Cahiers de
topologie et géométrie différentielle
catégoriques

Marta Bunge
Patrice Sawyer

On connections, geodesics and sprays in synthetic differential geometry

Cahiers de topologie et géométrie différentielle catégoriques, tome 25, nº 3 (1984), p. 221-258
<http://www.numdam.org/item?id=CTGDC_1984__25_3_221_0>

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

Vol. XXV-3 (1984)

ON CONNECTIONS, GEODESICS AND SPRAYS IN SYNTHETIC DIFFERENTIAL GEOMETRY

by Marta BUNGE and Patrice SAWYER

Résumé. Cet article traite de la théorie des connexions en Géométrie différentielle synthétique. Il possède deux principales facettes. Nous discutons d'abord de plusieurs notions classiques de façon synthétique. Nous utilisons ensuite ces préliminaires pour établir l'équivalence, sous certaines conditions des concepts suivants : connexion et fonction de connexion, connexion et gerbe, et leurs géodésiques respectives. Nous comparons enfin deux notions synthétiques de gerbes dues à Joyal et à Lawvere.

0. INTRODUCTION.

A synthetic definition of what a connexion is has already been given by Kock and Reyes [4]; they also partially compare it therein with the notion of a connection map in the sense of Dombrowski and Patterson [7]. Other (combinatorial) notions of a connection were later proposed by Kock and are still being systematically exploited. In this paper we pursue the definitions given in [4], and push them further in order to include the relationship between connections, geodesics and sprays ; a topic not yet touched upon in any detail by the various synthetic treatments of this subject. The definition of spray we use was proposed by Lawvere (lecture, Topos meeting, McGill, October 10, 1981) ; it is the synthetic version of the usual classical notion.

After showing the equivalence (under some extra assumption ; the "short path lifting property") of connections in the two senses mentioned above ; i.e. that of Kock and Reyes on the one hand and the infinitesimal form of the Dombrowski-Patterson connection maps on the other, we proceed to the study of prolongations (cf. also [7]). We then define covariant differentiation of vector fields over a curve, and use it to give a basic definition of geodesic for a connection.

The main theorems in this paper refer to the passages : connection - geodesic spray and spray - torsion-free connection, which we prove synthetically. Both of these proofs, we feel, provide ample evidence of the power and simplifying nature of the synthetic (versus the classical) point of view.

In the process of establishing the Ambrose-Palais-Singer Theorem this way, we need to assume a "property of existence of exponential maps". This is done in two ways, a local and an infinitesimal one, according to which notion of spray we are considering at the moment. Indeed, alongside the synthetic version of spray proposed by Lawvere, we consider yet another synthetic version proposed by Joyal (lecture, McGill University, November 23, 1981) and then compare the two. We find that Joyal sprays, although simpler from the point of view of stating the homogeneity condition, are a bit more special than the Lawvere sprays in the synthetic context. However, the assumptions needed to render them equivalent are mild, and certainly true of classical manifolds. Among these is a property which we label the "iterated tangent bundle property" ; of this we prove it holds of any locally parallelizable object in any model.

We assume familiarity with portions of Kock's book [3] as well as with the article of Kock and Reyes [4].

The contents of this paper were delivered by the first author at a workshop which took place in Aarhus in June 1983. A preliminary version of this paper appears in the corresponding volume, edited by A. Kock : Category Theoretic Methods in Geometry, Aarhus Var. Pub. Ser. 35, 1983. Thanks are given to the Danish Science Research Council of Canada for the financial support which made it possible for the first author to attend this workshop. The second author is presently holding a scholarship of the National Sciences and Engineering Research Council of Canada.

Thanks are also due to Anders Kock and Bill Lawvere whose remarks were very useful when preparing the final version of this paper.

1. CONNECTIONS VERSUS CONNECTION MAPS.

A synthetic definition of a connection is given in [4] as the infinitesimal germ of parallel transport. An alternative interpretation of what connections are used for, leads to the notion of a connection map. Connection maps (also in their infinitesimal guise) provide a simple approach to prolongations and to Koszul derivatives. For this reason, it is important that we begin by establishing the equivalence of the two data provided by a connection and by a connection map. This is partially carried out in [4] and completed below.

As in [4], we assume that $p : E \to M$ is a vector bundle, i.e., an R-module in $\mathcal{E}/M$, with E, M infinitesimally linear. The basic example will be the tangent bundle

$$\pi_M : M^D \to M,$$

but it will be important to use two different notations for it in order to deal with parallel transport while distinguishing the vector being transported from that which effects the transport.

On E^D there are two linear structures. Since E is infinitesimally linear, there is the tangential addition on the tangent bundle $\pi_E : E^D \to E$. We shall denote it by $\oplus$. Corresponding to it is the scalar multiplication given by

$$(\lambda \odot f)(d) = f(\lambda d) .$$

E^D is also equipped with $p^D : E^D \to M^D$. For f, g in the same fiber

$$(\text{i.e., } p^D(f) = p^D(g), \quad \text{or} \quad p \circ f = p \circ g)$$

the following addition is defined :

$$(f + g)(d) = f(d) + g(d).$$

Together with

$$(\lambda \cdot f)(d) = \lambda \cdot f(d),$$

this is an R-module structure on the fibers.

Consider now the map

$$K = \langle p^D, \pi_E \rangle : E^D \to M^D \times_M E ;$$

notice that the diagrams

$$\begin{array}{ccc} E^D & \xrightarrow{K} & M^D \times_M E \\ & & \downarrow{\scriptstyle proj_2} \\ {\scriptstyle \pi_E}\downarrow & & E \\ & & \downarrow{\scriptstyle p} \\ E & \xrightarrow{p} & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} E^D & \xrightarrow{K} & M^D \times_M E \\ & & \downarrow{\scriptstyle proj_1} \\ \downarrow{\scriptstyle p^D} & & M^D \\ & & \downarrow{\scriptstyle \pi_M} \\ M^D & \xrightarrow{\pi_M} & M \end{array}$$

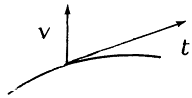
commute. Using the vertical maps on the right to define $\oplus$, respectively $+$, the linear structures on $M^D \times_M E$ in the obvious fashion, K ends up being linear with respect to the two structures (cf [4]).

Definition. A *connection on* $p : E \to M$ is a splitting ∇ of K, which is linear with respect to both the $\oplus$ and the $+$ structures.

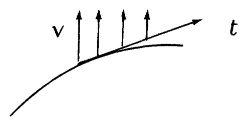
Take the case of the tangent bundle $\pi_M : M^D \to M$, where we assume furthermore that it is a trivial bundle (i.e., M is "parallelizable"), i. e., that a Euclidean R-module V exists with

$$\begin{array}{ccc} M^D & \simeq & M \times V \\ & {\scriptstyle \pi_M}\searrow \quad \swarrow{\scriptstyle proj_1} & \\ & M & \end{array}$$

In this case, a connection on $\pi_M : M^D \to M$ is a rule which, given *(t, v)* in $M^D \times V$,



allows for v to be transported in parallel fashion along the (infinitesimal portion of the curve with velocity) vector *t* , in pictures : the infinitesimal diagram can be completed as follows :



The importance of parallel transport is that it allows for the comparison of velocity functors v, *w* attached to different (nearby) positions along a curve. This transport must therefore be required to be a linear map between the tangent spaces involved ($\oplus$-structures). The other linearity will find justification later, when we deal with sprays.

Now, one reason for the necessity of comparing nearby velocity vectors is the need for computing acceleration in terms of small changes in the velocity vectors. Given a motion $\xi(t)$ in M, we may always consider the iterated vector field $\xi''(t)$ of the velocity field $\xi'(t)$, but in order to interpret it as acceleration, it is necessary to have a rule which allows for the reduction of second order data to first order data. In other words, what is needed in general is a map $C : E^D \to E$ with some good properties.

First of all, there is a map $\nu : E \rightarrowtail E^D$ which identifies the fiber E_m for $m \in M$ with the tangent space to this fiber at 0_m, and is given by the rule

$$v \longmapsto [d \mapsto d \cdot v] .$$

It is natural to require that this lifting of first order data in a trivial way to second order data gives back the original data when a connection map be applied ; in other words, one should have

$$E \xrightarrow{\nu} E^D \xrightarrow{C} E = id_E .$$

Secondly, some linearity assumptions are in order to insure good behavior. Following [7], let us give the following

Definition. A *connection map on p :* $E \to M$ is a map

$$C : E^D \to E, \quad \text{with} \quad C \circ \nu = id_E$$

and C linear with respect to the two structures $\oplus$, + on E^D (and the only available structure on $p : E \to M$).

In this definition, the map C is also supposed to commute with the appropriate maps, without which it makes no sense to speak of linearity. The diagrams in question are :

$$\begin{array}{ccc} E^D & \xrightarrow{C} & E \\ \downarrow{\scriptstyle \pi_E} & & \downarrow{\scriptstyle p} \\ E & \xrightarrow{p} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} E^D & \xrightarrow{C} & E \\ \downarrow{\scriptstyle p^D} & & \downarrow{\scriptstyle p} \\ M^D & \xrightarrow{\pi_M} & M \end{array}$$

and the commutativity of any one of these insures that of the other, on account of the naturality of π ("base point").

