) $$\underline{i}(a) = a_P(1_{A'}) = a\tilde{a} \in \mathrm{Prj}(A''),$$

which simulate, respectively, the coimage and the image of a (see 3.6.8).

The transfer mappings preserve $\leqq$ and $\propto$: if $e = ef$ in Prj(A') then

$$a_P(e).a_P(f) = ae\tilde{a} \, . \, af\tilde{a} = aef(\tilde{a}a)f(\tilde{a}a)\tilde{a} = aef\tilde{a}a\tilde{a} = ae\tilde{a} \, .$$

If a and b are composable

(5) $$(ba)_P = b_P a_P, \quad (ba)^P = a^P b^P \, .$$

1.4. Owing to the regularity of the involution, it is easy to see that, for any morphism a :

a) a is monic iff it is a coretraction, iff $\tilde{a}a = 1$ $(\underline{c}(a) = 1)$,
b) a is epi iff it is a retraction, iff $a\tilde{a} = 1$ $(\underline{i}(a) = 1)$,
c) a is monic and epi iff it is iso, iff $\tilde{a}$ and a are reciprocal isos.

It follows that epi-monic factorizations $a = a_2 a_1$ when existing, are unique up to isomorphism.

1.5. One verifies easily that the mapping (1) (resp. (2)) :

(1) $$\mathrm{Sub}_{\underline{A}}(A) \to \mathrm{Prj}_{\underline{A}}(A), \qquad h \mapsto h\tilde{h}$$

(2) $$\mathrm{Quo}_{\underline{A}}(A) \to \mathrm{Prj}_{\underline{A}}(A), \qquad k \mapsto \tilde{k}k$$

is an embedding (resp. anti-embedding) of ordered sets with regard to $\propto$: in particular $\underline{A}$ is well-powered and well-copowered. The mappings (1) and (2) are consistent with the anti-isomorphism

(3) $$\mathrm{Sub}_{\underline{A}}(A) \to \mathrm{Quo}_{\underline{A}}(A), \qquad h \mapsto \tilde{h}$$

and will be seen (3.3) to be surjective (for every object A) iff $\underline{A}$ has epi-monic factorizations.

In any case, the projections of an object in a RO-category $\underline{A}$ (more generally in any category provided with a regular involution) substitute advantageously the subobjects also when (1) is a bijection, because the projections do not present any problem of choice of representants or of quotientation.

1.6. A *restriction* of A will be an endomorphism $e : A \to A$ such that $e \leqq 1$: it is hence a projection, as

$$e = e\tilde{e}e \leqq e\tilde{e} \leqq e, \quad \text{and} \quad e = e\tilde{e}).$$

All restrictions of A build a small set Rst(A), which is a semilattice with regard to composition: if $e, f \in$ Rst(A) then $ef \leqq 1$ is a restriction, hence a projection and

$$ef = (ef)^{\sim} = fe \quad ;$$

moreover $\leqq$ and $\propto$ coincide on Rst(A) : if $e \leqq f$ then

$$ef \leqq e.1 = e = e.e \leqq e.f$$

conversely, if $e \propto f$ then

$$e = ef \leqq 1.f = f .$$

Analogously, the *corestrictions* $e : A \to A$, $e \geqq 1$, form a semilattice Crs(A) $\subset$ Prj(A), in which $e \leqq f$ iff $e \infty f$.

1.7. A morphism $u: A' \to A''$ of the RO-category $\underline{A}$ is *proper* if :

(1) $\qquad \underline{c}(u) = \tilde{u}u \geqq 1_{A'}, \qquad \underline{i}(u) = u\tilde{u} \leqq 1_{A''}.$

These morphisms form a subcategory Prp($\underline{A}$) of $\underline{A}$, non-closed under the involution, whose induced order is trivial :

(2) $\qquad$ if $u, v \in$ Prp(A) and $u \leqq v$ then $u = v$

because

$$v = v.1 \leqq v(\tilde{u}u) \leqq (v\tilde{v})u \leqq 1.u = u.$$

In the above example $\underline{A}$ = Rel($\underline{E}$) (1.2), one recovers the first category : $\underline{E}$ = Prp($\underline{A}$) [3]).

1.8. A morphism $a : A' \to A''$ is said to be *null* if, for every $a': A'' \to A'$, $aa'a = a$. Null morphisms form an ideal $\underline{N}$ = Nul($\underline{A}$) of $\underline{A}$ (the composition of any morphism with a null one is null). Other trivial properties ((7) follows from (5), (6) and (2)) :

(1) $a \in \underline{N}$ iff $\tilde{a} \in \underline{N}$, iff $\underline{c}(a) \in \underline{N}$, iff $\underline{i}(a) \in \underline{N}$,

(2) if $a, a' \in \underline{N}(A', A'')$, $\underline{c}(a) = \underline{c}(a')$, $\underline{i}(a) = \underline{i}(a')$, then $a = a'$,

(3) if $a \in \underline{N}(A, A)$, then $a^2 = a$,

(4) if $a, a' \in \underline{N}(A, A)$ and $aa' = a'a$, then $a = a'$,

([3]) We shall always suppose that the construction of Rel($\underline{E}$) is carried out so that this *equality* holds.

(5) if $e_0 \in \mathrm{Rst}(A) \cap \underline{N}$, then $e_0 \leqq a$, for any $a : A \to A$,

(6) if $e_0 \in \mathrm{Crs}(A) \cap \underline{N}$, then $e_0 \geqq a$ for any $a : A \to A$,

(7) if $u, v : A' \to A''$ are proper and null, then $u = v$.

1.9. An object A_0 is said to be *null* if it has a unique endomorphism, that is if its identity is null ; if A_0 and A_1 are null objects, *connected* in $\underline{A}$ [4]), there exist unique isomorphisms $A_0 \rightleftarrows A_1$: actually, two such morphisms are necessarily reciprocal.

Any morphism which factorizes through a null object (hence by a null identity) is a null morphism.

2. The 2-category RO.

2.1. A *RO-functor* $F : \underline{A} \to \underline{B}$ is a functor between RO-categories, which preserves involution and order. It also preserves : projections and their canonical order $\propto$, transfer mappings

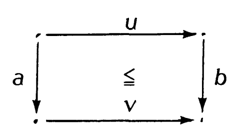
$$(F(a_P\, e) = (Fa)_P (Fe)_P),$$

the operators c and *i*, monics and epis, restrictions and corestrictions, proper morphisms. It need not preserve null morphisms and null objects (a counterexample with $\underline{A} = \underline{1}$ can be easily given).

A faithful RO-functor F reflects projections and their canonical order, monics, epis, isos, null morphisms and null objects.

A RO-functor $F : \underline{A} \to \underline{B}$ has a restriction Prp F: Prp $\underline{A} \to$ Prp $\underline{B}$; on the other hand, a zero-preserving functor $F_0 : \underline{E} \to \underline{E}'$ between exact categories extends to a RO-functor $F : \mathrm{Rel}(\underline{E}) \to \mathrm{Rel}(\underline{E}')$ iff F_0 is exact ([5], Theorem 6.15); in such a case F is (trivially) uniquely determined, and will be written $\mathrm{Rel}(F_0)$.

2.2. A *RO-square* of $\underline{A}$ will be a square diagram in $\underline{A}$, of the following type :

(1) 

$$u, v \in \mathrm{Prp}\, \underline{A}, \qquad va \leqq bu, \quad u\tilde{a} \leqq \tilde{b}v,$$

where the second and third condition are equivalent (when the first holds) : if $va \leqq bu$, then

([4]) In the *involutive* category $\underline{A}$ this simply means that $\underline{A}(A_0, A_1)$ is not empty.

$$u\tilde{a} \leq u\tilde{a}(\tilde{v}v) = u(\tilde{a}\tilde{v})v \leq u(\tilde{u}\tilde{b})v = (u\tilde{u})\tilde{b}v \leq \tilde{b}v.$$

Obviously, RO-squares can be composed, horizontally and vertically ; the vertical composition has a *vertical involution,* which is regular [5].

By 1.7.2, the RO-square (1) is commutative when a and b are proper.

2.3. A *RO-transformation*

$$\alpha : F \rightarrow G : \underline{A} \rightarrow \underline{B}$$

between parallel RO-functors F and G is a family $(\alpha A)_{A \in \mathrm{Ob}\underline{A}}$ satisfying :

a) for any $A \in \mathrm{Ob}\, \underline{A}$, $\alpha A : FA \rightarrow GA$ is a *proper* morphism,

b) for every $a : A' \rightarrow A''$ in $\underline{A}$, the following square is RO[6])

(1)
$$\begin{array}{ccc} FA' & \xrightarrow{\alpha A'} & GA' \\ {\scriptstyle Fa}\downarrow & \leq & \downarrow{\scriptstyle Ga} \\ FA'' & \xrightarrow{\alpha A''} & GA'' \end{array}$$

$$(\alpha A'')(Fa) \leq (Ga)(\alpha A').$$

Remark that the square (1) is commutative when $a \in \mathrm{Prp}\, \underline{A}$ (2.2) : thus the RO-transformation α determines a *natural* transformation

$$\mathrm{Prp}\alpha : \mathrm{Prp}\, F \rightarrow \mathrm{Prp}\, G : \mathrm{Prp}\, \underline{A} \rightarrow \mathrm{Prp}\, \underline{B},$$

with $(\mathrm{Prp}\, \alpha)A = \alpha A$, on any object A.

2.4. RO-transformations have an obvious *vertical composition :* given another RO-transformation

$$\beta : G \rightarrow H : \underline{A} \rightarrow \underline{B},$$

we get

(1) $$\beta . \alpha : F \rightarrow H : \underline{A} \rightarrow \underline{B}, \quad (\beta . \alpha) A = (\beta A)(\alpha A)$$

by vertical composition of RO-squares. The vertical composition of RO-transformations is associative, and has obvious identities

$$1_F : F \rightarrow F : \underline{A} \rightarrow \underline{B}.$$

Remark that the RO-transformation

([5]) The category of RO-squares of $\underline{A}$ and vertical composition has an obvious RO-structure, which yields the cotensor product $2 \pitchfork A$ ([13] ; see also 9.6).

([6]) In other words α is lax-natural according to [14, 12] ("quasi-natural" according to [11]).

$$\alpha : F \to G : \underline{A} \to \underline{B}$$

is an isomorphism between F and G (with regard to the vertical composition) iff the following conditions hold :

(2) for any object A, αA is *iso*, in $\underline{B}$,

(3) for any morphism $a : A' \to A''$ the square 2.3.1 is *commutative* in $\underline{B}$.

Actually, the sufficiency of these being obvious, suppose that

$$\beta : G \to F : \underline{A} \to \underline{B}$$

is reciprocal to α ; then, for any object A, αA and βA are reciprocal isos, and for any morphism $a : A' \to A''$ the commutativity of 2.3.1 follows from :

$$(\alpha A'')(Fa) = (\alpha A'')(Fa)(\beta A')(\alpha A') \geq (\alpha A'')(\beta A'')(Ga)(\alpha A') = (Ga)(\alpha A').$$

2.5. RO-transformations have also a *horizontal composition :* if

$$\gamma : F' \to G' : \underline{B} \to \underline{C}$$

is RO take for any A :

(1) $$(\gamma\alpha)A = G'(\alpha A).\gamma(FA) = \gamma(GA).F'(\alpha A) : F'FA \to G'GA$$

where the second equality comes from the square

(2) $$\begin{array}{ccc} F'FA & \xrightarrow{\gamma(FA)} & G'FA \\ \downarrow{\scriptstyle F'(\alpha A)} & & \downarrow{\scriptstyle G'(\alpha A)} \\ F'GA & \xrightarrow{\gamma(GA)} & G'GA \end{array}$$

which is commutative because γ is RO and αA is proper. Now

$$\gamma.\alpha : F'F \longrightarrow G'G : \underline{A} \to \underline{C}$$

is RO, by the horizontal composition of RO-squares.

The horizontal composition of RO-transformations is associative, and has identities

$$1_{1_{\underline{A}}} : 1_{\underline{A}} \longrightarrow 1_{\underline{A}} : \underline{A} \to \underline{A}.$$

2.6. RO-categories, RO-functors and RO-transformations, with the horizontal and vertical compositions, build obviously a 2-category **RO** (or also U-**RO**) : the "interchange law" and other axioms are easily verified, in the same way that in **CAT.**

There is a canonical 2-functor

(1) $$\text{Prp} : \mathbf{RO} \to \mathbf{CAT}$$

which has already been defined in 1.7, 2.1 and 2.3.

2.7. Let **EX** (or U-**EX**) denote the sub-2-category of **CAT** consisting of exact categories (0.6), exact functors and natural transformations. There is also a 2-functor

(1) $$\text{Rel} : \mathbf{EX} \to \mathbf{RO}$$

whose composition with Prp : **RO** → **CAT** is the inclusion. It has already been considered on objects (1.2) and morphisms (2.1). Now, if

$$\alpha : F \to G : \underline{E} \to \underline{E}'$$

is a 2-cell in **EX**, take

(2) $$\text{Rel}\,\alpha : \text{Rel}\,F \to \text{Rel}\,G : \text{Rel}\,\underline{E} \to \text{Rel}\,\underline{E}'$$

(3) $$(\text{Rel}\,\alpha)A = \alpha A\,, \quad \text{for any} \quad A \in \text{Ob}(\underline{E}),$$

so that (Rel α)A is always proper ; for any $a \in \text{Rel}\,\underline{E}(A', A'')$ the "quaternary factorization" $a = n\tilde{q}p\tilde{m}$ ([5], § 1.5), yields the following *commutative* diagram of $\underline{E}'$:

(4) $$\begin{array}{ccccc}
A' & & FA' & \xrightarrow{\alpha A'} & GA' \\
m\uparrow & & Fm\uparrow & & \uparrow Gm \\
A_1 & & FA_1 & \xrightarrow{\alpha A_1} & GA_1 \\
p\downarrow & & Fp\downarrow & & \downarrow Gp \\
A_2 & & FA_2 & \xrightarrow{\alpha A_2} & GA_2 \\
q\uparrow & & Fq\uparrow & & \uparrow Gq \\
A_3 & & FA_3 & \xrightarrow{\alpha A_3} & GA_3 \\
n\downarrow & & Fn\downarrow & & \downarrow Gn \\
A'' & & FA'' & \xrightarrow{\alpha A''} & GA''
\end{array}$$

a fortiori all the squares are RO-squares ; by vertical involution and vertical composition (2.2), one gets the RO-square 2.3.1.