Let us consider now the map

$$E \times_M E \xrightarrow{H} E^D$$

given by

$$(u, v) \mapsto [\, d \mapsto u + d\, v\,].$$

This map is linear with respect to $\oplus$ and +, if these structures are put on

$$E \times_M E \xrightarrow{proj_1} E, \qquad E \times_M E \xrightarrow[p \circ proj_2]{p \circ proj_1 =} M$$

respectively, in the obvious fashion, i.e.,

$$(u, v_1) + (u, v_2) = (u, v_1 + v_2), \qquad \lambda \odot (u, v) = (u, \lambda v),$$

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2), \qquad \lambda \bullet (u, v) = (\lambda \cdot u, \lambda \cdot v).$$

Assume now that $p : E \to M$ is a *Euclidean* R-module in *E*/M. This implies (cf. [4]) that $H = Ker(p^D)$. Thus, for the $\oplus$-structure, $H = Ker(K)$. Using this fact, the map C may be defined out of ∇, as in the diagram

$$\begin{array}{ccc} E^D & \xrightarrow{id \ominus \nabla \circ K} & E^D \\ & {\scriptstyle \langle \pi_E, C \rangle} \searrow & \uparrow{\scriptstyle H} \\ & & E \times_M E \end{array}$$

and is uniquely determined. It is also shown in [4] that C is linear with respect to $\oplus$, + provided ∇ is a connection (affine, in the terminology of [4]). This almost constitutes a proof of the following :

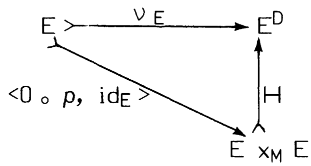
Proposition 1. *Let* $p : E \to M$ *be a Euclidean* R*-module in* **E**/M *with* E , M *infinitesimally linear. Let* ∇ *be a connection on* $p : E \to M$. *Then, there exists a unique connection map* C *on* $p : E \to M$, *satisfying the equation*

$$H \circ \langle \pi_E, C \rangle = id \ominus \nabla \circ K.$$

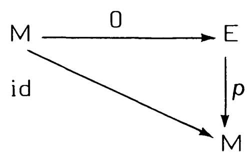
Proof. All that remains to be verified is that

$$C \circ \nu = id_E .$$

For this notice that



commutes, where



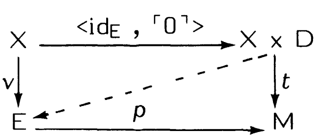
is the zero of the R-module structure of $p : E \to M$ in *E*/M. It follows that for any $v \in E$,

$$H(0_{p(v)}, C(\nu(v))) = H(\nu(v)(0), C(\nu(v)))$$
$$= H(\Pi_E(\nu(v)), C(\nu(v))) = \nu(v) \ominus \nabla \circ K(\nu(v))$$
$$= \nu(v) \ominus \nabla \circ K \circ H \circ \langle 0 \circ p, id \rangle (v)$$
$$= \nu(v) = H(0_{p(v)}, v) .$$

Use now the fact that H is mono. #

Is the data provided by a connection map equivalent to the data provided by a connection? For this we need to retrieve a suitable ∇ (i.e., one satisfying the equation involving ∇ and C) out of a given C. A further assumption is needed for this, as follows :

Definition. Say that $p : E \to M$ has the *short path lifting property* if, given any X and any commutative diagram as shown below, a diagonal fill-in exists (not necessarily unique) :



Remark. If a connection exists on $p: E \to M$, or simply, if K is split epic, then $p: E \to M$ has the SPLP. For example, this is the case for M parallelizable, if we consider the tangent bundle $\pi_M : M^D \to M$, since in this case (identifying it with

$$M \times V \xrightarrow[proj_1]{} V, \text{ V Euclidean R-module)}$$

a splitting for

$$M{\times}V{\times}V{\times}V \xrightarrow{K} M{\times}V{\times}V : (m, u, v, w) \mapsto (m, u, v)$$

is simply given by

$$M{\times}V{\times}V \to M{\times}V{\times}V{\times}V : (m, u, v) \mapsto (m, u, v, 0).$$

We can prove :

Proposition 2. *Let $p : E \to M$ be a Euclidean* R*-module in* ***E***/M *with* E, M *infinitesimally linear. Assume that $p : E \to M$ has the short path lifting property. Then, given any connection* C *on $p : E \to M$, a connection ∇ on $p : E \to M$ exists such that*

$$H \circ < \pi_E, C > = id \ominus V \circ K.$$

Proof. Let $(t, v) \in_X M^D \times_M E$. Consider $f \in_X E^D$ as the exponential adjoint of any diagonal fill-in arising from the assumption that $\pi_M(t) = p(v)$. Let

$$\nabla (t, v) = f \ominus H(f(0), C(f)).$$

The main thing here is to show this is well defined. So, let $g \in_X E^D$ be any other diagonal fill-in. To show

$$f \ominus H(f(0), C(f)) = g \ominus H(g(0), C(g)).$$

Equivalently, show that

$$f \ominus g = H(v, C(f \ominus g)).$$

Now

$$(f \ominus g)(0) = v \quad \text{and} \quad p^D(f \ominus g) = 0_{p(v)},$$

thus,

$$K(f \ominus g) = <p^D(f \ominus g), (f \ominus g)(0)> = <0_{p(v)}, v>$$

which is the zero of the $\oplus$-structure on the fiber of

$$M^D \times_M E \xrightarrow{proj_2} E$$

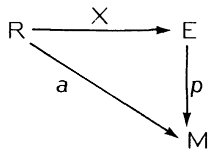
above $p(v)$. Since $H = Ker(K)$ for this structure, we must have that there exists $\tilde{v} \in E$ with $p(v) = p(\tilde{v})$, such that

$$H(v, \tilde{v}) = f \ominus g .$$

Let us employ the notation $f' : R \to V^D$ for just the composite

$$R \xrightarrow{\hat{+}} R^D \xrightarrow{f^D} V^D .$$

This is always available, whether or not V is Euclidean. Assume now that $C : E^D \to E$ is a connection map on $p : E \to M$. Let $a : R \to M$ be a map (curve) and



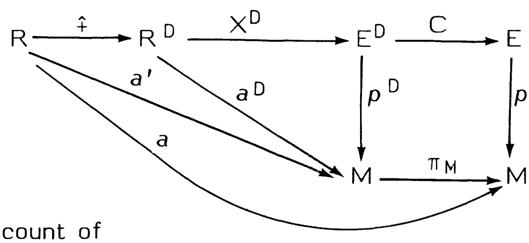
a vector field on E above a. As above, we can always form

$$R \xrightarrow{X'} E^D = R \xrightarrow{\hat{+}} R^D \xrightarrow{X^D} E^D,$$

and this is a vector field on E^D above $a' : R \to M$. In the presence of a connection map C, we define

$$R \xrightarrow{\frac{DX}{dt}} E = R \xrightarrow{\hat{+}} R^D \xrightarrow{X^D} E^D \xrightarrow{C} E$$

and this is now a vector field on E above $a : R \to M$:

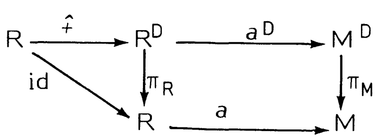


The latter on account of

$$\pi_M \circ a^D \circ \hat{+} = a \circ \pi_R \circ \hat{+} = a.$$

Remark. This is indeed a generalization of the derivative of $f : R \to V$, regarding $V \to 1$ as a vector bundle (with one fiber ; but V Euclidean R-module) and $\gamma : V^D \longrightarrow V$ as a connection map on $V \to 1$. Any map f: $R \longrightarrow V$ may be regarded as a vector field on V along the unique curve $R \to 1$.

Given $a : R \to M$, there is a canonical vector field on $\pi_M : M^D \to M$ over a , namely $a' : R \to M^D$, on account of



Hence, if C is a connection on $\pi_M : M^D \to M$, the covariant derivative $\frac{Da'}{dt}$ is defined and is a vector field over a .

Definition. $a : R \to M$ is a *geodesic* (with respect to a connection map C) if

$$\frac{Da'}{dt} : R \to M^D$$

is the zero map, i.e. if it is given by the map

$$R \xrightarrow{a} M \xrightarrow{0} M^D.$$

We shall simply write $\frac{Da'}{dt} = 0$.

An alternative notion of geodesic can be given in terms of the connection map C itself ; equivalently in terms of its associated connection ∇. We recall from [4] that $X : R \to M^D$ is ∇-*parallel* provided

$$X' = \nabla((\pi_M \circ X)', X),$$

and that $a : R \to M$ is said to be a *geodesic* for ∇ provided a' is ∇-parallel.

Similarly, in terms of C, a is called a *geodesic* for C (cf. [7]) if $C \circ a'' = 0$, i.e., iff

$$R \xrightarrow{a''} (M^D)^D \xrightarrow{C} M^D = R \xrightarrow{a} M \xrightarrow{0} M .$$

We show below that all three notions are equivalent.

Proposition 1. *Let* ∇ *be a connection on* $\pi_M : M^D \to M$, C *a corresponding connection map and* D/dt *the covariant differentiation arising from* C. *Let* $X : R \to M$. *Then the following are equivalent :*

(1) X *is* ∇ *-parallel, i.e.,* $\nabla((\pi_M \circ X)', X) = X'$.

(2) $DX/dt = 0$.

(3) $C \circ X' = 0$.

Proof. Let

$$R \xrightarrow{a} M = R \longrightarrow M^D \xrightarrow{\pi_M} M.$$

Now

(1) X is ∇-parallel iff $\nabla((\pi_M \circ X)', X) = X'$

iff $\nabla(a', X) = X'$ iff $X' \ominus \nabla(a', X) = 0$

iff $X' \ominus \nabla(\pi_{M^D} \circ X', \pi_{M^D} X') = 0$ iff $X' \ominus \nabla \circ K(X') = 0$

iff $H(\pi_M \circ K(X'), C(X')) = 0.$

(2) iff $C(X') = 0$ iff $C(X^D \circ \hat{+}) = 0$

(3) iff $\frac{DX}{dt} = 0.$

We have used, aside from the basic definitions and relationships, that H is linear for the ⊕-structure, and that it is monic. #

Covariant differentiation is a variation of the Koszul derivative in the presence of a connection ∇. For a tangent vector field $\xi : M \to M^D$ and an E-vector field $X : M \to E$, $\nabla_\xi X : M \to E$ is an E-vector field given by

$$\nabla_\xi X = M \xrightarrow{\xi} M^D \xrightarrow{X^D} E^D \xrightarrow{C} E.$$

The following proposition establishes the relationship between the two and gives the additivity and Leibniz rule for covariant differentiation. The proofs are similar but not identical to those given in Propositions 3.1 and 3.2 of [4], as modifications are necessary.

Proposition 2. *Let* C *be a connection map on* $\pi_M : M^D \to M$.

(1) Let $a : R \to M$ *be given and let* $X : R \to M^D$ *be a vector field of* $\pi_M : M^D \to M$ *along a, of the form*

$$R \xrightarrow{a} M \xrightarrow{Y} M^D,$$

where Y *is a vector field on* $\pi_M : M^D \to M$. *Then*

$$\frac{D(Y \circ a)}{dt} = \nabla_{a'} Y .$$

(2) Let $X_1, X_2 : R \to M^D$ *be vector fields of* $\pi_M : M^D \to M$ *along* $a : R \to M$ *(so that* $\pi_M \circ X_1 = \pi_M \circ X_2$ *). Then*

$$\frac{D(X_1 + X_2)}{dt} = \frac{DX_1}{dt} + \frac{DX_2}{dt} .$$

(3) Assuming furthermore that M^D *is parallelizable (i.e.,* M^D *trivial, cf.* [4]*), if* $X : R \to M^D$ *is a vector field of* $\pi_M : M^D \to M$ *along* $a : R \to M$ *and* $f : R \to R$, *then*

$$\frac{D(f \cdot X)}{dt} = \frac{df}{dt} \cdot X + f \cdot \frac{DX}{dt} .$$

Proof. (1)

$$\nabla_{a'} Y = R \xrightarrow{a'} M^D \xrightarrow{Y^D} (M^D)^D \xrightarrow{C} M^D$$

$$= R \xrightarrow{\hat{+}} R^D \xrightarrow{a^D} M^D \xrightarrow{Y^D} (M^D)^D \xrightarrow{C} M^D$$

$$= R \xrightarrow{\hat{+}} R^D \xrightarrow{(Y \circ a)^D} (M^D)^D \xrightarrow{C} M^D = \frac{D(Y \circ a)}{dt} .$$

(2) Since $\pi_M \circ X_1 = \pi_M \circ X_2$, for each $r \in R$, $X_1(r) + X_2(r)$ makes sense in M^D. The result is denoted $(X_1 + X_2)(r)$. Now

$$\frac{D}{dt}(X_1 + X_2) = C \circ (X_1 + X_2)^D \circ \hat{+}.$$

We have for $t \in R^D$:

$$(X_1 + X_2)^D (t)(d) = (X_1 + X_2)(t(d)) = X_1(t(d)) + X_2(t(d))$$
$$= (X_1^D(t))(d) + (X_2^D(t))(d) = (X_1^D(t) + X_2^D(t))(d).$$

Hence :

$$C \circ (X_1 + X_2)^D(r) \circ \hat{+}(r) = C \circ (X_1^D(\hat{+}(r)) + X_2^D(\hat{+}(r)))$$
$$= (C \circ X_1^D)(\hat{+}(r)) + (C \circ X_2^D)(\hat{+}(r)).$$

Hence :

$$\frac{D}{dt}(X_1 + X_2)(r) = \frac{DX_1}{dt}(r) + \frac{DX_2}{dt}(r) = (\frac{DX_1}{dt} + \frac{DX_2}{dt})(r) .$$

(3) We assume then that

$$M^D \approx M \times V \xrightarrow{proj_1} M$$

is the tangent bundle on M. We have

$$(M^D)^D \approx (M \times V)^D \approx M^D \times V^D \approx M^D \times V \times V.$$

Under this identification we get $\oplus$ and $+$ as follows

$$(t_1, u, v_1) \oplus (t_2, u, v_2) = (t_1 \oplus t_2, u, v_1 + v_2),$$
$$(t, u_1, v_1) + (t, u_2, v_2) = (t, u_1 + u_2, v_1 + v_2).$$

The map H becomes

$$M \times V \times V \xrightarrow{H} M^D \times V \times V : (m, u, v) \longmapsto (0_m, u, v).$$

If

$$X : R \longrightarrow M^D \approx M \times V$$

let us denote by $X^1 : R \to M$, $X^2 : R \to V$ the two components. Now

$$X^D \circ \hat{+} : R \to (M^D)^D \approx M^D \times V \times V$$

is given by

$$r \longmapsto \langle (X^1)^D(r), X^2(r), (X^2 \circ \hat{+}(r))'(0) \rangle .$$

We have

$$f \cdot X = \langle X^1, f \cdot X^2 \rangle .$$

Let us begin by calculating

$$\frac{D(f\cdot X)}{dt}(r) = C\circ(f\cdot X)^D\circ\hat{+}(r) =$$

$$= C(X^{1D}\circ\hat{+}(r),\ f(r)\cdot X^2(r),\ ((f\cdot X^2)\circ\hat{+}(r))'(0)).$$

Now

$$((f\cdot X^2)\circ\hat{+}(r))(d) = f((r+d)\cdot X^2(r+d)$$

$$= (f(r)+d\cdot f'(r)) = (X^2(r)+d\cdot X^2(r))$$

$$= (f(r)+d\cdot f'(r))\cdot(X^2(r)+d\cdot(X^2\circ\hat{+}(r))'(0))$$

$$= f(r)\cdot X^2(r)+d\cdot[f'(r)\cdot X^2(r)+f(r)\cdot(X^2\circ\hat{+}(r))'(0)].$$

Hence

$$((f\cdot X^2)\circ\hat{+}(r))'(0) = f'(r)\cdot X^2(r)+f(r)\cdot(X^2\circ\hat{+}(r))'(0).$$

So

$$C\circ(f\cdot X)^D\circ\hat{+}(r) =$$

$$C(X^{1D}\circ\hat{+}(r),\ f(r)\cdot X^2(r),\ f'(r)\cdot X^2(r)+f(r)\cdot(X^2\circ\hat{+}(r))'(0)).$$

Using the $\oplus$-linearity of C, the above is equal to

$$C(0_{X^1(r)},\ f(r)\cdot X^2(r),\ f'(r)\cdot X^2(r))\ \oplus\ C(X^{1D}\circ\hat{+}(r),\ f(r)\cdot X^2(r),\ f(r)\,(X^2\circ\hat{+}(r))'(0))$$

$$= (C(0_{X^1(r)},\ f(r)\cdot X^2(r),\ 0)\ \oplus C(0_{X^1(r)},\ 0,\ f'(r)\cdot X^2(r)))$$
$$\oplus\ f(r)\ C(X^{1D}\circ\hat{+}(r),\ X^2(r),\ (X^2\circ\hat{+}(r))'(0))$$

$$= C(0_{X^1(r)},\ 0,\ f'(r)\cdot X^2(r))\ \oplus\ f(r)\cdot C(X^{1D}(\hat{+}(r)),\ X^2(r),\ (X^2\circ\hat{+}(r))'(0)).$$

Since

$$f(r)\cdot C(X^{1D}(\hat{+}(r)),\ X^2(r),\ (X^2\circ\hat{+}(r))'(0)) = (f\cdot\frac{DX}{dt})(r).$$

it remains to show that, in turn

$$C(0_{X^1(r)},0,\ f'(r)\cdot X^2(r)) = (\frac{df}{dt}\cdot X)(r).$$

Now : the map $\nu: M^D \to (M^D)^D$ may be identified with the map

$$\nu: M\times V\to M^D\times V\times V:\ (m,\ u)\mapsto(0_m,\ 0,u)$$

and recalling that $C\circ\nu = id$, we have

$$C(0_{X^1(r)},\ 0,\ f'(r)\cdot X^2(r)) = C(\nu(X^1(r),\ f'(r)\cdot X^2(r)))$$

$$= (X^1(r),\ f'(r)\cdot X^2(r)) = f'(r)\cdot(X^1(r),\ X^2(r)) = f'(r)\cdot X(r) = (\frac{df}{dt}\cdot X)(r),$$

as required. Hence

$$\frac{D}{dt}(f\cdot X)(r) = (f\cdot\frac{DX}{dt})(r)+(\frac{df}{dt}\cdot X)(r) = (f\cdot\frac{DX}{dt}+\frac{df}{dt}\cdot X)(r)$$

as claimed. #

3. THE SYMMETRY MAP OF THE ITERATED TANGENT BUNDLE.

Out of $p: E \to M$, a vector bundle with E, M infinitesimally linear, we can consider $(E^D)^D$ with several bundle structures :

$$(E^D)^D \xrightarrow{\pi_{E^D}} E^D, \quad (E^D)^D \xrightarrow{\pi_{E}^D} E^D \quad \text{and} \quad (E^D)^D \xrightarrow{(p^D)^D} (M^D)^D .$$

We wish to investigate how does the symmetry map

$$\Sigma_E : (E^D)^D \xrightarrow[\simeq]{} (E^D)^D$$

given as the composite

$$(E^D)^D \xrightarrow{\varphi} E^{D\times D} \xrightarrow{E^\tau} E^{D\times D} \xrightarrow{\varphi^{-1}} (E^D)^D$$

(with $\tau = \langle proj_2, proj_1 \rangle$ the twist map and φ the canonical isomorphism) behave with respect to these structures. We have :

Lemma 1. *(i)*

$$\begin{array}{ccc} (E^D)^D & \xrightarrow{\Sigma_E} & (E^D)^D \\ & \searrow^{\pi_{E^D}} \quad = \quad \swarrow^{\pi_{E}^D} & \\ & E^D & \end{array}$$

and addition is preserved, i.e., if $f, g \in (E^D)^D$ such that $\pi_{E^D}(f) = \pi_{E^D}(g)$, then

$$\Sigma_E(f \oplus g) = \Sigma_E(f) + \Sigma_E(g),$$

as well as scalar multiplication, i.e., $\Sigma_E(\lambda \cdot f) = \lambda \cdot \Sigma_E(f)$.