2.8. Remark. In any RO-category *the order ≦ between morphisms is determined by its restriction to projections,* in the following way : $a \leq b$ iff

(1) $$a\tilde{b} = a\tilde{b}.a\tilde{b}, \qquad \underline{c}(a) \leq \underline{c}(b), \qquad \underline{i}(a\tilde{b}) \leq \underline{i}(b).$$

Actually, if $a \leq b$:

$$a\tilde{b} = a\tilde{a}a\tilde{b} \leq (a\tilde{b}.a\tilde{b}) \leq a\tilde{b}b\tilde{b} = a\tilde{b}\ ;$$

conversely, if (1) holds :

$$a = a\tilde{a}a \leqq a\tilde{b}b = (a\tilde{b})(b\tilde{a})(a\tilde{b})b = (a\tilde{b}b\tilde{a}).(b\tilde{a}a\tilde{b})b$$
$$\leqq \underline{i}(a\tilde{b}).(b\tilde{b}b\tilde{b})b \leqq \underline{i}(b)b = b.$$

Analogously : $a = b$ iff

(2) $$a\tilde{b} = a\tilde{b}.a\tilde{b}, \quad \underline{c}(a) = \underline{c}(b), \quad \underline{i}(a) = \underline{i}(b).$$

2.9. It follows that an involution preserving functor $F : \underline{A} \to \underline{B}$ between RO-categories is a RO-functor iff it preserves $\leqq$ between projections ; moreover a RO-functor is faithful iff it reflects idempotent endomorphisms (from endomorphisms) and is faithful on projections.

3. Factorizing RO-categories.

$\underline{A}$ is always a RO-category.

3.1. We say that the RO-category $\underline{A}$ is *factorizing* (or *FRO-category*) if any morphism a has an epi-monic factorization $a = a_2 a_1$ (necessarily unique up to isomorphism, by 1.4) ; then

(1) $$\underline{c}(a_1) = \underline{c}(a), \qquad \underline{i}(a_1) = 1,$$

(2) $$\underline{c}(a_2) = 1, \qquad \underline{i}(a_2) = \underline{i}(a)$$

which also proves that if a is proper, we have an epi-monic factorization of a in Prp $\underline{A}$.

These categories determine a sub-2-category **FRO** of **RO**.

3.2. If $e : A \to A$ is an endomorphism in the RO-category $\underline{A}$, with epi-monic factorization $e = mp$, it is easy to verify that :

a) e is idempotent iff $pm = 1$,
b) e is a projection iff $p = \tilde{m}$,
c) e is a restriction iff $p = \tilde{m}$ and $m \in$ Prp $\underline{A}$,
d) e is a corestriction iff $p = \tilde{m}$ and $p \in$ Prp$\underline{A}$.

3.3. Proposition. The following conditions are equivalent :

a) $\underline{A}$ is factorizing,
b) for any object A, the embedding 1.5.1 :

$$\mathrm{Sub}_{\underline{A}}(A) \to \mathrm{Prj}_{\underline{A}}(A)$$

is an isomorphism of ordered sets (with regard to $\propto$).
c) for any object A, the anti-embedding 1.5.2 :

$$\mathrm{Quo}_{\underline{A}}(A) \longrightarrow \mathrm{Prj}_{\underline{A}}(A)$$

is an anti-isomorphism of ordered sets (with regard to $\propto$).

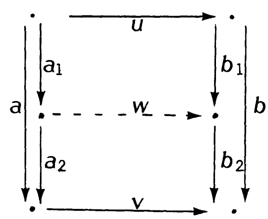
Proof. a $\Rightarrow$ b by 3.2 ; b $\Rightarrow$ c by the anti-isomorphism 1.5.3 ; c $\Rightarrow$ a : if $a : A \to A'$ is a morphism and $\underline{c}(a) = \tilde{p}p$ ($p \in \mathrm{Quo}(A)$), then

$$a = a\tilde{a}a = (a\tilde{p})p,$$

and $a\tilde{p}$ is monic since

$$(p\tilde{a})(a\tilde{p}) = p(\tilde{p}p)\tilde{p} = (p\tilde{p})(p\tilde{p}) = 1\,.$$

3.4. Lemma. Consider the (possibly non-commutative) diagram (1) in the RO-category $\underline{A}$:

(1) 

where u, v are proper and $a = a_2a_1$, $b = b_2b_1$ are epi-monic factorizations. Then the outer square is RO (i.e., $va \leq bu$) iff there exists a proper morphism w such that the inner squares are so ; in such a case

$$w = b_1u\tilde{a}_1 = \tilde{b}_2va_2.$$

Proof. First suppose that the outer square is RO, and take $w = b_1u\tilde{a}_1$; then

$$wa_1 = b_1u\tilde{a}_1a_1 = b_1u\tilde{a}a \leqq b_1\tilde{b}va \leqq b_1\tilde{b}bu = b_1\tilde{b}_1b_1u = b_1u,$$
$$b_2w = b_2b_1u\tilde{a}_1 \geqq va\tilde{a}_1 = va_2a_1\tilde{a}_1 = va_2\,,$$
$$\tilde{w}w = \tilde{w}\tilde{b}_2b_2w \geqq (\tilde{a}_2\tilde{v})(va_2) \geqq \tilde{a}_2a_2 = 1,$$
$$w\tilde{w} = wa_1\tilde{a}_1\tilde{w} \leqq (b_1u)(\tilde{u}\tilde{b}_1) \leqq b_1\tilde{b}_1 = 1\,.$$

Conversely, if there exists a proper morphism w such that the inner squares are RO, the global square is so and :

$$w = wa_1\tilde{a}_1 \leqq (b_1u\tilde{a}_1) \leqq b_1\tilde{b}_1w = w,$$
$$w = w\tilde{a}_2a_2 \leqq (\tilde{b}_2va_2) \leqq \tilde{b}_2b_2w = w.$$

3.5. We associate to any RO-category $\underline{A}$ a FRO-category $\mathrm{Fct}(\underline{A})$, which will be seen (3.8) to satisfy the obvious i-universal problem. The objects of Fct $\underline{A}$ are the projections of $\underline{A}$, its morphisms are the triples :

(1) $$(a\ ;\ e,\ f) : e \to f,$$

(2) $$a \in \underline{A}(\mathrm{Dom}\ e,\ \mathrm{Dom}\, f),\qquad a = ae = fa$$ [7]).

The composition, involution and order in Fct $\underline{A}$ are obviously :

(3) $$(b\ ;\ f,\ g).(a\ ;\ e,\ f) = (ba\ ;\ e,\ g),$$

(4) $$(a\ ;\ e,\ f)^{\sim} = (\tilde{a};\ f,\ e),$$

(5) $$(a\ ;\ e,\ f) \leqq (a'\ ;\ e,\ f) \quad \text{iff} \quad a \leqq a' \ \text{in}\ \underline{A}.$$

3.6. So Fct $\underline{A}$ is a RO-category, and :

(1) $$1_e = (e\ ;\ e,\ e),$$

(2) $(a\ ;\ e,\ f)$ is a projection iff $e = f$, a is a projection and $a \propto e$,

(3) $\underline{c}(a\ ;\ e,\ f) = (\underline{c}(a)\ ;\ e,\ e)\ ;\quad \underline{i}(a\ ;\ e,\ f) = (\underline{i}(a)\ ;\ f,\ f),$

(4) $(a\ ;\ e,\ f)$ is monic iff $\underline{c}(a) = e$, epi iff $\underline{i}(a) = f$,

(5) the projection $(e\ ;\ f,\ f)$ is | a restriction iff $e \leqq f$,
| a corestriction iff $e \geqq f$,

(6) $(a\ ;\ e,\ f)$ is proper iff $\underline{c}(a) \geqq e$ and $\underline{i}(a) \leqq f$,

(7) $(a\ ;\ e,\ f)$ is null iff a is null in $\underline{A}$.

Moreover Fct $\underline{A}$ has epi-monic factorizations :

(8) $$\begin{array}{ccc} e & \xrightarrow{\ a\ } & f \\ {\scriptstyle \tilde{a}a}\downarrow & & \uparrow{\scriptstyle a\tilde{a}} \\ \tilde{a}a & \xrightarrow{\ a\ } & a\tilde{a} \end{array}$$

where we write $e \xrightarrow{a} f$ for $(a\ ;\ e,\ f) : e \to f$.

3.7. It is now easy to define a 2-functor

(1) $$\mathrm{Fct} : \mathbf{RO} \to \mathbf{FRO}$$

so that, for any RO-transformation $\alpha : F \to G : \underline{A} \to \underline{B}$,

(2) $$(\mathrm{Fct}\ F)e = Fe\ ;\qquad (\mathrm{Fct}\ F)(a\ ;\ e,\ f) = (Fa\ ;\ Fe,\ Ff),$$

(3) $$(\mathrm{Fct}\ \alpha)\, e = (Ge.\alpha A.Fe\ ;\ Fe,\ Ge),\quad \text{where}\quad A = \mathrm{Dom}\ e\ .$$

[7] Equivalently : $a = fae$.

We verify only that Fct α is really a RO-transformation from Fct F to Fct G ; actually the morphism (3) is proper in Fct $\underline{B}$ by 3.6.6, and

(4) $\underline{c}(Ge.\alpha A.Fe) = Fe.(\alpha A)^{\sim}(Ge.\alpha A)Fe \geqq Fe.(\alpha A)^{\sim}.(\alpha A.Fe)Fe \geqq Fe,$

(5) $\underline{i}(Ge.\alpha A.Fe) = Ge(\alpha A.Fe)(\alpha A)^{\sim}.Ge \leqq Ge(Ge.\alpha A)(\alpha A)^{\sim}Ge \leqq Ge$

while every $(a\ ;\ e,\ f)$ in $\mathrm{Fct}(\underline{A})$ supplies the following RO-square in Fct $\underline{B}$ (where $A' = \mathrm{Dom}\ e$, $A'' = \mathrm{Dom}\ f$) :

(6)
$$\begin{array}{ccc} Fe & \xrightarrow{Ge.\alpha A'.Fe} & Ge \\ \Big\downarrow Fa & \leqq & \Big\downarrow Ga \\ Ff & \xrightarrow[Gf.\alpha A''.Ff]{} & Ff \end{array}$$

(7) $(Gf.\alpha A''.Ff)Fa = Gf(\alpha A''.Fa)Fe \leqq Gf(Ga.\alpha A')Fe = Ga(Ge.\alpha A'.Fe)$

3.8. The canonical full embedding of any RO-category $\underline{A}$ in Fct $\underline{A}$

(1) $\eta\underline{A} : \underline{A} \to \mathrm{Fct}\ \underline{A} :\ A \mapsto 1_A\ ;\ a \mapsto (a\ ;\ 1_{\mathrm{Dom}\ a}, 1_{\mathrm{Cod}\ a})$

gives a 2-natural transformation $\eta : 1 \to T.\mathrm{Fct} : \mathbf{RO} \to \mathbf{RO}$, where $T : \mathbf{FRO} \longrightarrow \mathbf{RO}$ is the inclusion.

Moreover (1) is an i-universal arrow (0.6) from the object $\underline{A}$ to the 2-functor T.

Actually, every RO-functor $F : \underline{A} \to T(\underline{B})$ towards a FRO-category extends to a RO-functor $G : \mathrm{Fct}(\underline{A}) \to \underline{B}$, in the following way. For any projection e' of $\underline{B}$, choose a monomorphism $m_{e'}$ so that $e' = m_{e'}(m_{e'})^{\sim}$ (3.2 b) and take :

(2) $$G(e) = \mathrm{Dom}(m_{Fe}),$$

(3) $$G(a\ ;\ e,\ f) = (m_{Ff})^{\sim}(Fa)m_{Fe} : Ge \to Gf .$$

Thus, G is a RO-functor

(4) $$G(b\ ;\ f,\ g).G(a\ ;\ e,\ f)) = \tilde{m}_{Fg}(F\ b)m_{Ff}.\tilde{m}_{Ff}(Fa)m_{Fe} =$$
$$= \tilde{m}_{Fg}(Fb.Ff.Fa)m_{Fe} = \tilde{m}_{Fg}.F(bfa).m_{Fe} = G(ba\ ;\ e,\ g)$$

which extends F, as we assume that $m_{1_B} = 1_B$ for each object B of $\underline{B}$.