(ii)

$$\begin{array}{ccc} (E^D)^D & \xrightarrow{\Sigma_E} & (E^D)^D \\ {\scriptstyle (p^D)^D}\downarrow & & \downarrow{\scriptstyle (p^D)^D} \\ (M^D)^D & \xrightarrow{\Sigma_M} & (M^D)^D \end{array}$$

and addition is preserved, i.e.: if $f, g \in (E^D)^D$ are such that $(p^D)^D(f) = (p^D)^D(g)$, then

$$\Sigma_E(f + g) = \Sigma_E(f) + \Sigma_E(g),$$

as well as scalar multiplication.

Proof. Commutativity : For $f \in (E^D)^D$, $d \in D$,

$$(\pi_{E^D} \circ \Sigma_E)(f)(d) = \pi_E(\Sigma_E(f)(d)) = \Sigma_E(f)(d)(0) = f(0)(d) = (\pi_{E^D}(f))(d).$$

Let

$$f,\ g \in (E^D)^D \quad \text{with} \quad \pi_{E^D}(f) = \pi_{E^D}(g).$$

So $f \oplus g \in (E^D)^D$ is defined by means of a unique $l : D(2) \to E^D$ such that

$$l(d, 0) = f(d), \quad l(0, d) = g(d)$$

as $(f \oplus g)(d) = l(d, d)$. Consider now

$$\Sigma_E(f \oplus g)(d_1)(d_2) = (f \oplus g)(d_2)(d_1) = l(d_2, d_2)(d_1).$$

We wish to prove that

$$\Sigma_E(f \oplus g)(d_1)(d_2) = (\Sigma_E(f)(d_1) \oplus \Sigma_E(g)(d_1))(d_2)$$

and this will follow provided we prove that, for each d_1,

$$\psi_{d_1} : D(2) \to E \quad \text{defined by} \quad \psi_{d_1}(d, d) = l(d, d)(d_1)$$

has the properties

$$\psi_{d_1}(d, 0) = \Sigma_E(f)(d_1)(d), \quad \psi_{d_1}(0, d) = \Sigma_E(g)(d_1)(d)$$

and

$$\psi_{d_1}(d, d) = \Sigma_E(f \oplus g)(d_1)(d).$$

Now :

$$\psi_{d_1}(d, 0) = l(d, 0)(d_1) = f(d)(d_1) = \Sigma_E(f)(d_1)(d),$$

$$\psi_{d_1}(0, d) = l(0, d)(d_1) = g(d)(d_1) = \Sigma_E(g)(d_1)(d)$$

and

$$\psi_{d_1}(d, d) = l(d, d)(d_1) = (f \oplus g)(d)(d_1) = \Sigma_E(f \oplus g)(d_1)(d)$$

as required. It remains to prove that $f \in (E^D)^D$, $\lambda \in R$,

$$\Sigma_E(\lambda \odot f) = \lambda \cdot \Sigma_E(f).$$

For $d, \tilde{d} \in D$, we have

$$(\Sigma_E(\lambda \odot f))(d)(\tilde{d}) = (\lambda \odot f)(\tilde{d}, d) = (\lambda \odot f)(\tilde{d})(d) = f(\lambda \tilde{d})(d),$$

and

$$(\lambda \cdot \Sigma_E(f))(d)(\tilde{d}) = (\lambda \odot \Sigma_E(f)(d))(\tilde{d}) = (\Sigma_E(f))(d)(\lambda \tilde{d}) = f(\lambda \tilde{d})(d).$$

(ii) The commutativity follows readily from

$$M^\tau \cdot p^{D \times D} = p^\tau = p^{D \times D} \circ E^\tau.$$

Next, if $(p^D)^D(f) = (p^D)^D(g)$,

$$(f + g)(d_1)(d_2) = f(d_1)(d_2) \oplus g(d_1)(d_2) \ ;$$

addition in $p : E \to M$. Now

$$(\Sigma_E(f + g))(d_1)(d_2) = (f + g)(d_2)(d_1) = f(d_2)(d_1) \oplus g(d_2)(d_1) =$$

$$= (\Sigma_E(f))(d_1)(d_2) \oplus (\Sigma_E(g))(d_1)(d_2) = (\Sigma_E(f) + \Sigma_E(g))(d_1)(d_2).$$

It remains to prove that for $f \in (E^D)^D$, $\lambda \in R$,

$$\Sigma_E(\lambda \cdot f) = \lambda \cdot \Sigma_E(f)$$

where for any $f \in (E^D)^D$ with $(p^D)^D : (E^D)^D \to (M^D)^D$, the meaning of scalar multiplication is

$$(\lambda \cdot f)(d_1)(d_2) = \lambda \cdot f(d_1)(d_2).$$

Hence :

$$(\Sigma_E(\lambda \cdot f))(d_1)(d_2) = (\lambda \cdot f)(d_2)(d_1) = \lambda \cdot f(d_2)(d_1)$$

and

$$(\lambda \cdot \Sigma_E(f))(d_1)(d_2) = \lambda \cdot (\Sigma_E(f))(d_1)(d_2) = \lambda \cdot f(d_2)(d_1). \qquad \#$$

Suppose a connection on a vector bundle $p: E \to M$ is given. Can a connection be naturally defined on the iterated tangent bundles

$$E^D \xrightarrow{p^D} M^D, \quad (E^D)^D \xrightarrow{(p^D)^D} (M^D)^D,$$

etc.? This becomes a simple matter if we use the connection map associated to a connection (the approach in [10] uses the connections themselves and is rather more involved), as was discovered by Patterson [7]. What is given below is the synthetic analogue of the treatment given in [7], but in this case, our treatment is basically the same as that of [7].

Definition. Let $C : E^D \to E$ be a connection map on $p: E \to M$. A family

$$E^{D^{j+1}} \xrightarrow{C_j} E^{D^j}$$

of connection maps on the bundles

$$E^{D^j} \xrightarrow{p^{D^j}} M^{D^j}, \quad 0 \leq j \leq k$$

is said to be a *k-th prolongation of* C if $C_0 = C$ and for each $1 \leq j \leq k$, the diagram

$$\begin{array}{ccc}
(E^{D^{j-1}})^D \simeq E^{D^j} & \xrightarrow{C_{j-1}} & E^{D^{j-1}} \\
{\scriptstyle (0_{E^{D^{j-1}}})^D} \downarrow & & \downarrow {\scriptstyle 0_{E^{D^{j-1}}}} \\
(E^{D^j})^D \simeq E^{D^{j+1}} & \xrightarrow{C_j} & E^{D^j}
\end{array}$$

commutes, where

$$0_{E^{D^{j-1}}}: E^{D^{j-1}} \to E^{D^j} = (E^{D^{j-1}})^D$$

is the zero map given by

$$0(e) = 0_e = [\, d \mapsto e \,] \,.$$

In order that k-th prolongations always exist, we need another lemma referring to the symmetry map :

Lemma 2. *The following diagrams are commutative :*

(i)

$$\begin{array}{ccc} E^D & \xrightarrow{(0_E)^D} & (E^D)^D \\ & {\scriptstyle 0_{E^D}} \searrow & \downarrow {\scriptstyle \Sigma_E} \\ & & (E^D)^D \end{array}$$

(ii)

$$\begin{array}{ccc} E^D & \xrightarrow{\nu_{E^D \xrightarrow{p^D} M^D}} & (E^D)^D \\ & {\scriptstyle (\nu_{E \xrightarrow{p} M})^D} \searrow & \downarrow {\scriptstyle \Sigma_E} \\ & & (E^D)^D \end{array}$$

Proof. (i)

$$\Sigma_E \circ (0_E)^D(t)(d_1)(d_2) = (0_E)^D(t)(d_2)(d_1) =$$
$$= 0_E(t(d_2))(d_1) = t(d_2)$$

whereas

$$(0_{E^D}(t)(d_1))(d_2) = t(d_2).$$

(ii) $$(\Sigma_E \circ \nu_{E^D \xrightarrow{p^D} M^D})(t)(d_1)(d_2) = (\nu_{E^D \xrightarrow{p^D} M^D}(t)(d_2))(d_1)$$

$$= (d_2 \cdot t)(d_1) = d_2 \cdot t(d_1)$$

whereas

$$((\nu_{E \xrightarrow{p} M})^D(t))(d_1))(d_2) = (\nu_{E \xrightarrow{p} M})(t(d_1))(d_2) = d_2 \cdot t(d_1). \qquad \#$$

We can now prove that :

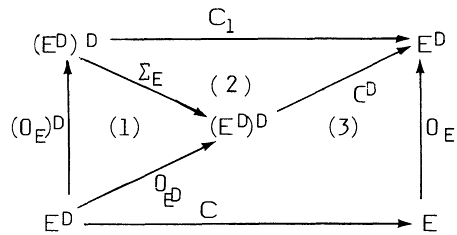
Proposition 3. *Let* $p : E \to M$ *be a vector bundle, with* E, M *infinitesimally linear. Let* C *be a connection map on* $p : E \to M$ *. Then, for any* $k \geq 0$ *, a* k*-th prolongation of* C *exists.*

Proof. The definition is by induction. For $i = 1$, define C_1 by

$$\begin{array}{ccc} (E^D)^D & \xrightarrow{C_1} & E \\ {\scriptstyle \Sigma_E} \searrow & & \nearrow {\scriptstyle C^D} \\ & (E^D)^D & \end{array}$$

and, given C_{j-1} ; C_j is defined similarly out of C_{j-1}.

We begin by showing that these maps prolong C in the required manner. This amounts to showing that the diagrams involving zero maps into the tangent bundles commute. The proof is the same for the case $j = 1$ as it is in general ; hence we show that the diagram below is commutative :



(1) is true by Lemma 2 (i) ; (2) is the definition of C_1, and (3) can be shown as follows (naturality of the zero map) :

$$(C^D \circ 0)(t)(d) = C(0(t)(d)) = C(t) ,$$
$$(0 \circ C)(t)(d) = 0(C(t))(d) = C(t) .$$

Next, we prove that C_1 is a connection map (again the proof depends on general properties of the connection map C and of Σ_E, hence is the same in the general case).