Suppose now that

$$\varphi : F_1 \to F_2 : \underline{A} \to T(\underline{B})$$

is a RO-transformation, and that $G_i : \mathrm{Fct}(\underline{A}) \to \underline{B}$ extends F_i ($i = 1, 2$). For each $e \in \mathrm{Prj}(A)$ the epi-monic factorization of $(e\ ;\ 1_A, 1_A)$ in $\mathrm{Fct}(\underline{A})$:

$$(e;\, 1_A,\, 1_A) = (e\,;e,\, 1_A).(e\,;\, 1_A,\, e) \tag{5}$$

is transformed by G_i into an epi-monic factorization of

$$G_i(e;\, 1_A,\, 1_A) = G_i(\eta\, A(e)) = F_i(e)\,;$$

therefore, by 3.4, there is a unique proper morphism of $\underline{B}$

$$\gamma_e : G_1(e) \to G_2(e) \tag{6}$$

such that the inner squares of (7) are RO-squares :

$$\begin{array}{ccccc} F_1(A) = G_1(1_A) & \xrightarrow{G_1(e\,;\,1,\,e)} & G_1(e) & \xrightarrow{G_1(e\,;\,e,\,1)} & G_1(1_A) = F_1(A) \\ \Big\downarrow \varphi_A & \Vert\wedge & \Big\downarrow \gamma_e & \Vert\wedge & \Big\downarrow \varphi_A \\ F_2(A) = G_2(1_A) & \xrightarrow{G_2(e\,;\,1,\,e)} & G_2(e) & \xrightarrow{G_2(e\,;\,e,\,1)} & G_2(1_A) = F_2(A) \end{array} \tag{7}$$

hence a unique FRO-transformation $\gamma : G_1 \to G_2$ such that $\gamma.\eta\,\underline{A} = \varphi$.

3.9. It is easy to see that this functor G is faithful iff F is iso.

Now, if the RO-category $\underline{A}$ is factorizing it follows (by taking $F = 1_{\underline{A}}$) that there is a faithful functor $G : \mathrm{Fct}\,\underline{A} \to \underline{A}$ which is a retraction ($G.\eta\underline{A} = 1_{\underline{A}}$) : $\underline{A}$ and Fct $\underline{A}$ are equivalent categories.

4. RE-categories.

We introduce here our generalization of (the categories of relations on) exact categories, essentially based on the fact that in a RO-category Rel($\underline{E}$) every projection $e : A \to A$ is associated with a subquotient H/K of A with regard to $\underline{E}$ [5], hence it has a numerator

$$\underline{n}(e) = (A \leftarrow\!\!\prec H \rightarrowtail A)$$

and a denominator

$$\underline{d}(e) = (A \leftarrow\!\!\prec K \rightarrowtail A)\,;$$

the latter is also determined by the associated corestriction

$$\underline{d}^c(e) = (A \twoheadrightarrow A/K \twoheadleftarrow A).$$

The null restriction

$$\omega_A = (A \leftarrow\!\!\prec 0 \rightarrowtail A)$$

and the null corestriction

$$\Omega_A = (A \twoheadrightarrow 0 \twoheadleftarrow A)$$

will also be of interest.

4.1. Definition. A *RE-category* is a triple $\underline{A} = (\underline{A}, \sim, \leqq)$ satisfying :

(RE.0) $\underline{A}$ is a RO-category,
(RE.1) for every projection e
a) there exists exactly one restriction $\underline{n}(e)$ (the *numerator* of e) such that $e \propto \underline{n}e \leqq e$.
b) there exists exactly one corestriction $\underline{d}^c(e)$ (the ^{c}*- denominator* of e [8])) such that $e \propto \underline{d}^c e \geqq e$.

(RE.2) Every object A has a null restriction ω_A and a null corestriction Ω_A (unique by 1.8.5-6).

The order duality (1.2) turns numerators into ^{c}-denominators and null restrictions into null corestrictions.

4.2. Lemma. In the RO-category $\underline{A}$ satisfying (RE.1), with

$$e, e', e'', f \in \text{Prj}(A) :$$

(1) $e = \underline{n}e.\underline{d}^c e = \underline{d}^c e.\underline{n}e$,
(2) if $e' \leqq 1 \leqq e''$ and $e = e'.e''$ then $e' = \underline{n}e$, $e'' = \underline{d}^c e$,
(3) if $f \geqq 1$ then $\underline{n}(efe) = \underline{n}e$,
(4) if $f \leqq 1$ then $\underline{d}^c(efe) = \underline{d}^c(e)$.

Proof. (1)

$$e = \underline{n}e.e \leqq (\underline{n}e.\underline{d}^c e) \leqq e.\underline{d}^c e = e ;$$

(2) : obvious ; (3) : if $f \geqq 1$, $\underline{n}e$ is the numerator of the projection efe, since $efe \propto \underline{n}e \leqq e \leqq efe$.

4.3. Lemma. In the RO-category $\underline{A}$ satisfying (RE.1), the following conditions on $a, b \in \underline{A}(A', A'')$ are equivalent :

a) $a \leqq b$,
b) there exist $e \in \text{Rst}(A')$ and $f \in \text{Crs}(A'')$ such that $fa = be$,
c) there exist $e \in \text{Rst}(A')$ and $f \in \text{Crs}(A'')$ such that

$$fa = be, \quad ae = a, \quad fb = b,$$

d) $\underline{d}^c(b\tilde{b}).a = b.\underline{n}(\tilde{a}a)$.
In particular :

(1) if $a \leqq b$, $\underline{n}(\tilde{a}a) = \underline{n}(\tilde{b}b)$, $\underline{d}^c(a\tilde{a}) = \underline{d}^c(b\tilde{b})$ then $a = b$.

([8]) The denominator, a restriction, will be defined in 4.8.

Proof. It is obvious that d $\Rightarrow$ c $\Rightarrow$ b $\Rightarrow$ a ; a $\Rightarrow$ d :

$$\underline{d}^C (b\tilde{b}).a = \underline{d}^C(b\tilde{b}).a.\underline{n}(\tilde{a}a) \leqq \underline{d}^C(b\tilde{b}).b.n(\tilde{a}a) = b.\underline{n}(\hat{a}a),$$

$$\underline{d}^C (b\tilde{b}).a \geqq \text{[illegible]}\ b(\hat{a}a) \geqq b.\underline{n}(aa).$$

4.4. Proposition. If $\underline{A}$ is a RO-category satisfying (RE.1), and e, f in Prj(A) :

a) $e \propto f$ iff ($\underline{n}e \propto \underline{n}f$ and $\underline{d}^C e \propto \underline{d}^C f$) iff ($\underline{n}e \leqq \underline{n}f$ and $\underline{d}^C e \geqq \underline{d}^C f$),

b) $e \leqq f$ iff ($\underline{n}e \propto \underline{n}f$ and $\underline{d}^C e \infty \underline{d}^C f$) iff ($\underline{n}e \leqq \underline{n}f$ and $\underline{d}^C e \leqq \underline{d}^C f$).

Proof. The right-hand equivalences follow from 1.6.

a) If $e \propto f$ the projections (1.6) :

(1) $$e' = \underline{n}f.\underline{n}e \leqq 1, \quad e'' = \underline{d}^C e.\underline{d}^C f \geqq 1$$

verify

(2) $$e'.e'' = \underline{n}f.\underline{n}e.\underline{d}^C e.\underline{d}^C f = \underline{n}f.e.\underline{d}^C f = \underline{n}f.fef.\underline{d}^C f = fef = e,$$

hence (4.2.2)

$$\underline{n}e = e', \quad \underline{d}^C e = e'' \quad \text{and so} \quad \underline{n}e \propto \underline{n}f, \quad \underline{d}^C e \propto \underline{d}^C f.$$

Conversely, when these conditions hold :

(3) $$fef = \underline{n}f(\underline{d}^C f.\underline{d}^C e)(\underline{n}e.\underline{n}f)\underline{d}^C f = \underline{n}f.\underline{d}^C e.\underline{n}e.\underline{d}^C f =$$
$$= \underline{n}f.\underline{n}e.\underline{d}^C e.\underline{d}^C f = \underline{n}e.\underline{d}^C e = e.$$

b) By 4.3, if $e \leqq f$ then $\underline{d}^C f.e = f.\underline{n}e$ and :

(4) $$\underline{n}e(\underline{d}^C f.\underline{d}^C e)\underline{n}e = \underline{n}e.\underline{d}^C f.e = \underline{n}e.f.\underline{n}e = (\underline{n}e.\underline{n}f)\underline{d}^C f(\underline{n}e.\underline{n}f)$$

and applying 4.2.3 to the first and last term of (4) one gets :

(5) $$\underline{n}e = \underline{n}(\underline{n}e(\underline{d}^C f.\underline{d}^C e)\underline{n}e) = \underline{n}(\underline{n}e.\underline{n}f(\underline{d}^C f)\underline{n}e.\underline{n}f) = \underline{n}e.\underline{n}f,$$

that is $\underline{n}e \propto \underline{n}f$; by order duality, $\underline{d}^C e \infty \underline{d}^C f$. The converse property follows at once from 4.2.1.

4.5. A RO-category $\underline{A}$ satisfies (RE.2) iff it satisfies the following condition :

(RE.2') for every object A there exist endomorphisms ω_A, Ω_A : A $\to$ A such that :

a) for every endomorphism a : A $\to$ A, $\omega_A \leqq a \leqq \Omega_A$

b) $\omega_A\, \Omega_A\, \omega_A = \omega_A$; $\Omega_A\, \omega_A\, \Omega_A = \Omega_A$.

Actually, (RE.2) implies (RE.2') via 1.8.5-6 ; conversely if a and b hold, then $\omega_A \leqq 1_A \leqq \Omega_A$ and for any $a' : A \to A$:

(1) $$\omega_A \leqq (\omega_A a' \omega_A) \leqq \omega_A \Omega_A \omega_A = \omega_A,$$

(2) $$\Omega_A = \Omega_A \omega_A \Omega_A \leqq (\Omega_A a' \Omega_A) \leqq \Omega_A.$$

We notice also that the object A is null (1.9) iff $1_A = \omega_A$, iff $1_A = \Omega_A$, iff $\omega_A = \Omega_A$.

4.6. From now on $\underline{A}$ is a RE-category. If Nrp(A) is the set of null projections z of the object A, ordered by $\leqq$, there are biunivocal correspondences :

(1) $$\mathrm{Rst}(A) \rightleftarrows \mathrm{Npr}(A) : e \mapsto e\Omega_A e, \quad z \mapsto \underline{n}(z),$$

(2) $$\mathrm{Crs}(A) \rightleftarrows \mathrm{Npr}(A) : f \mapsto f\omega_A f, \quad z \mapsto \underline{d}^c(z),$$

which, by 4.4, preserve the orders $\leqq$.

Actually, for $e \in \mathrm{Rst}(A)$ and $z \in \mathrm{Npr}(A)$:

$$\underline{n}(e\Omega_A e) = \underline{n}(e) = e \quad (4.2.3),$$

$$\underline{n}(z) . \Omega_A . \underline{n}(z) \leqq z\Omega_A z = z = \underline{n}(z).z.\underline{n}(z) \leqq \underline{n}(z). \Omega_A . \underline{n}(z).$$

4.7. By composing 4.6, 1-2, one gets a biunivocal correspondence :

(1) $$\mathrm{Rst}(A) \longrightarrow \mathrm{Crs}(A), \quad e \mapsto e^c = \underline{d}^c(e\Omega_A e),$$

(2) $$\mathrm{Crs}(A) \longrightarrow \mathrm{Rst}(A) : f \mapsto f_c = \underline{n}(f\omega_A f)$$

which preserves $\leqq$ (hence reverses $\propto$). In particular :

(3) $$1_c = \omega, \qquad 1^c = \Omega.$$

It should be noticed that this ^{c}*-duality* between restrictions and corestrictions of a RE-category $\underline{A}$ is given by the whole RE-structure, and has little to do with the *order duality* (which turns restrictions of $\underline{A}$ into corestrictions of its order-opposite RO-category, preserving the order $\propto$). It will be seen (6.1.5) that the ^{c}-duality extends the (ker-cok)-duality between subobjects and quotients, in an exact category.

4.8. This duality supplies, for any projection $e \in \mathrm{Prj}(A)$, a *denominator* $\underline{d}(e) \in \mathrm{Rst}(A)$ and a ^{c}*-numerator* $\underline{n}^c(e) \in \mathrm{Crs}(A)$:

(1) $$\underline{d}e = (\underline{d}^c e)_c = \underline{n}(e\omega_A e),$$

(2) $$\underline{n}^c e = (\underline{n}e)^c = \underline{d}^c(e\,\Omega_A e)$$

where the right equality in (1) follows from 4.7.1, 4.6.1, 4.2.4 :

$$(\underline{n}(e\omega e))^C = \underline{d}^C(\underline{n}(e\omega e).\Omega.\underline{n}(e\omega e))) = \underline{d}^C(e\omega e) = \underline{d}^C e.$$

Thus

(3) $$\underline{d}e \propto \underline{n}e, \quad \underline{n}^C e \propto \underline{d}^C e,$$

(4) $$\text{if } e \in \mathrm{Rst}(A), \quad e^C = \underline{n}^C e \quad ,$$

(5) $$\text{if } f \in \mathrm{Crs}(A), \quad f_C = \underline{d}f .$$

4.9. Proposition. For $e \in \mathrm{Prj}(A)$ and $a : A' \to A''$:

a) $$e.\underline{d}e = e\omega e, \quad e.\underline{n}^C e = e\Omega e,$$

b) e is null iff $\underline{n}e = \underline{d}e$ iff $\underline{n}^C e = \underline{d}^C e$,

c) the projections $e' = \underline{d}e$ and $e'' = \underline{n}^C e$ are respectively characterized by (1) and (2) :

(1) $$e' \leqq 1, \quad e' \leqq e, \quad e'.e = e.e' \in \mathrm{Nul}\,\underline{A},$$

(2) $$e'' \geqq 1, \quad e'' \geqq e, \quad e''.e = e.e'' \in \mathrm{Nul}\,\underline{A}$$

d) $$a\,\omega_{A'}\tilde{a} = (a\tilde{a})\omega_{A''}(a\tilde{a}), \quad a\,\Omega_{A'}\tilde{a} = (a\tilde{a})\,\Omega_{A''}(a\tilde{a}).$$

Proof. a)

$$e.\underline{d}e = \underline{n}e.\underline{d}^C e.\underline{d}e = \underline{n}e.\underline{d}^C(e\omega e).\underline{n}(e\omega e) = \underline{n}e.e\omega e = e\omega e.$$

b) If e is null, $\underline{d}e = \underline{n}(e\omega e) = \underline{n}e$; conversely if $\underline{d}e = \underline{n}e$, then

$$e = e.\underline{d}e = e\omega e \in \mathrm{Nul}(\underline{A}) \quad .$$

c) $e' = \underline{n}e$ verifies (1) by a ; conversely any projection e' satisfying (1) coincides with $\underline{d}e = \underline{n}(e\omega e)$ since (4.2.2) its product with $\underline{d}^C(e\omega e) =$ $= e\omega e$ is $e\omega e$

$$e'.e\omega e \leqq (e\omega e) = e\omega\,\omega e \leqq ee'\omega e = e'(e\omega e).$$

d) $$a\,\omega_{A'}\tilde{a} \leqq a(\tilde{a}\,\omega_{A''}a)\tilde{a} \leqq a\tilde{a}(a\,\omega_{A'}\tilde{a})a\tilde{a} = a\,\omega_{A'}\tilde{a}.$$

4.10. Every morphism $a : A' \to A''$ of the RE-category $\underline{A}$ determines the following *restrictions* of its domain and codomain, which we call *definition, values, indetermination :*

(1) $$\underline{def}(a) = \underline{n}(\underline{c}(a)) \in \mathrm{Rst}(A'),$$

(2) $$\underline{ann}(a) = \underline{d}(\underline{c}(a)) = \underline{n}(\tilde{a}a\omega\tilde{a}a) = \underline{n}(\tilde{a}\omega a) \in \mathrm{Rst}(A'),$$

(3) $$\underline{val}(a) = \underline{n}(\underline{i}(a)) \in \mathrm{Rst}(A''),$$

(4) $$\underline{ind}(a) = \underline{d}(\underline{i}(a)) = \underline{n}(a\tilde{a}\omega a\tilde{a}) = \underline{n}(a\omega\tilde{a}) \in \mathrm{Rst}(A'')$$

so that

(5) $\underline{def}(\tilde{a}) = \underline{val}(a), \quad \underline{ann}(\tilde{a}) = \underline{ind}(a),$

(6) $\underline{ann}(a) \leqq \underline{def}(a), \quad \underline{ind}(a) \leqq \underline{val}(a),$

(7) a is monic iff $\underline{def}(a) = 1$, $\underline{ann}(a) = \omega$,

(8) a is proper iff $\underline{def}(a) = 1$, $\underline{ind}(a) = \omega$,

(9) a is null iff $\underline{ann}(a) = \underline{def}(a)$, iff $\underline{ind}(a) = \underline{val}(a)$.

Moreover, for every projection e :

(10) $\underline{n}(e) = \underline{def}(e) = \underline{val}(e), \quad \underline{d}(e) = \underline{ann}(e) = \underline{ind}(e).$

It will be seen in 6.2 that, when $\underline{A}$ is a category of relations Rel($\underline{E}$), $\underline{def}(a)$ simulates the $\underline{E}$-subobject def (a): Def$(a) \rightarrowtail A'$ [5], and so on.

4.11. Let $\underline{A}$ be a RE-category : for every connected objects A', A" there exist unique $\omega_{A'A''}$, $\Omega_{A'A''}$, $0_{A'A''} \in \underline{A}(A', A'')$ such that :

(1) for each $a \in \underline{A}(A', A'')$, $\omega_{A'A''} \leqq a$,

(2) for each $a \in \underline{A}(A', A'')$, $a \leqq \Omega_{A'A''}$,

(3) $0_{A'A''}$ is null and proper.

Equivalent characterizations are :

(1') $\underline{c}(\omega_{A'A''}) = \omega_{A'}, \quad \underline{i}(\omega_{A'A''}) = \omega_{A''},$

(2') $\underline{c}(\Omega_{A'A''}) = \Omega_{A'}, \quad \underline{i}(\Omega_{A'A''}) = \Omega_{A''},$

(3') $\underline{c}(0_{A'A''}) = \Omega_{A'}, \quad \underline{i}(0_{A'A''}) = \omega_{A''}.$

Indeed, take some $a_0 \in \underline{A}(A', A'')$ and define :

(4) $\omega_{A'A''} = \omega_{A''} a_0 \omega_{A'}, \quad \Omega_{A'A''} = \Omega_{A''} a_0 \Omega_{A'}, \quad 0_{A'A''} = \omega_{A''} a_0 \Omega_{A'}.$

Then the properties (1)-(3') are easily verified, while the uniqueness is obvious or follows from 1.8.

4.12. As a consequence of 1.9 and 4.11, any null object in a *connected* RE-category $\underline{A}$ is a *zero-object* (i.e., initial and terminal) *for* Prp $\underline{A}$. We shall see that such objects necessarily exist when $\underline{A}$ is also factorizing and non-empty.

5. RE-functors and RE-transformations.

$\underline{A}$ and $\underline{B}$ are RE-categories.

5.1. Definition. A *RE-functor* will be a RO-functor $F : \underline{A} \to \underline{B}$ between

RE-categories, which satisfies the following equivalent conditions :

a) F preserves null morphisms,
b) for any object A, $F(\omega_A) = \omega_{FA}$,
c) for any object A, $F(\Omega_A) = \Omega_{FA}$,
d) for any connected pair A', A'' : $F(\omega_{A',A''}) = \omega_{FA',FA''}$;
e) for any connected pair A', A'' : $F(\Omega_{A'A''}) = \Omega_{FA',FA''}$,
f) for any connected pair A', A'' : $F(0_{A'A''}) = 0_{FA',FA''}$,
g) F preserves the operators $\underline{n}$, $\underline{d}^c$, $\underline{d}$, $\underline{n}^c$, $\underline{def}$, $\underline{ann}$, $\underline{val}$, $\underline{ind}$ and ^{c}-duality.

5.2. Let $F : \underline{A} \to \underline{B}$ be an involution-preserving functor (between RE-categories) ; it is easy to see that :

a) F preserves the order iff it preserves restrictions and corestrictions (from 4.3),
b) if $\underline{A}$ is factorizing, F preserves the order iff it preserves proper morphisms (from a and 3.2),
c) if $\underline{A}$ is connected, with a null object A_0, F preserves null morphisms iff $F(A_0)$ is null in $\underline{B}$.

5.3. Definition. A *RE-transformation* will be a RO-transformation

$$\alpha : F \to G : \underline{A} \to \underline{B}$$

between RE-functors. Thus, we have **RE**, a sub-2-category of **RO**.

5.4. Lemma. Let $F : \underline{A} \to \underline{B}$ be a RE-functor, and $a, b \in \underline{A}(A', A'')$; then $Fa \leqq Fb$ iff there exist $a', b' \in \underline{A}(A', A'')$ such that

$$(1) \qquad a \leqq a' \sim_F b' \leqq b \quad ^{9)}.$$

Proof. The condition (1) is clearly sufficient ; conversely, if $Fa \leqq Fb$ by 4.3 d :

$$F(\underline{d}^c(b\tilde{b})a) = \underline{d}^c(Fb.F\tilde{b}).F(a) = Fb.\underline{n}(F\tilde{a}.Fa) = F(b.\underline{n}(\tilde{a}a))$$

and therefore it suffices to take

$$a' = \underline{d}^c(b\tilde{b}).a \quad \text{and} \quad b' = b.\underline{n}(\tilde{a}a).$$

5.5. Corollary. A faithful RE-functor $F : \underline{A} \to \underline{B}$ reflects the order between parallel morphisms ; it also reflects proper morphisms and null morphisms ; moreover, when acting on endomorphisms, it reflects restrictions and corestrictions.

(9) Here $a' \sim_F b'$ means that a' and b' are parallel maps and $Fa' = Fb'$.

5.6. It follows from 5.2 a and 5.5 that, if

$$\underline{A}_1 = (\underline{A}, \sim, \leqq_1) \qquad \text{and} \qquad \underline{A}_2 = (\underline{A}, \sim, \leqq_2)$$

are RE-structures on the same involutive category $(\underline{A}, \sim)$ and moreover the restrictions and corestrictions of $\underline{A}_1$ are still so in $\underline{A}_2$, then $\underline{A}_1 = \underline{A}_2$. The same condition holds, a fortiori, if $\leqq_1$ implies $\leqq_2$.

It can be noticed that, if $\underline{A}_1 = (\underline{A}, \sim, \leqq)$ is a *non-null* RE-category (has some non-null object), then $\underline{A}_1$ itself and its order opposite $\underline{A}_2 = (\underline{A}, \sim, \geqq)$ are different RE-structures on $(\underline{A}, \sim)$, because $\omega_A \neq \Omega_A$ for every non-null object A (4.5).

5.7. A *RE-subcategory* $\underline{A}'$ of $\underline{A}$ is an involutive subcategory satisfying :

(1) for each $A \in \mathrm{Ob}\, \underline{A}'$ and each $e \in \mathrm{Prj}_{\underline{A}'}(A)$, the projections $\underline{n}(e)$, $\underline{d}^c(e)$, ω_A, Ω_A belong to $\underline{A}'$.

Then $\underline{A}'$ will be provided with the induced RE-structure, that is the only one which makes the inclusion $\underline{A}' \to \underline{A}$ a RE-functor.

It should be noticed that *any full subcategory* (more generally any *Prj-full involutive subcategory* (5.11)) *of a RE-category is a RE-subcategory,* and any intersection of RE-subcategories is so.

5.8. If Δ is a subgraph of $\underline{A}$, the *RE-subcategory of* $\underline{A}$ *spanned by* Δ is the intersection $\underline{A}'$ of all RE-subcategories of $\underline{A}$ containing Δ ; $\underline{A}'$ is given by :

(1) $$\mathrm{Ob}\, \underline{A}' = \mathrm{Ob}\, \Delta, \qquad \mathrm{Mor}\, \underline{A}' = \bigcup_{n \geqq 0} \Delta_n$$

where the sets $\Delta_n \subset \mathrm{Mor}\, \underline{A}$ are inductively defined as :

(2) $$\Delta_0 = \mathrm{Mor}\, \Delta \cup \{1_A, \omega_A, \Omega_A \mid A \in \mathrm{Ob}\, \Delta\},$$

(3') if $a \in \Delta_n$, then $\tilde{a} \in \Delta_{n+1}$,

(3") if $a, b \in \Delta_n$ are composable in $\underline{A}$, then $ba \in \Delta_{n+1}$,
(3"') if $e \in \Delta_n$ is a projection of $\underline{A}$, then $\underline{n}e, \underline{d}^c e \in \Delta_{n+1}$.

This proves that

$$\mathrm{card}(\mathrm{Mor}\, A') \leqq \max(\mathrm{card}(\mathrm{Ob}\, \Delta), \mathrm{card}(\mathrm{Mor}\, \Delta), \aleph_0).$$

Any two RE-functors $F, G : \underline{A} \to \underline{B}$ which coincide on Δ coincide also on $\underline{A}'$.

5.9. A RE-functor $F : \underline{A} \to \underline{C}$ is called a *RE-quotient* if it is bijective on the objects and full ; by 5.4 the RE-structure on $\underline{C}$ (i.e., composition, involution and order) is then determined by the one of $\underline{A}$, and by F.

Analogously, if $F : \underline{C} \to \underline{B}$ is a *faithful* RE-functor, the RE-structure of $\underline{C}$ is determined by the one of $\underline{B}$, by the mapping F and by the domain and codomain mappings of $\underline{C}$ (by 5.5).

5.10. Any RE-functor $F : \underline{A} \to \underline{B}$ has an essentially unique *RE-factorization*

$$(1) \qquad \underline{A} \xrightarrow{F_1} \underline{C} \xrightarrow{F_2} \underline{B}\,, \qquad F = F_2F_1$$

where F_1 is a RE-quotient and F_2 is a faithful RE-functor. To prove the existence, consider the (usual) CAT-factorization of F [10] : F_1 is a quotient and F_2 a faithful functor ; then define the involution on $\underline{C}$ via F_1 ($F_1(a)^{\sim} = F_1(\tilde{a})$) and the order via F_2:

$$c \leq c' \text{ iff they are parallel in } \underline{C} \text{ and } F_2(c) \leq F_2(c') \quad \text{in } \underline{B} \quad ;$$

$\underline{C}$ is thus a RO-category and (1) a factorization in **RO.** Now, any e in $\text{Prj}_{\underline{C}}(F_1A)$ is the F_1-image of some $e \in \text{Prj}_{\underline{A}}(A)$: actually, if

$$e' = F_1(a) \quad \text{with} \quad a \in \underline{A}(A, A),$$

then

$$F_1(\tilde{a}a) = \tilde{e}'e' = e' \quad ;$$

thus the existence of numerators and c-denominators of projections of $\underline{C}$ comes from $\underline{A}$, while their uniqueness comes from $\underline{B}$; last F_1 supplies the null restrictions (ω) and the null corestrictions (Ω) of $\underline{C}$, and F_2 preserves them.

5.11. The RE-functor $F : \underline{A} \to \underline{B}$ will be said to be *Prj-full* (resp. *Prj - faithful*) if for any $A \in \text{Ob}\,\underline{A}$ the mapping

$$(1) \qquad \text{Prj}_{A}(A) \longrightarrow \text{Prj}_{B}(FA), \quad e \mapsto Fe$$

is surjective (resp. injective) ; any full (resp. faithful) functor is so. Analogously one can define *Rst-full* RE-functors, and so on.

However, the following conditions on the RE-functor $F : \underline{A} \to \underline{B}$, having RE-factorization $F = F_2F_1$, are equivalent :

a) F is Prj-full,
b) F is Rst-full,
c) F_2 is Prj-full (and Prj-faithful),
d) F_2 is Rst-full (and Rst-faithful).

In fact, a $\Leftrightarrow$ c and b $\Leftrightarrow$ d are obvious, while c $\Leftrightarrow$ d follows from 5.5 and 4.7. Analogous results hold for the "local" faithfulness of F and F_1.