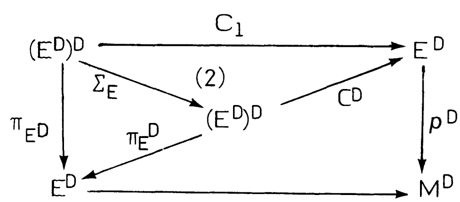
We begin with the identity

$$C_1 \circ \nu_{E^D \xrightarrow{p^D} M^D} = id_{E^D} .$$

By Lemma 2 (ii) and the definition of C_1,

$$C_1 \circ \nu_{E^D \xrightarrow{p^D} M^D} = C^D \circ \Sigma_E \circ \nu_{E^D \xrightarrow{p^D} M^D} =$$
$$= C^D \circ (\nu_{E \xrightarrow{p} M})^D = (C \circ \nu_{E \xrightarrow{p} M})^D = id_E^D = id_{E^D} .$$

Next, we must show that C_1 is linear with respect to both the ⊕ and the +-structures, and this requires establishing first the commutativity of any of the two relevant diagrams in the definition of a connection map. We try the one below :



Here, (1) was shown in Lemma 1 (i), (2) is the definition of C_1 and (3) is true since C is assumed to be a connection map on $p : E \to M$.

By Lemma 1 (i),

$$\Sigma_E(f \oplus g) = \Sigma_E(f) + \Sigma_E(g).$$

In turn, for any $f, g \in (E^D)^D$ with $\pi_E^D(f) = \pi_E^D(g)$,

$$(C^D(f + g))(d) = C((f + g)(d)) = C(f(d) \oplus g(d)) =$$

$$C(f(d)) \oplus C(g(d)) = C^D(f)(d) \oplus C^D(g)(d) = (C^D(f) + C^D(g))(d).$$

Similarly, by Lemma 1 (i),

$$\Sigma_E(\lambda \odot f) = \lambda \cdot \Sigma_E(f) .$$

Now, for any $f \in (E^D)^D$, $\lambda \in R$,

$$C^D(\lambda \cdot f)(d) = C((\lambda \cdot f)(d)) = C(\lambda \cdot f(d)) = \lambda \cdot C(f(d)) = \lambda \cdot C^D(f)(d) .$$

The commutativity of the other diagram is automatic ; let us show linearity with respect to +, ., where

$$\begin{array}{ccc} (E^D)^D & \xrightarrow{C_1} & E^D \\ {\scriptstyle (p^D)^D}\downarrow & & \downarrow{\scriptstyle p^D} \\ (M^D)^D & \xrightarrow{\pi_{M^D}} & M^D \end{array}$$

By Lemma 1 (ii), Σ_E preserves the structure, and C^D does too, hence C_1 is +-linear and a connection map. #

Denote by

$$TM \rightarrowtail (M^D)^D \underset{\pi_M^D}{\overset{\pi_{M^D}}{\rightrightarrows}} M^D$$

the subobject of $(M^D)^D$ given by the equalizer diagram. This map is not linear for any of the structures on $(M^D)^D$.

Definition. M *has the iterated tangent bundle property* if the diagram

$$\begin{array}{ccc} TM & \rightarrowtail & (M^D)^D \\ & \searrow & \downarrow{\scriptstyle \Sigma_M} \\ & & (M^D)^D \end{array}$$

commutes, where

$$TM \rightarrowtail M = \text{equalizer } (\pi_{M^D}, \pi_M^D).$$

Remark. Classically a smooth manifold has the iterated tangent bundle property as can be seen in [2], Chapter IX. The reader will also find there a coordinate-free definition of the "classical" symmetry map.

We prove now :

Proposition 4. *If* M *is locally parallelizable, i.e., there exists some epic etale morphism* $M' \twoheadrightarrow M$ *with* M' *parallelizable, then* M *satisfies the iterated tangent bundle property, under the additional assumptions that* M' *has property* W *and is infinitesimally linear. Also, 2 is assumed invertible in* R.

N. B. In order to use property W to prove that $\Sigma_M(f) = f$ for $f \in (M^D)^D$, we need

$$f(d, 0) = f(0, d) = f(0, 0) \quad \text{for each } d \in D .$$

Only the first equality is warranted by $\pi_{M^D}(f) = \pi_M^D(f)$. Therefore to assume Property W for M does not seem enough to insure that it has the iterated tangent bundle property.

Proof. (1) Assume M parallelizable, infinitesimally linear, with Property W and 2 invertible. Let $M^D \simeq M\times V$, V Euclidean R-module. We have

$$(M^D)^D \simeq (M\times V)^D \simeq M^D \times V^D \simeq (M\times V)\times(M\times V)$$

and $f \in (M^D)^D$ will be identified with

$$f = (t_M, t_V), \quad t_M \in M^D, \quad t_V \in V^D;$$

in turn, we set

$$t_M = (m, v_1) \in M\times V , \quad t_V = (v_2, v_3) \in V\times V .$$

We have

$$\pi_{M^D}(f) = f(0) = (t_M(0), t_V(0)) = (m, v_2) \qquad (*)$$

$$(\pi_M)^D(f) = t_M = (m, v_1) \qquad (**)$$

this is because

$$(\pi_M)^D(f)(d) = \pi_M(f(d)) = proj_M(t_M(d), t_V(d)) = t_M(d).$$

For $t = (m, v) \in M^D$ denote $m + d \cdot v := (m, v)(d)$. We prove

$$m + d \cdot (v_1 + v_2) = (m + d \cdot v_1) + d \cdot v_2 \qquad (***)$$

Let $t_i = (m, v_i)$, $i = 1, 2$; since $\pi_M: M^D \to M$ has the same bundle structure as $proj_1 : M \times V \to M$, we have

$$t_1 \oplus t_2 = (m, v_1 + v_2), \qquad (t_1 \oplus t_2)(d) = m + d \cdot (v_1 + v_2).$$

Also

$$(t_1 \oplus t_2)(d) = l(d, d)$$

for the unique

$$l : D(2) \to M \quad \text{with} \quad l(d, 0) = t_1(d), \quad l(0, d) = t_2(d).$$

Now

$$l(d_1, d_2) = (m + d_1 \cdot v_1) + d_2 \cdot v_2$$

because this satisfies

$$l(d, 0) = m + d \cdot v_1 = t_1(d) \quad \text{and} \quad l(0, d) = m + d \cdot v_2 = t_2(d)$$

hence the result. Also,

$$(m + d \cdot v) + \tilde{d} \cdot v = (m + \tilde{d} \cdot v) + d \cdot v \qquad (****)$$

We prove this by considering

$$X_v : M \to M^D \quad \text{given by} \quad X_v(m)(d) = m + d \cdot v .$$

Using Exercise 9.1 of [3] we obtain

$$[X_v, \check{X}_v](d) = ([\check{X}_v, X_v](d))^{-1} \quad \text{for each} \quad d \in D$$

or equivalently

$$([X_v, \check{X}_v](d))^2 = [\check{X}_v, X_v](2d) = id_M \quad \text{for each} \quad d \in D.$$

Since 2 is invertible this implies

$$[X_v, \check{X}_v](d) = id_M \quad \text{for each} \quad d \in D$$

and by Exercice 9.5 of [3] we get the result.

Finally, if $f = (m, v_1, v_2, v_3)$,

$$f(d, \tilde{d}) = ((m, v_1)(d), (v_2, v_3)(d)(\tilde{d}) = (m + d \cdot v_1) + \tilde{d} \cdot (v_2 + d \cdot v_3)$$
$$= (m + d \cdot v_1 + \tilde{d} \cdot v_2) + (\tilde{d}d) \cdot v_3$$

using (***) and

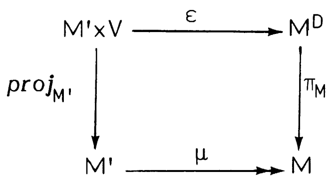
$$(\tilde{d} \cdot t)(d) = t(\tilde{d}d) = t(\tilde{d}d).$$

In particular, if f satisfies

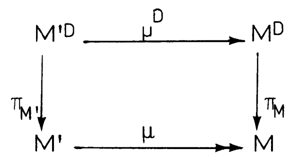
$$\pi_M^D(f) = \pi_{M^D}(f), \qquad f = (m, v, v, \bar{v})$$

(by (*) and (**)) so that $f(d, \tilde{d}) = f(\tilde{d}, d)$ (using (****)), i.e., $f = \Sigma_M(f)$.

(2) Assume that the diagram below is a pullback

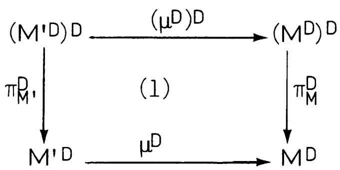


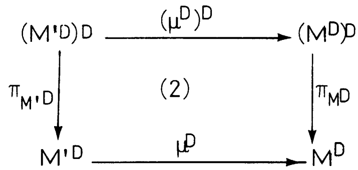
with μ etale epic. Equivalently, we have the pullback (μ etale) :



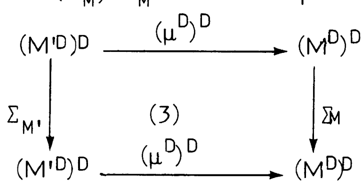
with M parallelizable.

Since $(\ )^D$ preserves pullbacks and since μ^D is etale, we have that both diagrams below are pullbacks :



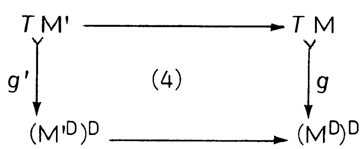


It is easy to see that (Σ_M, $\Sigma_{M'}$ are isomorphisms)

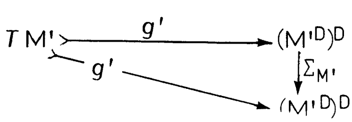


is also a pullback, since it commutes.

Using pullbacks (1) and (2), it follows that



is a pullback (since pulling back commutes with equalizer diagrams). Since M' is parallelizable, it has the ITBP. Hence,



The equation $\Sigma_M{}' \circ g' = g$ can be written using pullbacks (3) and (4) :

$$((\mu^D)^D)^*(\Sigma_M) \circ ((\mu^D)^D)^*(g) = ((\mu^D)^D)^*(g).$$

Now, $(\mu^D)^D$ is epic : hence pulling back along it is faithful. Thus

$$\Sigma_M \circ g = \Sigma_M \quad . \qquad /\!/$$

4. GEODESIC SPRAYS.

Let M be infinitesimally linear. The following definition is the synthetic analogue of the usual classical notion and was proposed by Lawvere. In order to distinguish it from yet another synthetic notion proposed by Joyal, we shall employ the expressions "Lawvere spray" and "Joyal spray" throughout this paper. Their exact relationship is established below, in Proposition 1.

Definition. A *Lawvere spray* on the tangent bundle $\pi_M : M^D \to M$ is a map

$$S : M \longrightarrow (M^D)^D$$

satisfying

(1)

$$\begin{array}{ccc} M^D & \xrightarrow{\quad S \quad} & (M^D)^D \\ & {\scriptstyle \langle id_{M^D},\, id_{M^D} \rangle} \searrow \quad = & \downarrow {\scriptstyle K = \langle \pi_M^D ,\, \pi_{M^D} \rangle} \\ & & M^D \times_M M^D \end{array}$$

(2) For any $t \in M^D, \lambda \in R$,

$$S(\lambda \odot t) = \lambda \cdot (\lambda \odot S(t)),$$

where $\odot$ refers to scalar multiplication in $\pi_M : M^D \to M$ as well as in

$$\pi_{M^D} : (M^D)^D \to M^D,$$

while $\cdot$ is that of $(\pi_M)^D : (M^D)^D \to M^D$.

Definition. A *Joyal spray* on the tangent bundle $\pi_M : M^D \to M$ is a map

$$\sigma : M^D \longrightarrow M^{D_2}$$

such that

(i) $M^D \xrightarrow{\sigma} M^{D_2} \xrightarrow{M^u} M^D = id_{M^D}$, where $u : D \rightarrowtail D_2$;

(ii) $\sigma(\lambda \odot t) = \lambda \odot \sigma(t)$ for every $t \in M^D$, $\lambda \in R$.

Proposition 1. *Let* M *be infinitesimally linear. Then, any Joyal spray on the tangent bundle of* M *gives rise to a Lawvere spray in a canonical* *way. The process is reversible also, provided* M *satisfies the iterated tangent bundle property, the symmetric tangent bundle property, and perceives that* $+ : D^2 \to D_2$ *is surjective (i.e.,* M^+ *is monic).*

Proof. Given $\sigma : M^D \to M^{D_2}$, a Joyal spray, define

$$S = M^D \xrightarrow{\sigma} M^{D_2} \xrightarrow{M^+} (M^D)^D,$$

where

$$M^+(t)(d_1, d_2) = t(d_1 + d_2).$$

Lemma. *The diagram*

$$\begin{array}{ccc} M^{D_2} & \xrightarrow{\quad M^+ \quad} & (M^D)^D \\ {\scriptstyle M^u}\downarrow & & \downarrow{\scriptstyle K} \\ M^D & \xrightarrow[\langle id_{M^D}, id_{M^D}\rangle]{} & M^D \times_M M^D \end{array}$$

commutes.

Proof. For $t \in M^D$,

$$(K \circ M^+)(t) = K(M^+(t)) = \langle \pi_M^D(M^+(t)), \pi_{M^D}(M^+(t))\rangle$$
$$\overset{(*)}{=} \langle M^u(t), M^u(t)\rangle = (\langle id_{M^D}, id_{M^D}\rangle \circ M^u)(t).$$

We justify (*) as follows :

$$(\pi_{M^D}(M^+(t))(d) = M^+(t)(0)(d) = t(0+d) = t(d) = M^u(t)(d),$$

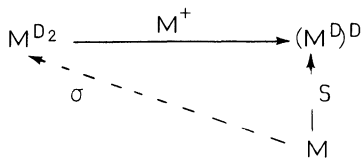
$$((\pi_{M^D}(M^+(t)))) (d) = \pi_M(M^+(t)(d)) = M^+(t)(d)(0) =$$
$$= t(d+0) = t(d) = M^u(t)(d). \qquad \#$$

S is a spray :

(1) $\quad K \circ S = K \circ M^+ \circ \sigma = \langle id_{M^D}, id_{M^D}\rangle \circ M^u \circ \sigma = \langle id_{M^D}, id_{M^D}\rangle.$

(2)
$$(S(\lambda \odot t)(d_1))(d_2) = \sigma(\lambda \odot t)(d_1 + d_2) =$$
$$= (\lambda \odot \sigma(t))(d_1 + d_2) = \sigma(t)(\lambda(d_1 + d_2))$$
$$= \sigma(t)(\lambda d_1 + \lambda d_2) = (S(t)(\lambda d_1))(\lambda d_2) =$$
$$= (\lambda \odot [S(t)(\lambda d_1)])(d_2) = (\lambda \cdot S(t)(d_1))(d_2) =$$
$$= (\lambda \cdot (\lambda \odot S(t)))(d_1)(d_2).$$

Assume now that additional hypotheses on M have been made as in the statement of the proposition and let S be a Lawvere spray on the tangent bundle of M. On account of the iterated tangent bundle property and of (1), S is symmetric, hence by the symmetric functors property, a factorization



exists and is unique since M^+ is monic on account of M perceiving that + is surjective. It remains to prove that σ satisfies both conditions.

(i) $$\langle id_{M^D}, id_{M^D}\rangle \circ M^u \circ \sigma = K \circ M^+ \circ \sigma =$$

$$= K \circ S = \langle id_{M^D}, id_{M^D}\rangle,$$

where we have used the Lemma above and (1) for S. Since $\langle id_{M^D}, id_{M^D}\rangle$ is monic, $M^+ \circ \sigma = id$.

(ii) It is enough to check that

$$M^+(\sigma(\lambda \odot t)) = M^+(\lambda \odot \sigma(t)).$$

But, the left hand side is $S(\lambda \odot t)$ and the right hand side is

$$M^+(\lambda \odot \sigma(t))(d_1)(d_2) = (\lambda \odot \sigma(t))(d_1+d_2) =$$
$$= \sigma(t)(\lambda(d_1+d_2)) = \sigma(t)(\lambda d_1 + \lambda d_2) =$$
$$= M^+(\sigma(t))(\lambda d_1, \lambda d_2) = S(t)(\lambda d_1, \lambda d_2) = (\lambda\cdot(\lambda \odot S(t)))(d_1)(d_2)$$

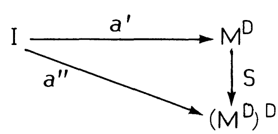
as shown earlier. But now, use (2) in the definition of a Lawvere spray, and M^+ monic. #

Remark. Joyal sprays are not as general as Lawvere sprays. However, on the one hand, the assumptions needed for their equivalence are classically true of manifolds, and on the other, the former may be preferable to deal with in generalizations to higher dimensions. The symmetry condition inherent in any spray in this sense (i.e., $S = \Sigma_M \circ S$), and which follows on account of the equation $M^+ = \Sigma_M \circ M^+$, can easily be guaranteed by assuming M satisfies the iterated tangent bundle property. Indeed, by (1)

$$S \circ \pi_M^D = S \circ \pi_{M^D}$$

hence, by the ITBP, $S = \Sigma_M \circ S$.

Definition. A curve $a : I \to M$, where I is a subobject of R closed under addition of elements of D, is said to be an *integral curve for the Lawvere spray* S, on $\pi_M : M^D \to M$ if



commutes.

Definition. Given a connection ∇ on $\pi_M: M^D \to M$ a Lawvere spray S is said to be a *geodesic spray for* ∇ provided the following holds : for any curve $a : I \to M$, a is a geodesic for ∇ iff a is an integral curve for S.

We now prove :

Proposition 2. *Let* ∇ *be a connection on a tangent bundle* $\pi_M : M^D \to M$. *Then, there exists a geodesic Lawvere spray for* ∇.

Proof. Given ∇, define S by :

$$M^D \xrightarrow{\langle id_{M^D}, id_{M^D}\rangle} M^D \times_M M^D \xrightarrow{\nabla} (M^D)^D .$$

We verify that S is a spray :

(1) $$K \circ S = K \circ \nabla \circ \langle id_{M^D}, id_{M^D}\rangle = id_{M^D \times_M M^D} \circ \langle id_{M^D}, id_{M^D}\rangle =$$
$$= \langle id_{M^D}, id_{M^D}\rangle .$$

(2) For $t \in M^D$, $\lambda \in R$,

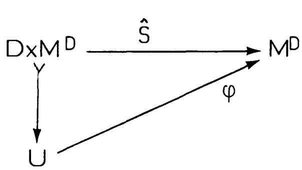
$$S(\lambda \odot t) = \nabla(\lambda \odot t, \lambda \odot t) = \lambda \cdot \nabla(\lambda \odot t, t) =$$
$$= \lambda \cdot (\lambda \odot \nabla(t, t)) = \lambda \cdot (\lambda \odot S(t)).$$

Moreover, S is a geodesic spray for ∇ : let $a : I \to M$, $I \rightarrowtail R$, a curve. Then, a is a geodesic for ∇ iff a' is ∇-parallel, i.e., iff $a'' = \nabla(a', a')$ iff $a'' = S(a')$, i.e., iff a is an integral curve for S. #

In order to reverse the process which leads to a Lawvere spray from a connection, we must integrate (locally) the spray.

Definition. Let $S : M^D \to (M^D)^D$ be a Lawvere spray on $\pi_M : M^D \to M$. A *(local) flow for* S is a pair (U, φ) where $U \rightarrowtail R \times M$ is a Penon open containing $D \times M^D$ and $\varphi : U \to M$ such that

(1)



$[S(d, t) = S(t)(d)]$.