We also remark that a Rst-faithful RE-functor reflects the order $\propto$ of projections, hence also their order $\leqq$ (4.4) ; it also reflects monics, epis, isos, proper morphisms, null morphisms (4.10.7-9).

6. Exact categories and RE-categories.

6.1. Main Theorem. Let $\underline{A} = (\underline{A}, \sim, \leqq)$ be a RO-category and $\underline{E} = \mathrm{Prp}\ \underline{A}$. Then the following conditions are equivalent :

a) $\underline{A}$ is a factorizing, connected, non-empty RE-category.

b) $\underline{E}$ is exact and the embedding $\underline{E} \to \underline{A}$ is (isomorphic to) the canonical symmetrization of $\underline{E}$, that is the embedding $\underline{E} \to \mathrm{Rel}(\underline{E})$.

If these conditions hold, for any $e \in \mathrm{Prj}_{\underline{A}}(A)$ (notations as in [5]) :

(1) $\underline{n}(e) = \underline{i}(\mathrm{val}\ e), \quad \underline{d}(e) = \underline{i}(\mathrm{ind}\ e),$

(2) $\underline{n}^c(e) = \underline{c}(\mathrm{ann}^*\ e), \quad \underline{d}^c(e) = \underline{c}(\mathrm{def}^*\ e),$

(3) $\omega_A = 0_{0A}\, \tilde{0}_{0A}\,, \quad \Omega_A = \tilde{0}_{A0}\, 0_{A0}\,,$

(4) A_0 is a zero object for $\underline{E}$ iff it is a null object for $\underline{A}$.

(5) there are commutative squares of order isomorphisms ($\rightarrow$) and anti-isomorphisms ($\dashrightarrow$), with regard to $\propto$:

(6)
$$\begin{array}{ccc} \mathrm{Sub}_{\underline{E}}(A) & \xrightarrow{\ \underline{i}_A\ } & \mathrm{Rst}_{\underline{A}}(A) \\ \mathrm{cok} \downarrow \uparrow \mathrm{ker} & & \underline{c} \downarrow \uparrow \underline{c} \\ \mathrm{Quo}_{\underline{E}}(A) & \xrightarrow{\ \underline{c}_A\ } & \mathrm{Crs}_{\underline{A}}(A) \end{array}$$

Proof. First we prove b $\Rightarrow$ a , as well as properties (1)-(5). We can suppose that $\underline{A} = \mathrm{Rel}(\underline{E})$ is precisely the RO-category of relations on the exact category $\underline{E} = \mathrm{Prp}\ \underline{A}$: thus $\underline{A}$ is trivially connected and non-empty (so is $\underline{E}$) and factorizing [5]. As regards (RE.1) : if $e \in \mathrm{Prj}(A)$, then ([5], § 5.21),

$$e = (m\tilde{m})(\tilde{p}p) \quad \text{where} \quad m = \mathrm{val}(e) \quad \text{and} \quad p = \mathrm{def}^*(e)$$

are proper morphisms, so that

$$m\tilde{m} \leqq 1 \quad \text{and} \quad \tilde{p}p \geqq 1\,;$$

conversely, if

$$e = e_1 e_2 \qquad \text{and} \qquad e_1 \leqq 1 \leqq e_2\,,$$

then (3.2)

$$e_1 = n\tilde{n}\,, \qquad e_2 = \tilde{q}q$$

where n is monic and p is epi in $\underline{E}$ (and in $\underline{A}$) ; now,

$$e = n\tilde{n}\tilde{q}q = n(qn)^{\sim}q$$

is a coternary factorization of e ([5], § 5.15) so that

$$n \sim \text{val}(e) \quad \text{and} \quad q \sim \text{val}^*(e) = \text{def}^*(e) :$$

in other words

$$e_1 = n\tilde{n} = m\tilde{m}\ (= \underline{i}(\text{val}\, e)) \quad \text{and} \quad e_2 = \tilde{q}q = \tilde{p}p\ (= \underline{c}(\text{def}^*(e))$$

are uniquely determined, and two formulas in (1)-(2) are checked.

Now, the axiom (RE.2) is trivially satisfied by assuming (3), and (4) is a consequence of 4.12. Let us verify (5) : let

$$m \in \text{Sub}_{\underline{E}}(A) \quad \text{and} \quad p = \text{cok}\, m \in \text{Quo}_{\underline{E}}(A) ;$$

then

$$e = \underline{i}(m) \in \text{Rst}(A) \quad \text{and} \quad f = \underline{c}(p) \in \text{Crs}(A)$$

commute ([5], § 5.21.1-2) and

$$fe = \tilde{p}pm\tilde{m} = \tilde{p}\,0\,\tilde{m} \in \text{Nul}(\underline{A}) :$$

by 4.9.1, $e = \underline{d}(f) = f_C$. The other two formulas in (1)-(2) follow easily ; for example :

$$\underline{d}(e) = (\underline{d}^{\mathcal{C}}(e))_C = (\underline{c}(\text{def}^*\, e))_C = \underline{i}(\text{ker def}^*\, e) = \underline{i}(\text{ind}\, e).$$

Conversely, let a be assumed. First we prove that $\underline{A}$ has some null object; if A is an object, consider the epi-monic factorization of 0_{AA} in $\underline{A}$:

(7) $$A \xrightarrow{p} A_0 \xrightarrow{m} A, \qquad 0_{AA} = mp,$$

then

$$1_{A_0} = (\tilde{m}m)(p\tilde{p}) = \tilde{m}(mp)\tilde{p} = \tilde{m}\,0\,\tilde{p} \in \text{Nul}\ \underline{A},$$

hence A_0 is null in $\underline{A}$ and a zero-object in $\underline{E}$ (4.12). We proceed now to verify Puppe's axioms (K1-3) [20], to ensure b.

The first follows trivially from (RE.0), 4.11 and the above arguments. As to (K 2), let

$$a, b \in \underline{A}(A, B) \quad \text{and} \quad a\,\omega_{0A} \leqq b\,\omega_{0A}$$

(i.e., Ia ≦ I b, in Puppe's notation). By 4.9 d :

(8) $$a\tilde{a}\omega_B a\tilde{a} = a\,\omega_A\tilde{a} \leqq b\,\omega_A\tilde{b} = b\tilde{b}\omega_B b\tilde{b}$$

and, by 4.8, 4.4 and 4.7 :

(9) $$\underline{d}(a\tilde{a}) = \underline{n}(a\tilde{a}\,\omega\, a\tilde{a}) \leqq \underline{n}(b\tilde{b}\,\omega\, b\tilde{b}) = \underline{d}(b\tilde{b}),$$

(10) $$\underline{d}^{\mathcal{C}}(a\tilde{a}) \propto \underline{d}^{\mathcal{C}}(b\tilde{b}),$$

(11) $$\tilde{a}ab \leqq \underline{d}^{\mathcal{C}}(a\tilde{a}).\underline{d}^{\mathcal{C}}(b\tilde{b}).b = \underline{d}^{\mathcal{C}}(b\tilde{b}).b = b.$$

By duality on the order $\leqq$, one also has that

$$a\,\Omega_{OA} \geqq b\,\Omega_{OA} \quad \text{implies} \quad a\tilde{a}b \geqq b \ .$$

Last, for (K 3), let $a \in \underline{A}(O, A)$: a is a null morphism, hence (4.9 and 3.2 c) :

(12) $$\underline{n}(a\tilde{a}) = \underline{d}(a\tilde{a}) = m\tilde{m}$$

where $m : A' \to A$ is a proper monic of $\underline{A}$ (hence monic in $\underline{E}$) ; moreover $a = m\,\Omega_{OA'}$ by 1.8.2, as :

(13) $$\underline{c}(a) = 1_O = \underline{c}(m\,\Omega_{OA'}),$$

(14) $$\underline{n}(\underline{i}(a)) = m\tilde{m} = \underline{n}(m\tilde{m}\,\Omega_A\, m\tilde{m}) = \underline{n}(m\Omega_{A'}\,\tilde{m}) = \underline{n}(\underline{i}(m\,\Omega_{OA'})),$$

where the third equality in (14) follows from 4.9 d.

6.2. Let $\underline{E}$ be an exact category and $\underline{A}$ = Rel($\underline{E}$). If

$$a \in \underline{A}(A', A'') \quad \text{and} \quad u \in \underline{E}(A', A''),$$

it is easy to derive from 6.1 that :

(1) $\underline{i}(\mathrm{def}\, a) = \underline{def}\ a$, $\underline{i}(\mathrm{ann}\, a) = \underline{ann}\ a$, $\underline{i}(\mathrm{val}\ a) = \underline{val}\ a$, $\underline{i}(\mathrm{ind}\ a) = \underline{ind}\ a$,

(2) $\underline{i}(\ker u) = \underline{ann}\ u$, $\underline{c}(\mathrm{cok}\, u) = \underline{val}^{\,c}\, u$.

Moreover the restriction

$$e_o = \underline{ann}\ u = \underline{d}(\tilde{u}u) \in \mathrm{Rst}(A')$$

is characterized by :

a) ue_o is null,
b) if v is a proper morphism and uv is null, then $v = e_o v$,

and can be called the ~ *-kernel* of u ; analogously one characterizes the ~ *-cokernel* of u, $f_o = \underline{c}(\mathrm{cok}\, u) = (\underline{val}\ u)^c$.

6.3. The above Theorem 6.1 supplies a 2-adjoint 2-equivalence :

(1) $$\mathbf{REX} \underset{\mathrm{Rel}}{\overset{\mathrm{Prp}}{\rightleftarrows}} \mathbf{EX} : \quad \eta : 1 \overset{=}{\to} \mathrm{Prp.Rel}, \quad \varepsilon : \mathrm{Rel.Prp} \longrightarrow 1$$

where :

- **REX** is the full sub-2-category of **RE** determined by connected, non-empty, factorizing RE-categories ;
- Rel is the restriction of the 2-functor Rel : **EX** $\to$ **RO** (2.7) according to 6.1 and 5.2 c) ;
- Prp is the restriction of the 2-functor Prp : **RO** $\to$ **CAT** (2.6), according to 6.1 and to the following fact : if F : $\underline{A} \to \underline{B}$ is a REX-functor, then it is the symmetrization (i.e. the involution-preserving exten-

sion of

$$F_0 = \text{Prp}\, F : \text{Prp}\, \underline{A} \to \text{Prp}\, \underline{B},$$

a zero-preserving functor which is exact ([5], Th. 6.15);

- the equality Prp Rel($\underline{E}$) = $\underline{E}$ has already been considered (1.7) ;
- for each REX-category $\underline{A}$

(2) $$\varepsilon \underline{A} : \text{Rel}(\text{Prp}(\underline{A})) \to \underline{A}$$

is the unique isomorphism of RE-categories extending the identity

$$\text{Prp}(\text{Rel}(\text{Prp}(\underline{A}))) = \text{Prp}(\underline{A}) ;$$

the family $\varepsilon = (\varepsilon \underline{A})$ is natural ;

-last, the composite transformations

(3) $$\text{Prp} \xrightarrow{\eta\, \text{Prp}} \text{Prp.Rel.Prp} \xrightarrow{\text{Prp}\, \varepsilon} \text{Prp},$$

(4) $$\text{Rel} \xrightarrow{\text{Rel}\, \eta} \text{Rel.Prp.Rel} \xrightarrow{\varepsilon\, \text{Rel}} \text{Rel}$$

are identities, since η, Prp ε and ε Rel are so.

6.4. More generally, call a category $\underline{E}$ *component-wise exact* if its connected components are exact (the empty category, having no connected components, is allowed). Call **EX'** the (obvious) 2-category of these categories.

Then 6.3.1 trivially extends to a 2-adjoint 2-equivalence

(1) $$\mathbf{FRE} \xrightarrow{\text{Prp}} \mathbf{EX'} \xrightarrow{\text{Rel}} \mathbf{FRE}$$

where **FRE** is the 2-category of factorizing RE-categories.

6.5. Theorem. For any RE-category $\underline{A}$, $\underline{B}$ = Fct $\underline{A}$ is also RE; the former is connected and non-empty iff the latter is so. Moreover, for

$$x \in \text{Prj}\,(A'), \quad y \in \text{Prj}\,(A'') \quad \text{and} \quad e' = (e\ ;\ x, x) \in \text{Prj}\,(x)$$

(that is $e \propto x$ in Prj ($\underline{A}'$)) :

(1) $\underline{n}_{\underline{B}}(e') = (xe_1\ ;\ x, x) = (e_1x\ ;\ x, x)$ where $e_1 = \underline{n}_{\underline{A}}(e)$,

(2) $\underline{d}^C_{\underline{B}}(e') = (xe_2\ ;\ x, x) = (e_2x\ ;\ x, x)$ where $e_2 = \underline{d}^C_{\underline{A}}(e)$,

(3) $\underline{d}_{\underline{B}}(e') = (xe_3\ ;\ x, x) = (e_3x\ ;\ x, x)$ where $e_3 = \underline{d}_{\underline{A}}(e)$,

(4) $\underline{n}^C_{\underline{B}}(e') = (xe_4\ ;\ x, x) = (e_4x\ ;\ x, x)$ where $e_4 = \underline{n}^C_{\underline{A}}(e)$,

(5) $\omega_{xy} = (y\omega_{A'A''}x;\ x, y), \qquad \Omega_{xy} = (y\,\Omega_{A'A''}x\ ;\ x, y).$

Proof. We know from 3.6 that $\underline{B}$ = Fct $\underline{A}$ is a factorizing RO-category. Now $e'_1 = (xe_1x\ ;\ x, x)$ trivially satisfies the conditions of (RE.1 a) :

(6) $$e' \propto e'_1 \ , \quad e'_1 \leqq 1_x \ , \quad e'_1 \leqq e' \quad (\text{in } \underline{B}).$$

Conversely, if $e'_1 = (\bar{e}_1 \ ; x, x)$ verifies (6) then :

(7) $$e \propto \bar{e}_1 \propto x \ , \quad \bar{e}_1 \leqq x \ , \quad \bar{e}_1 \leqq e$$

so that, by 4.4,

$$\underline{n}(\bar{e}_1) = \underline{n}(e) \propto \underline{n}(x) \ , \quad \underline{d}^C(e_1) = \underline{d}^C(x)$$

and

(8) $$\bar{e}_1 = \underline{n}(e).\underline{d}^C(x) = \underline{n}e.\underline{n}x.\underline{d}^C x = e_1.x \quad (= x.e_1).$$

In the same way

$$e'_2 = (xe_2 \ ; x, x) = (e_2 x \ ; x, x)$$

is the unique projection of x satisfying

$$e' \propto e'_2 \ , \quad e'_2 \geqq 1 \quad , \quad e'_2 \geqq e'.$$

Thus we have proved the axiom (RE.1), together with the properties (1) and (2), while the axiom (RE.2) is clearly satisfied by assuming (5).