(2) For any (λ, t), (ξ, t), $(\lambda + \xi, t) \in U$, then also

$$(\lambda, \varphi(\xi, t)) \in U \quad \text{and} \quad \varphi(\lambda + \xi, t) = \varphi(\lambda, \varphi(\xi, t)).$$

(3) For any λ, $\xi \in R$, $t \in M^D$, if

$$(\lambda\xi, t), (\lambda, \xi \odot t) \in U,$$

then

$$\varphi(\lambda, \xi \odot t) = \xi \cdot \varphi(\lambda\xi, t).$$

Remark. Classically, condition (2) does not appear in the definition of a flow for a vector field ; it is rather a consequence of the uniqueness in the theorem about solutions of ordinary differential equations (cf. [5]). As for condition (3) in the definition of a Lawvere spray, it is classically stated as a property of all integral curves for the spray. Synthetically, on account of the representability of the tangent bundle, it is possible to state the condition already at the level of the vector field, without resorting to integral curves (cf. definition of a spray) ; the corresponding condition, for flows of sprays, is then the natural local extension of the infinitesimal version of it.

We shall now consider a property for M which guarantees the existence of local solutions of second order homogeneous differential equations. This assumption seems to be the natural one to consider in order to obtain the passage from a spray to a connection if the notion of spray used is that of a Lawvere spray. However, as will be seen later, an infinitesimal version of this property is all that is required if dealing with Joyal sprays instead. As both notions of spray have been discussed and compared here, we shall deal with the passage in question separately for both and comment on their relationship later.

Definition. M is said to have *the local property of existence of exponential maps* if the following holds : given any Lawvere spray S on the tangent bundle of M, there exists a local flow

$$(D(S), \varphi) \qquad \text{with} \qquad (1, 0_M) \in D(S) \rightarrowtail R\times M^D .$$

Let $D_S \rightarrowtail M^D$ be given by the pullback diagram :

$$\begin{array}{ccc} D_S & \longrightarrow & D(S) \\ \downarrow & & \downarrow \\ M^D & \xrightarrow{\langle 1, id_{M^D} \rangle} & R\times M^D \end{array} \qquad (1 = \text{constant map})$$

Notice that since $(1, 0_M) \in D(S)$, then $0_M \in D_S \rightarrowtail M$ is a Penon open neighborhood of 0_M, on which the *exponential map* can be defined : $\exp_S : D_S \to M$ is then given by $\exp_S(t) = \pi_M\varphi(1, t)$.

On account of our definition of flow, and unlike the classical situation, we immediately get here the

Proposition 3. *Let* S *be a Lawvere spray,* $\varphi : D(S) \to M^D$ *a local flow of* S*. Then, for any* $\lambda \in R$, $t \in M$ *such that* $(\lambda, t) \in D(S)$ *and* $\lambda \odot t \in D_S$, *we have* $\exp(\lambda \odot t) = \pi_M(\varphi(\lambda, t))$.

Proof. By assumption,

$$\lambda \odot \varphi(\lambda, t) = (1, \lambda \odot t).$$

Applying π_M and noticing that

$$\lambda \odot \varphi(\lambda, t) \quad \text{and} \quad \varphi(\lambda, t)$$

are on the same fiber, we get

$$\exp_S(\lambda \odot t) = \pi_M \varphi(1, \lambda \odot t) = \pi_M(\lambda \odot \varphi(\lambda, t)) = \pi_M(\varphi(\lambda, t)). \quad \#$$

Corollary. *For* $d \in D$, $\exp_S(d \odot t) = t(d)$ ($d \odot t \in D_S$ as can be seen in the next Lemma).

Armed with the exponential map, we can attack the question of recovering a connection given a Lawvere spray (cf. [9], Th. of Ambrose-Palais-Singer) without invoking limits. Before we proceed we will prove an auxiliary lemma.

Lemma. *Let* S *be a spray on* $\pi_M : M^D \to M$. *Then for any* $n \geq 1$, *given*

$$(t_1, \ldots, t_n) \in M^D \times_M \ldots \times_M M^D \quad \textit{and} \quad (d_1, \ldots, d_n) \in D_S^n,$$

$$d_1 \odot t_1 \oplus \ldots \oplus d_n \odot t_n \in D_S.$$

Proof. Consider

$$D^n \times M^D \times_M \ldots \times_M M^D \xrightarrow{\beta_n} M^D$$

$$(d_1, \ldots, d_n, t_1, \ldots, t_n) \mapsto d_1 \odot t_1 \oplus \ldots \oplus d_n \odot t_n.$$

Now, for any map $f : N \to P$, for any

$$y, z \in N : \neg\neg(y = z) \Rightarrow \neg\neg(fy = fz)$$

(monoticity of $\neg\neg$). Now, clearly since $\neg\neg$ commutes with $\wedge$

$$\neg\neg(d_1, \ldots, d_n, t_1, \ldots, t_n) = (0, \ldots, 0, t_1, \ldots, t_n)).$$

So,

$$\neg\neg(\beta_n(d_1, \ldots, d_n, t_1, \ldots, t_n) = \beta_n(0, \ldots, 0, t_1, \ldots, t_n))$$

i.e.,

$$\neg\neg(d_1 \odot t_1 \oplus \ldots \oplus d_n \odot t_n = 0_M);$$

but $0_M \in D_S$ which is a Penon open, therefore

$$d_1 \odot t_1 \oplus \ldots \oplus d_n \odot t_n \in D_S. \quad \#$$

And now, passage from a Lawvere spray to a connection :

Proposition 4. *Let* M *have the local property of existence of exponential maps. Let* S *be a Lawvere spray on* $\pi_M : M^D \to M$. *Then there exists a torsion-free connection on* $\pi_M : M^D \to M$ *of which* S *is its associated geodesic spray.*

Proof. Let $\varphi_S : D(S) \to M^D$ be a local flow of S ; exp_S the exponential map. Define

$$\nabla : M^D \times_M M^D \to (M^D)^D$$

as the exponential adjoint of the map :

$$D \times M^D \times_M M^D \xrightarrow{\alpha} M^D \times_M M^D \xrightarrow{H} (M^D)^D \xrightarrow{exp_S^D} M^D$$

where

$$\alpha(d, t_1, t_2) = (d \odot t_1, t_2) \quad \text{and} \quad H(t, s)(d) = t \oplus d \odot s.$$

This map is defined because

$$\nabla(t_1, t_2)(d_1)(d_2) = exp_S(d_1 \odot t_1 \oplus d_2 \odot t_2)$$

and the latter is in the domain of the exponential map (Lemma). We check that ∇ is a connection. First :

(1) $$(\pi_{M^D} \circ \nabla)(t_1, t_2)(d) = \nabla(t_1, t_2)(0)(d) = exp_S(0 \odot t_1 \oplus d \odot t_2)$$
$$= exp_S(d \odot t_2) = t_2(d) = proj_2(t_1, t_2)(d),$$

$$(\pi_M^D \circ \nabla)(t_1, t_2)(d) = \pi_M(\nabla(t_1, t_2)(d)) = \nabla(t_1, t_2)(d)(0)$$
$$= exp_S(d \odot t_1 \oplus 0 \odot t_2) = exp(d \odot t_1) = t_1(d) = proj_1(t_1, t_2)(d).$$

Hence, $K \circ \nabla = id$.

(2) ∇ is $\oplus$-linear :

(i) $$\nabla((t_1, s) \oplus (t_2, s)) = \nabla(t_1, s) \oplus \nabla(t_2, s) :$$

Let $l : D(2) \to M^D$ be given by :

$$l(d_1, d_2) = exp_S^D(H(d \odot t_1 \oplus d_2 \odot t_2, s)).$$

Clearly

$$l(d, d) = exp_S^D(H(d \odot (t_1 \oplus t_2), s)) = \nabla(t_1 \oplus t_2, s)(d) ;$$

on the other hand

$$l(d, 0) = exp_S^D(H(d \odot t_1, s)) = \nabla(t_1, s)(d),$$

$$l(0, d) = exp_S^D(H(d \odot t_1, s)) = \nabla(t_2, s)(d) ;$$

hence the result (again here l is well defined because of the Lemma and the fact that $D(2) \rightarrowtail D^2$).

(ii) $$\nabla(\lambda \odot (t, s)) = \lambda \odot \nabla(t, s) :$$

We have

$$(\lambda \odot (t, s))(d) = \nabla(\lambda \odot t, s)(d) = exp_S^D H(d \odot (\lambda \odot t), s)$$

$$= exp_S^D((d\lambda) \odot t, s) = exp_S^D(\lambda d \odot t, s) = \nabla(t, s)(\lambda d) = (\lambda \odot \nabla(t, s))(d).$$

(3) ∇ is +-linear :

(i) To show that

$$\nabla((t, s_1) + (t, s_2)) = \nabla(t, s_1) + \nabla(t, s_2) ;$$

equivalently, show that for every $d \in D$,

$$\nabla(t, s_1 + s_2)(d) = \nabla(t, s_1)(d) + \nabla(t, s_2)(d).$$

Let $l^*_d : D(2) \to M$ be given by :

$$l^*_d(d^*_1, d^*_2) = exp_S(d_1 \odot t \oplus d^*_1 \odot s_1 \oplus d^*_2 \odot s_2).$$

Notice that

$$l^*_d(d^*, d^*) = exp_S(d \odot t \oplus d^* \odot (s_1 \oplus s_2)) = \nabla(t, s_1 \oplus s_2)(d)(d^*)$$

on the other hand

$$l^*_d(d^*, 0) = exp_S(d \odot t \oplus d^* \odot s_1) = \nabla(t, s_1)(d)(d^*),$$
$$l^*_d(0, d^*) = exp_S(d \odot t \oplus d^* \odot s_2) = \nabla(t, s_2)(d)(d^*),$$

hence the result (again l^*_d is well defined since $D(2) \rightarrowtail D^2$ and then by the Lemma).