Last (3) and (4) follow from (1), (2), (5) ; e.g. :

(9) $$\underline{d}_B(e') = \underline{n}_B(e' \Omega_{xx} e') = (x.\underline{n}(ex \Omega xe) \ ; x, x) =$$
$$= (x.\underline{n}(e \Omega e) \ ; x, x) = (x.\underline{d}_A(e) \ ; x, x).$$

6.6. Corollary. For every RE-category $\underline{A}$ there is a full RE-embedding :

(1) $$\eta\underline{A} : \underline{A} \to \text{Rel}(\underline{E}) \ ; \quad \underline{E} = \text{Prp}(\text{Fct}(\underline{A}))$$

which is an i-universal arrow from $\underline{A}$ to the 2-functor Rel : **EX'** → **RE.** Thus, RE-categories coincide up to isomorphism with the *full subcategories of the categories of relations on componentwise exact categories.*

Proof. By 6.5, the i-universal arrow $\underline{A} \to \text{Fct}(\underline{A})$ in 3.8.1 embeds RE-categories into FRE-categories ; by composition with a suitable 2-universal arrow related to the 2-equivalence 6.4.1 one gets (1). The last assertion follows from the preceding one and from 5.7.

6.7. Every componentwise exact category $\underline{E}$ has a Sub-full exact embedding :

(1) $$\eta\underline{E} : \underline{E} \to Z(\underline{E})$$

where $Z(\underline{E})$ is the exact category obtained by adding to $\underline{E}$, for every pair A, B of disconnected objects of $\underline{E}$, one morphism 0_{AB}: A → B, with obvious compositions ; when $\underline{E}$ is empty, take $Z(\underline{E}) = \underline{1}$.

It is easy to see that (1) is an i-universal arrow from $\underline{E}$ to the 2-inclusion $\mathbf{EX} \to \mathbf{EX'}$.

6.8. By composing the i-universal arrows "n" in 6.6.1 and 6.7.1 one gets an i-universal arrow

$$(1) \qquad n\underline{A} : \underline{A} \to \mathrm{Rel}(\underline{E}) , \quad \underline{E} = Z(\mathrm{Prp}(\mathrm{Fct}(\underline{A})))$$

from the RE-category $\underline{A}$ to the 2-functor Rel : $\mathbf{EX} \to \mathbf{RE}$; (1) is a Prj-full RE-embedding.

Thus, RE-categories coincide also, up to isomorphism, with the *Prj-full involutive subcategories of the categories of relations on exact categories.*

6.9. Proposition. In any RE-category $\underline{A}$ the ordered sets Rst(A) and Crs(A) are modular lattices. If $e, f \in \mathrm{Prj}(A)$:

a) $\underline{n}(efe) = \underline{n}e \cap (\underline{n}f \cup \underline{d}e) = (\underline{n}e \cap \underline{n}f) \cup \underline{d}e,$

$\underline{d}(efe) = \underline{d}e \cup (\underline{d}f \cap \underline{n}e) = (\underline{d}e \cup \underline{d}f) \cap \underline{n}e,$

b) $(ef = fe)$ iff $(efe = fef)$ iff $(\underline{n}e \propto \underline{d}f$ and $\underline{n}f \propto \underline{d}e),$

c) if $ef = fe$: $\underline{n}(ef) = \underline{n}e \cap \underline{n}f, \ \underline{d}(ef) = \underline{d}e \cup \underline{d}f,$

d) if $e \leqq 1 \leqq f$:

$$(ef = fe) \ \text{iff} \ (f_C \propto e) \ \text{iff} \ (e^C \propto f),$$

and in this case

$$\underline{n}(ef) = e \quad \text{and} \quad \underline{d}^C(ef) = f.$$

Proof. By 6.6, $\underline{A}$ can be embedded in Rel $\underline{E}$ where $\underline{E}$ = Prp Fct $\underline{A}$ is componentwise exact, hence has modular lattices of subobjects and quotients ; therefore all lattices Rst (A) and Crs(A) are modular (6.1.6) and the property a is just a restatement of [8], § 2.13.

In b, the first condition implies the second one, which is equivalent to the third by a ; finally, if $efe = fef$ then :

$$ef = efef = feff = ffef = fefe = fe.$$

Last, c and d follow from a and b.

6.10. Let

$$\mathrm{Rst}_2(A) = \{(e_1, e_2) \in \mathrm{Rst}(A) \times \mathrm{Rst}(A) \mid e_1 \geqq e_2\} ;$$

the mapping

$$(1) \qquad \mathrm{Prj}(A) \to \mathrm{Rst}\ (A) : \ e \mapsto (\underline{n}e, \underline{d}e)$$

is an isomorphism of ordered sets, with regard to $\leqq$ (4.8.3, 6. 9d) ; therefore (Prj(A), $\leqq$) is a modular lattice.

If $e_1 \geqq e_2$ in Rst(A), we shall write e_1/e_2 the only projection e of A such that

$$\underline{n}(e) = e_1 , \quad \underline{d}(e) = e_2 ;$$

in other words :

(2) $$e_1/e_2 = e_1.e_2^{C} = e_2^{C}.e_1 .$$

7. Transfer functors and distributivity.

We extend here to RE-categories some notions concerning the transfer of subobjects (direct and inverse images) for exact categories [10].

7.1. Every RE-category $\underline{A}$ is provided with a canonical RE-functor, the *transfer-functor of* $\underline{A}$

(1) $$\mathrm{Rst}_{\underline{A}} : \underline{A} \to Mlr$$

into the REX-category of *modular lattices and modular relations* [10] § 3.3, associating to every object A the modular lattice $\mathrm{Rest}_{\underline{A}}(A)$ of its restrictions, and to every morphism $a : A' \to A''$ the modular relation :

(2) $$\mathrm{Rst}_{\underline{A}}(a) = (a_R , a^R),$$

(3) $$a_R(e) = \underline{n}(ae\tilde{a}) , \quad a^R(f) = \underline{n}(\tilde{a}fa).$$

Actually, the composed RE-functor :

(4) $$\underline{A} \xrightarrow{\eta A} \mathrm{Rel}\ \underline{E} \xrightarrow{S} \mathrm{Rel}(Mlc) \xrightarrow{\simeq} Mlr$$

(where $\underline{E}$ = Prp Fct $\underline{A}$ is componentwise exact and S = $\mathrm{Rel}(\mathrm{Sub}_{\underline{E}})$ is the symmetrized of the transfer functor of $\underline{E}$ ([10], § 4.1)) is transformed into the above mapping $\mathrm{Rst}_{\underline{A}}$ by the family of isomorphisms $i = (\underline{i}_A)$ (6.1.6). This is natural because if

$$a \in \underline{A}(A', A'') \quad \text{and} \quad x \in \mathrm{Sub}_{\underline{E}}(A'),$$

by 6.1.1 and [10], § 4.3 :

$$a_R(\underline{i}(x)) = \underline{n}(ax\tilde{x}\tilde{a}) = \underline{i}(\mathrm{val}\ (ax\tilde{x}\tilde{a})) = \underline{i}(\mathrm{val}\ (ax)) = \underline{i}(a_S(x)).$$

By 4.10 :

(5) $$a^R(1) = \underline{def}(a) , \quad a^R(\omega) = \underline{ann}(a) , \quad a_R(1) = \underline{val}(a) , \quad a_R(\omega) = \underline{ind}(a).$$

The functor $\mathrm{Rst}_{\underline{A}}$ is obviously Rst-faithful and Rst-full (5.11) ; in particular it reflects monics, epis, isos, proper morphisms, null morphisms.

Last, we remark that the transfer functor Rst : *Mlr* → *Mlr* of the RE-category *Mlr* is isomorphic to the identity functor 1 via :

(6) $$\iota : \mathrm{Rst} \to 1,$$

(7) $$\iota(X) : \mathrm{Rst}(X) \to X\ , \quad (\iota X)_\bullet(e) = e_\bullet(1)\ , \quad (\iota X)^\bullet(x) = x \wedge -\ .$$

7.2. Analogously to [10], § 4.7, every RE-functor $F : \underline{A} \to \underline{B}$ defines a *horizontal transformation of vertical functors* into the double category **Mhr** of modular lattices, their homomorphisms and their modular relations (or an **Mhr**-wise transformation according to [1], p. 251) :

(1) $$\mathrm{Rst}_F : \mathrm{Rst}_{\underline{A}} \to \mathrm{Rst}_{\underline{B}}.F : \underline{A} \to \mathbf{Mhr},$$

(2) $$\mathrm{Rst}_F(A) : \mathrm{Rst}_{\underline{A}}(A) \to \mathrm{Rst}_{\underline{B}}(FA)\ ,\ e \mapsto F(e)\ .$$

Actually, Rst_F is *the unique horizontal transformation*

$$\rho : \mathrm{Rst}_{\underline{A}} \to \mathrm{Rst}_{\underline{B}}.F$$

since, for any $e \in \mathrm{Rst}_{\underline{A}}(A)$, necessarily :

$$\rho(e) = \rho(e_R(1)) = (Fe)_R(1) = Fe\ .$$

F is Rst-faithful or Rst-full iff all the mapping (2) are respectively injective or surjective.

7.3. We say that the RE-category $\underline{A}$ is *transfer* if its transfer functor

$$\mathrm{Rst}_{\underline{A}} : \underline{A} \to Mlr$$

is faithful.

For every RE-category $\underline{A}$, the RE-factorization (5.10) of its transfer functor $\mathrm{Rst}_{\underline{A}}$ will be written

(1) $$\underline{A} \xrightarrow{R_1} \mathrm{Trn}(\underline{A}) \xrightarrow{R_2} Mlr\ .$$

It is easy to see that the (faithful) functor R_2 is isomorphic to $\mathrm{Rst}_{\mathrm{Trn}(\underline{A})}$: therefore $\mathrm{Trn}(\underline{A})$ will be called the *transfer RE-category*, associated to $\underline{A}$.

We say that $\underline{A}$ is *Rst-finite* whenever all the sets $\mathrm{Rst}_{\underline{A}}(A)$ are finite ; in this case the functor $\mathrm{Rst}_{\underline{A}}$ takes values in the full subcategory Mlr^f of finite modular lattices, which is clearly Hom-finite. Therefore every transfer Rst-finite RE-category is Hom-finite.

Obviously these notions agree with the analogous one for exact categories ([10], § 5.1, 5.2).

7.4. We say that the RE-category $\underline{A}$ is *distributive* (or also *orthodox*) if it satisfies the following conditions, equivalent by 6.1, 6.4, [7], Cor. 1.10, and [8], Thm. 2.8 :

a) for any object A, Rst(A) is a distributive lattice,
b) for any morphism $a : A' \to A''$, the mapping

$$a_R : \mathrm{Rst}(A') \to \mathrm{Rst}(A'')$$

is a lattice homomorphism,
c) the componentwise exact category Prp Fct $\underline{A}$ is distributive,
d) the category $\underline{A}$ is orthodox,
e) the category $\overline{\underline{A}}$ is quasi-inverse.

We only recall [6] that a category $\underline{A}$, provided with a regular involution, is *orthodox* when its idempotent endomorphisms are stable for composition.

Then $\underline{A}$ is provided with a canonical preorder $a \mathbb{C} b$ *(domination)* on parallel morphisms, consistent with composition and involution, defined by the following equivalent conditions :

(1) $$a = a\tilde{b}a,$$

(2) $$a = (a\tilde{a})b(\tilde{a}a),$$

(3) there exist idempotent endomorphisms e, f such that $a = fbe$,

(4) there exist projections e, f such that $a = fbe$.

The quotient of $\underline{A}$ modulo the associated congruence Φ is an inverse category.

For more informations about orthodox categories, inverse categories and their links with induction, canonical isomorphisms and distributive exact categories, see [6, 7, 8, 10] and their references.

7.5. Analogously, we say that $\underline{A}$ is *boolean* when, for every object A, the lattice Rst(A) is a boolean algebra ; i.e., when the associated componentwise exact category Prop(Fct(A)) is boolean (see [10], 6.1 and characterization 6.4).

7.6. If $F : \underline{A} \to \underline{B}$ is a RE-functor, it follows easily from 7.2 that :

a) if F is surjective on the objects and Rst-full, while $\underline{A}$ is distributive (resp. boolean), so is $\underline{B}$.

b) if F is Rst-faithful and $\underline{B}$ is distributive (resp. boolean), so is $\underline{A}$.

7.7. Each RE-category $\underline{A}$ has an associated *modular expansion* Mdl($\underline{A}$). The objects are the pairs (A, X), where A is an object of $\underline{A}$ and X is a (modular) sublattice of $\mathrm{Rst}_{\underline{A}}(A)$ containing its least and greatest elements (ω_A and 1_A). The morphisms

$$a = (a\,;\, X, X') : (A, X) \to (A', X')$$

are those morphisms $a \in \underline{A}(A, A')$ such that :

(1) $$a_R(X) \subset X', \quad a^R(X') \subset X.$$

In particular :

(2) $$\underline{def}\ a = a^R(1_{A'}) \in X, \quad \underline{ann}\ a = a^R(\omega_{A'}) \in X,$$

(3) $$\underline{val}\ a = a_R(1_A) \in X', \quad \underline{ind}\ a = a_R(\omega_A) \in X'.$$

The composition, involution and order of $\mathrm{Mdl}(\underline{A})$ are those of $\underline{A}$.

7.8. If (A, X) is in $\mathrm{Mdl}(\underline{A})$ and $e \in \mathrm{Prj}\,(A)$:

(1) $$\underline{e} \in \mathrm{Prj}(A, X) \quad \text{iff} \quad \underline{n}e \text{ and } \underline{d}e \text{ are in } X,$$

because

$$\underline{n}(e) = e_R(1) \quad \text{and} \quad \underline{d}(e) = e_R(\omega),$$

while for $x \in \mathrm{Rst}(A)$,

$$e_R(x) = \underline{n}(exe) = (\underline{n}e) \cap (x \cup \underline{d}e) \quad (6.9\ \text{a}).$$

This proves that $\mathrm{Mdl}(\underline{A})$ is a RE-category, and that the transfer functor $\mathrm{Rst} : \mathrm{Mdl}(\underline{A}) \to \mathit{Mlr}$ is described by :

(2) $$\mathrm{Rst}(A, X) = X,$$

(3) $$\mathrm{Rst}(a\,;\, X, X') = (a_R : X \to X',\ a^R : X' \to X).$$

7.9. There is an obvious faithful RE-functor

$$U : \mathrm{Mdl}\ \underline{A} \to \underline{A}, \quad (A, X) \mapsto A.$$

Every RE-functor $F : \underline{B} \to \underline{A}$ has a *unique Rst-full lifting*

(1) $$F^{\#} : \underline{B} \to \mathrm{Mdl}\ \underline{A},$$

(2) $$F^{\#}(B) = (F(B), X_B)\ ; \quad F^{\#}(b) = F(b),$$

(3) $$X_B = \mathrm{Im}(\mathrm{Rst}_F(B) : \mathrm{Rst}_{\underline{B}}(B) \to \mathrm{Rst}_{\underline{A}}(FB))$$

verifying $F = UF^{\#}$.

7.10. The *distributive expansion* $\mathrm{Dst}(\underline{A})$ of the RE-category $\underline{A}$ is the full subcategory of $\mathrm{Mdl}(\underline{A})$ having objects (A, X) where X is *distributive*. It is a distributive RE-category, by 7.8.2. The faithful RE-functor $U : \mathrm{Dst}(\underline{A}) \to \underline{A}$ solves the above lifting problem (7.9) whenever $\underline{B}$ is distributive. Analogously one defines the *boolean expansion* $\mathrm{Bln}(\underline{A})$.

7.11. Last we remark that larger modular, distributive and boolean expansions can be built like in [10], § 6.6, via horizontal comma squares of vertical RE-functors.

8. Idempotent RE-categories.

8.1. We say that the RE-category $\underline{A}$ is *idempotent* if all its endomorphisms are so.

In such a case, for parallel morphisms a, b (by 2.8) :

(1) $$a = b \text{ iff } \underline{c}(a) = \underline{c}(b) \text{ and } \underline{i}(a) = \underline{i}(b)$$

and a RE-functor $F : \underline{A} \to \underline{B}$ is faithful iff it is Prj-faithful (iff it is Rst-faithful).

8.2. Every idempotent RE-category is trivially orthodox (i.e., distributive), and also transfer, by the above remark. Every idempotent Rst-finite RE-category is Hom-finite, by 7.3.

8.3. If $F : \underline{A} \to \underline{B}$ is a RE-functor, it is easy to see that :

a) if F is a RE-quotient and $\underline{A}$ is idempotent, so is $\underline{B}$,
b) if F is faithful and $\underline{B}$ is idempotent, so is $\underline{A}$.

8.4. Theorem. Let $F : \underline{A} \to \underline{B}$ and $G : \underline{A} \to \underline{C}$ be RE-functors

(1) $$\begin{array}{ccc} \underline{A} & \xrightarrow{F} & \underline{B} \\ {\scriptstyle G}\downarrow & \nearrow {\scriptstyle H} & \\ \underline{C} & & \end{array}$$

If $\underline{A}$ is idempotent and G is a RE-quotient, the following conditions are equivalent :

a) F factors through G (via a unique RE-functor H),
b) for all morphisms a, a' of $\underline{A}$, if $G(a) = G(a')$ then
$$F(a) = F(a') ,$$
c) for all projections e, f of $\underline{A}$, if $G(e) = G(f)$ then $F(e) = F(f)$,
d) for all restrictions e, f of $\underline{A}$, if $G(e) = G(f)$ then $F(e) = F(f)$,
e) for every projection e of $\underline{A}$, if $G(e)$ is null so is $F(e)$.

If $\underline{A}$ is also factorizing, the above conditions are also equivalent to :

f) for each projection e of $\underline{A}$, if $G(e) = 1$ then $F(e) = 1$,

g) for each object A of $\underline{A}$, if G(A) is null so is F(A).

Proof. a ⇒ e and d ⇒ c are obvious.
e ⇒ d : Let $e, f \in \mathrm{Rst}(A)$ with $G(e) = G(f)$, and consider the projections (6.10) :

(2) $$e_1 = e/(e.f), \quad e_2 \doteq f/(e.f).$$

By 4.9, $G(e_i)$ is null, and so is $F(e_i)$ for $i = 1, 2$: in other words

$$Fe \propto Ff \propto Fe, \quad \text{and} \quad F(e) = F(f) .$$

c ⇒ b : We can suppose that F too is a RE-quotient (otherwise, use the RE-factorization of F (5.10)), so that $\underline{B}$ is idempotent. The conclusion follows from 8.1 : if $G(a) = G(a')$, then

$$G(\tilde{a}a) = G(\tilde{a}'a'), \quad \text{and} \quad \underline{c}(Fa) = F(\tilde{a}a) = F(\tilde{a}'a') = \underline{c}(Fa') ;$$

analogously, $\underline{i}(Fa) = \underline{i}(Fa')$.

b ⇒ a : Define :

(3) $$H(C) = F(A), \quad H(c) = F(a),$$

where $G(A) = C$ and $G(a) = c$. H is obviously a functor, which preserves involution and null morphisms. It also preserves the order by 5.2 : if $e = G(a)$ is a restriction of $\underline{C}$, then

$$e = G(\tilde{a}a) \text{ and also } \quad e = \underline{n}e = G(\underline{n}(\tilde{a}a)),$$

so that $H(e) = F(\underline{n}(\tilde{a}a))$ is a restriction.

Last, suppose that $\underline{A}$ is factorizing.
a ⇒ f is obvious. f ⇒ g : if G(A) is null, then $G(\omega_A) = 1_{GA}$, so that

$$\omega_{FA} = F(\omega_A) = 1_{FA}$$

and F(A) is null.

g ⇒ e : if $e : A \to A$ is a projection of $\underline{A}$ with epi-monic factorization

$$A \twoheadrightarrow A_0 \rightarrowtail A,$$

and $G(e)$ is null, so is the object $G(A_0)$, hence so are $F(A_0)$ and $F(e)$.

8.5. It follows immediately that the two RE-quotients $F : \underline{A} \to \underline{B}$ and $G : \underline{A} \to \underline{C}$ of an idempotent RE-category $\underline{A}$ are *equivalent* (that is, there exists an isomorphism of RE-categories H which makes 8.4.1 commutative) iff F and G annihilate the same projections of $\underline{A}$ (or also the same objects, provided that $\underline{A}$ is factorizing). In other words, a class of equivalent RE-quotients $F : \underline{A} \to \underline{B}$ is determined by "Ker F ", the subset of $\underline{A}$ containing the projections annihilated by F.

8.6. Theorem. Let $\underline{A}$ be an idempotent RE-category RE-spanned by its subgraph Δ (5.8). For every object A of $\underline{A}$ the distributive 0,1-lattice $Rst_{\underline{A}}(A)$ is spanned by its subset :

$$X^o_A = \{ \underline{val}(a), \underline{ind}(a) \mid a \in M_A \} \tag{1}$$

where M_A is the set of those morphisms a in $\underline{A}$ which can be written as a composition,

$$a = a_n \dots a_2 a_1 \qquad (\mathrm{Cod}\, a_n = A), \tag{2}$$

$$a_i \in \Delta \quad \text{or} \quad \tilde{a}_i \in \Delta , \quad \text{for every} \quad i, \tag{3}$$

$$\mathrm{Cod}\, a_i \neq \mathrm{Cod}\, a_j, \quad \text{for} \quad i \neq j \quad [10]. \tag{4}$$

If all these subsets X^o_A are finite, $\underline{A}$ is Hom-finite. If Δ is finite, so is $\underline{A}$; in other words : *a finitely generated idempotent RE-category is finite* [11].

Proof. First, notice that Δ and $\underline{A}$ have the same objects (5.8). Let $t : \Delta \to \underline{A}$ be the inclusion morphism, and consider the embedding

$$t_1 : \Delta \longrightarrow \mathrm{Dst}(\underline{A}), \tag{5}$$

$$t_1(A) = (A, X_A) , \quad t_1(d) = d , \tag{6}$$

where, for every A, X_A is the (distributive) sub-0, 1-lattice of Rst(A) spanned by X^o_A. This statement requires checking that, for $d \in \Delta(A, A')$,

$$d_R(X_A) \subset X_{A'} , \qquad d^R(X_{A'}) \subset X_A . \tag{7}$$

Since $\underline{A}$ is distributive, d_R and d^R are lattice-homomorphisms (7.4), generally not preserving the extremes ; thus we only need to verify that

$$d_R(X^o_A \cup \{0, 1\}) \subset X_{A'}, \qquad d^R(X^o_A \cup \{0, 1\}) \subset X_A . \tag{8}$$

For example, let $a \in M_A$ satisfy the conditions (2)-(4), and verify that $d_R(\underline{val}(a)) \in X_{A'}$. If

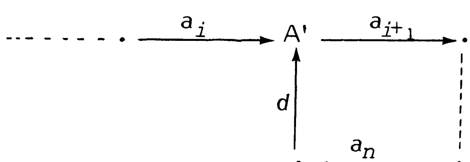
$$A' = \mathrm{Cod}\, d \neq \mathrm{Cod}\, a_i \quad \text{for every} \quad i = 1, 2, \dots, n,$$

then

$$d_R(\underline{val}(a)) = \underline{val}(da) \in X^o_{A'}.$$

Otherwise, $A' = \mathrm{Cod}\, a_i$ for one index i :

(9)



([10]) That is, the "path" a has at most one initial loop, when $\mathrm{Dom}\, a_1 = \mathrm{Cod}\, a_i$, for one index $i = 1, 2, \dots, n$.

([11]) Notice that a finitely generated idempotent semigroup (or category) is generally infinite [19].

Decompose $da = e.b$, where

$$b = a_i \dots a_1 \in M_{A'} \quad \text{and} \quad e = da_n \dots a_{i+1} \in M_{A'} ;$$

the last e is an idempotent, hence in the *quasi-inverse* (7.4) semigroup $\underline{A}(A', A')$:

$$(10) \qquad e(b\tilde{b})\tilde{e} = e\tilde{e}e(b\tilde{b})\tilde{e}e\tilde{e} = e\tilde{e}(b\tilde{b})e\tilde{e}.$$

Thus

$$(11) \qquad d_R(\underline{val}\ a) = \underline{d}_R a_R(1) = (eb)_R(1) = \underline{n}(eb\tilde{b}\tilde{e}) = \underline{n}(e\tilde{e}(b\tilde{b})e\tilde{e}) =$$
$$= (\underline{n}(e\tilde{e}) \wedge \underline{n}(b\tilde{b})!) \vee \underline{d}(e\tilde{e}) = (\underline{val}\ e \wedge \underline{val}\ b) \vee \underline{ind}\ e,$$

where the second equality follows from the functoriality of transfer (7.1), the third from its definition, the fourth from (10), the fifth from 6.9 a, and the last from 4.10. Since both b and e are in $M_{A'}$, we have $d_R(\underline{val}\ a) \in X_{A'}$ and our partial goal is reached.

Now, call $\underline{A}_1$ the RE-subcategory of Dst($\underline{A}$) RE-spanned by $t_1(\Delta)$, $U_1 : \underline{A}_1 \to \underline{A}$ the restriction of the forgetful functor $U : \text{Dst}(\underline{A}) \to \underline{A}$, and $t'_1 : \Delta \to \underline{A}$ the restriction of t_1. As $t = U\, t_1 = U_1 t'_1$, it follows that U_1 is bijective on the objects (so are t and t'_1) and full (because t is RE-spanning and by 5.7) ; therefore U_1 is Rst-full, and for every object A of $\underline{A}$:

$$(12) \qquad \varphi = \text{Rst}_{U_1}(A, X_A) : \text{Rst}_{\underline{A}_1}(A, X_A) \to \text{Rst}_{\underline{A}}(A)$$

is surjective : in other words, $\text{Rst}(A) = \text{Im}\ \varphi = X_A$.

Finally, a finitely generated distributive lattice is finite. Thus, if all the sets X^o_A are finite, $\underline{A}$ is Rst-finite, and also Hom-finite by 8.2. Now, if Δ is finite, so are the sets X^o_A and $\underline{A}$ is Hom-finite by the above argument ; moreover $\underline{A}$ has a finite set of objects (the same as Δ), hence it is finite.

8.7. We say that the (componentwise) exact category $\underline{E}$ is *pre-idempotent* when Rel($\underline{E}$) is idempotent ; a direct characterization will be given in 8.8 c.

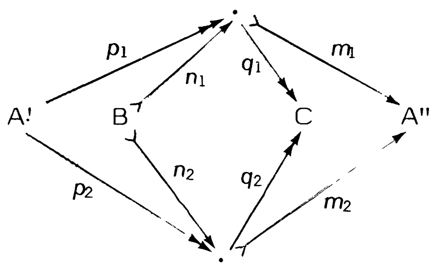
The paradigmatic example is the category $\mathcal{I}_0$ of *small sets and common parts* : a morphism $L : S \to T$ is any common subset L of the small sets S and T ; the composition is the intersection.

8.8. Theorem. Let $\underline{A}$ be a *factorizing* RE-category and $\underline{E}$ = Prp $\underline{A}$ the associated componentwise exact category. The following conditions are equivalent :

a) $\underline{A}$ is idempotent,
b) for every two parallel monics $h, k : L \to A$ in $\underline{A}$, $\tilde{k}h = 1_L$,
c) if

$$u_i = m_i p_i : A' \to A'' \qquad (i = 1, 2)$$

are canonical factorizations in $\underline{E}$ and in the (generally non commutative) diagram of $\underline{E}$

(1) 

the epi-square is a pushout and the monic-square a pullback, then q_1n_1 and q_2n_2 are equal isomorphisms [12]).

When these conditions hold :

d) every two parallel monics of $\underline{E}$ coincide.

Proof. Trivially a and b are eqiivalent.

b $\Rightarrow$ c : Let $v_i = q_i n_i$ in $\underline{E}$; since pushouts of epis are bicommutative in $\underline{A}$, we have :

$$\tilde{v}_2 v_1 = \tilde{n}_2\tilde{q}_2 q_1 n_1 = \tilde{n}_2 p_2 \tilde{p}_1 n_1 = (\tilde{p}_2 n_2)^{\sim} (\tilde{p}_1 n_1) = 1_B ,$$

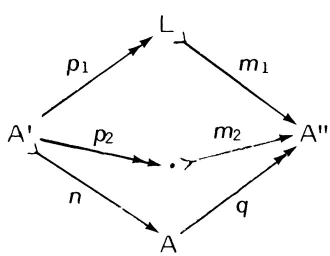
analogously $v_1 \tilde{v}_2 = 1_C$. Thus v_1 and $\tilde{v}_2$ are reciprocal isomorphisms, and $v_1 = v_2$.
c $\Rightarrow$ d : If $m_1, m_2 : M \to A''$ are parallel monics of $\underline{E}$, take

$$p_1 = p_2 = q_1 = q_2 = 1_M :$$

then n_1 and n_2 are equal isos and $m_1 = m_2$.
c $\Rightarrow$ b : Let $h, k : L \to A$ be parallel monics in $\underline{A}$, and consider the following factorizations

(2) $$h = n\tilde{p}_1 , \quad k = \tilde{q}m_1 ,$$

where p_1, q are $\underline{E}$-epis and n, m_1 are $\underline{E}$-monics :

(3) 

([12]) It can be seen that $m_1 n_1 (q_1 n_1)^{-1} q_1 p_1 = m_2 n_2 (q_2 n_2)^{-1} q_2 p_2$ is the intersection of u_1 and u_2 with respect to the canonical order $\mathfrak{C}$ on $\underline{E}$.

then factorize $qn = m_2p_2$ in $\underline{E}$, and apply the condition c to the inner square :

$$\tilde{k}h = \tilde{m}_1\, q n \tilde{p}_1 = \tilde{m}_1 m_2\, p_2 \tilde{p}_1 = n_1 \tilde{n}_2\, \tilde{q}_2 q_1 = n_1(q_2 n_2)^{-1} q_1$$

thus $\tilde{k}h$ is a proper morphism. Analogously, $(\tilde{k}h)^{\sim} = \tilde{h}k$ is proper : it follows that $\tilde{k}h : L \to L$ is iso in $\underline{E}$. Since it is parallel to 1_L, by the above argument (c $\Rightarrow$ d), $\tilde{k}h = 1_L$.

9. RE is complete.

We prove here that **RE** is 2-complete [13], while its sub-2-category **REX** (or the equivalent 2-category **EX**) is only 2-pseudocomplete.

9.1. **RE** has *small 2-products*. If $(\underline{A}_i)_{i\in I}$ is a family of RE-categories indexed on a small set, take $\underline{A} = \Pi\, \underline{A}_i$ the usual product in **CAT**, with the obvious involution and order ; $\underline{A}$ is a RE-category, the canonical functors $P_i : \underline{A} \to \underline{A}_i$ are RE-functors, and satisfy the 2-universal property : for any family

$$\alpha_i : F_i \to G_i : \underline{B} \to \underline{A}_i \qquad (i \in I)$$

of RE-transformations there is exactly one RE-transformation

$$\alpha : F \to G : B \to A \qquad \text{such that} \qquad P_i \circ \alpha = \alpha_i \qquad (i \in I) :$$

actually, for any object B of $\underline{B}$, $\alpha(B) = (\alpha_i B)_{i\in I}$.

9.2. It will be noticed that the terminal object of **RE** (the product of the empty family) is the RE-category $\underline{1}$ whose only object and morphism we write 0 and 1_0 , with trivial involution and order.

9.3. **RE** has *2-equalizers*. Let $F, G : \underline{A} \to \underline{B}$ be RE-functors, and $\underline{Z} \subset \underline{A}$ their usual equalizer in **CAT**, provided with the induced involution and order. By 5.1 and 5.7, $\underline{Z}$ is a RE-category and the inclusion $J : \underline{Z} \to \underline{A}$ a RE-functor. The 2-universal property is satisfied, in the same way as in **CAT**.

9.4. Thus **RE** has all conical 2-limits [13]. Analogously **RO**, and the inclusion 2-functor **RE** $\to$ **RO** creates these limits.

It can also be proved that the forgetful functor $\mathbf{RE}_1 \to \mathbf{CAT}_1$ between the underlying categories creates conical limits. For example, if $P_i : \underline{A} \to \underline{A}_i$ ($i \in I$) is the CAT-product of the family $(\underline{A}_i)$ of RE-categories, and

$$\underline{A}_1 = (\underline{A}, \sim_1, \leqq_1), \quad \underline{A}_2 = (\underline{A}, \sim_2, \leqq_2)$$

are RE-structures on $\underline{A}$ agreeing with all P_i, then trivially $\sim_1 = \sim_2$; moreover, if $e \in \mathrm{Rst}_{\underline{A}_1}(A)$ decomposes (according to $\underline{A}_2$) as

$$e = e'e'', \quad \text{with} \quad e' \leqq_2 1_A \leqq_2 e'',$$

then

$$P_i(e) = (P_i e').(P_i e'')$$

is a restriction-corestriction factorization in $\underline{A}$ of the restriction $P_i(e)$; therefore $P_i e = P_i e'$ for every i, that is $e = e' \in \mathrm{Rst}_{\underline{A}_2}(A)$. According to 5.6, $\underline{A}_1 = \underline{A}_2$.

9.5. RE has *2-comma squares*. Let

$$F : \underline{A} \to \underline{C}, \qquad G : \underline{B} \to \underline{C}$$

be RE-functors :

(1)
$$\begin{array}{ccc} \underline{A} & \xrightarrow{F} & \underline{C} \\ {\scriptstyle P_1}\uparrow & \overset{\alpha}{\searrow} & \uparrow{\scriptstyle G} \\ \underline{Z} & \xrightarrow{P_2} & \underline{B} \end{array}$$

and build the category $\underline{Z}$ whose objects are the triples

(2) $\quad (A, B, u : FA \to GB), \quad A \in \mathrm{Ob}\,\underline{A}, \quad B \in \mathrm{Ob}\,\underline{B}, \quad u \in \mathrm{Prp}\,\underline{C}$

while the morphisms are pairs

(3) $$(a, b) : (A, B, u) \to (A', B', u')$$

with

$$a \in \underline{A}(A, A'), \quad b \in \underline{B}(B, B'), \quad \text{and}$$

(4)
$$\begin{array}{ccc} FA & \xrightarrow{u} & GB \\ {\scriptstyle Fa}\downarrow & \leqq & \downarrow{\scriptstyle Gb} \\ FA' & \xrightarrow{u'} & GB' \end{array} \qquad \text{(RO-square)}$$

The RE-structure of $\underline{Z}$ is obvious ; P_1, P_2 and $\alpha : FP_1 \to GP_2$ are as usual :

(5) $\quad P_1(A, B, u) = A, \quad P_1(a, b) = a,$

(6) $\quad P_2(A, B, u) = B, \quad P_2(a, b) = b,$

(7) $\quad \alpha(A, B, u) = (u : FA \to GB),$

and α is a RE-transformation by (4). The 2-universal property in **RE** is trivially satisfied. We write $\underline{Z} = (F \downarrow G)$; it should be noticed that the category underlying $\underline{Z}$ is not even a subcategory of the CAT-comma category of F and G.

9.6. Theorem. The 2-category **RE** is complete with regard to limits indexed by 2-functors $J : \mathbf{D} \to \mathbf{Cat}$, where **D** is a small 2-category.

Proof. The 2-category **RE** is naturally enriched over **Cat'**, the cartesian closed category of U'-small categories, with U' some universe such that $U \in U'$. Its completeness with regard to the above considered indexed limits depends [21, 12] on the existence of small conical limits (proved in 9.1, 9.3) and of cotensor products of the form $\underline{2} \pitchfork \underline{A}$. This is the solution in **RE** of finding a natural isomorphism :

(1) $$\varphi_{A,B} : \mathbf{RE}(\underline{B}, \underline{2} \pitchfork \underline{A}) \to \mathbf{Cat'}(\underline{2}, \mathbf{RE}(\underline{B}, \underline{A})).$$

The solution is clearly

$$\underline{2} \pitchfork \underline{A} = (1_{\underline{A}} \downarrow 1_{\underline{A}}) \qquad (9.5)$$

the isomorphism being defined by the universal property of comma squares.

9.7. Last, we remark that the forgetful 2-functor $\mathbf{EX} \to \mathbf{CAT}$ creates 2-products (trivial), does not create equalizers or pullbacks which generally fail in **EX** [17], while it does create 2-pseudo-equalizers. Thus **EX** is 2-pseudocomplete, hence bicomplete, while it is not complete.

Istituto di Matematica
Università di Genova
Via L. B. Alberti, 4
I-16132 GENOVA. ITALIE

References.

1. A. (BASTIANI) & C. EHRESMANN, Multiple functors : I. Limits relative to double categories (Cahiers Top. et Géom. Diff. **XV**-2, 1974), re-edited in : "Charles Ehresmann: Oeuvres complètes et commentées", **IV**-2, Amiens, 1984.
2. H.B. BRINKMANN & D. PUPPE, Abelsche und exacte Kategorien, Korrespondenzen, Lecture Notes in Math. **96**, Springer (1969).
3. M.S. CALENKO, Correspondences over a quasi-exact category, Dkl. Akad. Nauk SSSR **155** (1964), 292-294. - Soviet Math. Dkl. **5** (1964), 416-418.
4. M.S. CALENKO, Correspondences over a quasi-exact category, Mat. Sbornik **73** (1967), 564-584. - Math. USSR Sbornik **2** (1967), 501-519.
5. M. GRANDIS, Symétrisations de catégories et factorisations quaternaires, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. **14**, Sez. 1 (1977), 133-207.
6. M. GRANDIS, Regular and orthodox involution categories (Orthodox categories 2), Boll. Un. Mat. Ital. **14**-A (1977), 39-48.
7. M. GRANDIS, Quaternary categories having orthodox symmetrizations (Orthodox Symmetrizations, I), Boll. Un. Mat. Ital. **14**-B (1977), 605-629.
8. M. GRANDIS, Exact categories and distributive lattices (Orthodox Symmetrizations, 2), Ann. Mat. Pura Appl. **118** (1978), 325-341.
9. M. GRANDIS, Canonical models in homological algebra, Internal Report, Genova, 1982.
10. M. GRANDIS, Transfer functors and projective spaces, Math. Nachr., to appear.
11. J.W. GRAY, Formal category theory : adjointness for 2-categories, Lecture Notes in Math. **391**, Springer (1974).
12. J.W. GRAY, The existence and construction of lax limits, Cahiers Top. et Géom. Diff. **XXI**-3 (1980), 277-304.
13. G.M. KELLY, Basic concepts of enriched category theory, Cambridge University Press, Cambridge, 1982.
14. G.M. KELLY & R. STREET, Review of the elements of 2-categories, Lecture Notes in Math. **420**, Springer (1974), 75-103.
15. S. MAC LANE, Categories for the working mathematician, Springer, 1971.
16. M. MAKKAI & G.E. REYES, First order categorical logic, Lecture Notes in Math. **611**, Springer (1977).
17. M. MARGIOCCO & F. MORA, Sulla categoria delle categorie esatte, Rend. Ist. Mat. Univ. Trieste, **13** (1981), 6-22.
18. B. MITCHELL, Theory of categories, Academic Press, 1965.
19. M. MORSE & G.A. HEDLUNG, Unending chess, symbolic dynamics and a problem in semigroups, Duke Math. J. **11** (1944), 1-7.
20. D. PUPPE, Korrespondenzen in abelschen Kategorien, Math. Ann. **148** (1962), 1-30.
21. R. STREET, Limits indexed by category-valued 2-functors, J. Pure Appl. Alg. **8** (1976), 149-181.
22. R. STREET, Fibrations in bicategories, Cahiers Top. et Géom. Diff. **XXI**-2 (1980), 111-160.
23. E.C. ZEEMAN, On the filtered differential group, Ann. Math. **66** (1957), 557-585.