(ii) To show that

$$\nabla(\lambda.(t, s)) = \lambda . \nabla(t, s)$$

i.e. to show that

$$\nabla(t, \lambda \odot s)(d) = \lambda \odot (\nabla(t, s)(d))$$

for every $d \in D$. For $d \in D$,

$$(\lambda . \nabla(t, s))(d) = \lambda \odot \nabla(t, s)(d) = \lambda \odot exp_S^D(H(d \odot t, s))$$

and

$$\nabla(t, \lambda \odot s)(d) = exp_S^D(H(d \odot t, \lambda \odot s)$$
$$\overset{(*)}{=} \lambda \odot exp_S^D H(d \odot t, s) = \lambda \odot \nabla(t, s)(d)$$

where (*) uses that H and (exp_S^D) are both $\oplus$-linear.

(4) S is a geodesic spray for this ∇:

$$\nabla(t, t)(d_1)(d_2) = exp_S(d_1 \odot t \oplus d_2 \odot t) = exp_S((d_1 + d_2) \odot t)$$
$$= \pi_M \varphi_S(d_1 + d_2, t) = \pi_M \varphi_S(d_2, \varphi_S(d_1, t)) = \pi_M S(d_2, S(d_1, t))$$
$$= (\pi_M^D \circ S)(S(d_1, t))(d_2) = S(d_1, t)(d_2) \text{ since } \pi_M^D \circ S = id_{M^D}$$
$$= S(t)(d_1)(d_2).$$

Hence, $\nabla(t, t) = S(t)$ as required. Note that

$$\urcorner\urcorner((d_1 + d_2, t) = (0, t))$$

and $(0, t) \in D(S)$, a Penon open, hence

$$(d_1 + d_2, t) \in D(S)$$

and $\varphi_S(d_1 + d_2, t)$ makes sense.

(5) ∇ is torsion-free : We will prove for now that

$$\nabla \circ \tau = \Sigma_M \circ \nabla$$

where

$$\tau : M^D \times_M M^D \to M^D \times_M M^D$$

sends (t_1, t_2) to (t_2, t_1). The next proposition will clarify the relation between this equation and the definition of the torsion map associated to a connection.

$$\nabla \circ \tau = \Sigma_M \circ \nabla :$$

$$\nabla(\tau(t, s))(d_1)(d_2) = \nabla(s, t)(d_1)(d_2) = exp_S(d_1 \odot s \oplus d_2 \odot t),$$

while

$$(\Sigma_M \circ \nabla)(t, s)(d_1, d_2) = \nabla(t, s)(d_2)(d_1) = exp_S(d_2 \odot t \oplus d_1 \odot s). \qquad \#$$

The torsion θ associated to a connection ∇ is defined as

$$\theta = C - C \circ \Sigma_M$$

where C is the connection map associated to ∇ (cf. [4]). The following result permits us to verify when a connection is torsion-free without using its associated connection map.

Proposition 5. *Let ∇ be a connection on $\pi_M^D : M \to M$ and C its associated connection map. Assume that M is infinitesimally linear and that $\pi_M : M^D \to M$ is Euclidean in E/M. Then*

$$\nabla \circ \tau = \Sigma_M \circ \nabla \text{ iff } C = C \circ \Sigma_M .$$

Proof. We will write 0 and $\underline{0}$ respectively for the zero sections of

$$(M^D)^D \xrightarrow[\pi_{M^D}]{} M^D \quad \text{and} \quad (M^D)^D \xrightarrow[\pi_M^D]{} M^D.$$

It is easy to verify that

$$\Sigma_M \circ 0 = \underline{0} \quad \text{and} \quad \Sigma_M \circ \underline{0} = 0 .$$

Concerning the linearity of Σ_M, we refer the reader to Lemma 1 (i), Section 3.

(1) $\nabla \circ \tau = \Sigma_M \circ \nabla$ implies $C = C \circ \Sigma_M$: $f \in (M^D)^D$,

(a) $$f \ominus \nabla \circ K(f) = H(f(0), C(f))$$

(b) $$\Sigma_M f \ominus \nabla \circ K \circ \Sigma_M(f) = H(\Sigma_M(f)(0), C(\Sigma_M(f)))$$

(definition of C from ∇).

Now

$$\nabla \circ K \circ \Sigma_M = \nabla \circ \tau \circ \quad = \Sigma_M \circ \nabla \circ K,$$

the first equality is straightforward and the second is by hypothesis. We have using (b) :

$$\Sigma_M(f) \ominus \Sigma_M \circ \nabla \circ K(f) = H(\Sigma_M(f)(0), C(\Sigma_M(f)))$$

$$\Leftrightarrow \quad \Sigma_M(f - \nabla \circ K(f)) = H(\Sigma_M(f)(0), C(\Sigma_M(f)))$$

$$\Leftrightarrow \quad f - \nabla \circ K(f) = \Sigma_M(0(\Sigma_M(f)(0)) + \nu(C(\Sigma_M(f))))$$

$$\Leftrightarrow \quad f - \nabla \circ K(f) = \underline{0}(\Sigma_M(f)(0)) \oplus \Sigma_M \circ \nu \circ C \circ \Sigma_M(f)$$

$$= \underline{0}(\Sigma_M(f)(0)) \oplus \nu(C(\Sigma_M(f))).$$

Here we have used that : Σ_M is linear and an involution, the definition of H and the fact (easy to verify) that $\Sigma_M \circ \nu = \nu$.

Applying C to the last equation we obtain

$$C(f) \ominus C(\nabla \circ K(f)) = C(\Sigma_M(f)) ;$$

C is linear and $C \circ \nu = id_{M^D}$. If we apply C to (a) we get :

$$C(f) \ominus C(\nabla \circ K(f)) = C(f) : C \circ \nabla \circ K = 0 .$$

Looking at the previous equation, we get the result.

(2) $C = C \circ \Sigma_M$ implies $\nabla \circ \tau = \Sigma_M \circ \nabla$

Equivalently : $\nabla = \Sigma_M \circ \nabla \circ \tau$. Let $\nabla' = \Sigma_M \circ \nabla \circ \tau$.

That ∇' is a connection on $\pi_M : M^D \to M$ is a straightforward verification. We want to show that $\nabla' = \nabla$. Let C' be the connection map associated to ∇'. We will show that $C = C'$; it clearly implies $\nabla = \nabla'$ (using the fact that $\pi_M : M^D \to M$ has the short path lifting property since it has a connection). Let $f \in (M^D)^D$:

$$f \ominus \nabla' \circ K(f) = H(f(0), C'(f)) :$$

definition of C'. Applying C to this equation :

$$C(f) \ominus C \circ \nabla' \circ K(f) = C'(f)$$

(it is clear from (1) that $C \circ H = proj_2$).

$$C \circ \nabla' \circ K = C \circ \Sigma_M \circ \nabla \circ \tau \circ K = C \circ \nabla \circ \tau \circ K \quad (C = C \circ \Sigma_M)$$
$$= C \circ \nabla \circ K \circ \Sigma_M = 0 \qquad (C \circ \nabla \circ K = 0).$$

Hence $C(f) = C'(f)$. #

From now on, we shall shift our attention to Joyal sprays. In this case, as pointed out earlier, a weaker assumption for the passage spray-connection is needed, as follows.

Definition. M is said to have *the infinitesimal property of existence of exponential maps* if the following holds : Given $\sigma : M^D \to M^{D_2}$, a spray on $\pi_M : M^D \to M$ in the sense of Joyal, there exists a map

$$e_\sigma : D(M) \to M \qquad (\text{where } D(M) = \bigcup_{n \geq 1} D_n(M)$$

and $D_n(M)$ is the image of the map β_n of the Lemma preceding Proposition 4 of this section

$$(D_n(M) = [\![d_1 \odot t_1 \oplus \ldots \oplus d_n \odot t_n \mid d_1, \ldots, d_n \in D \text{ and}$$
$$(t_1, \ldots, t_n) \in M^D \times_M \ldots \times_M M^D]\!]))$$

satisfying

$$d_1, d_2 \in D , \quad e_\sigma((d_1 + d_2) \odot t) = \hat{\sigma}(d_1 + d_2, t).$$

Proposition 6. *Let* M *have the infinitesimal property of existence of exponential maps. Given a Joyal spray* $\sigma : M^D \to M^{D_2}$ *there exists a torsion-free connection* ∇ *such that* σ *is the associated geodesic spray. We assume here that* S *comes from* σ *a Joyal spray* (as in Proposition 1 of this Section).

Proof. We will give only an outline as the proof is very similar to that of Proposition 4, although the basic assumption made is notably weaker here.

$$\nabla(t_1, t_2)(d_1, d_2) = e_\sigma(d_1 \odot t_1 \oplus d_2 \odot t_2).$$

(1) $$(\pi_{M^D} \circ \nabla)(t_1, t_2)(d) = \nabla(t_1, t_2)(0, d) = e_\sigma(0 \odot t_1 \oplus d_2 \odot t_2)$$
$$= e_\sigma(d \odot t_2) = \sigma(d, t_2) = t_2(d) = proj_2(t_1, t_2)(d).$$

We have used the property assumed and the definition of a spray σ .

(2) $$(\pi_M^D \circ \nabla)(t_1, t_2)(d) = \nabla(t_1, t_2)(d, 0) = e_\sigma(d \odot t_1 \oplus 0 \odot t_2)$$
$$= e_\sigma(d \odot t_1) = t_1(d) = proj_1(t_1, t_2)(d).$$

(3) ∇ is $\oplus$-linear. To show :

$$\nabla(t_1 \oplus t_2, s)(d_1, d_2) = (\nabla(t_1, s) \oplus \nabla(t_2, s))(d_1, d_2)$$

it suffices to take

$$l(d_1, d_2)(\tilde{d}) = e_\sigma (d_1 \odot t_1 \oplus d_2 \odot t_2 \oplus \tilde{d} \odot s)$$

and do as in Proposition 4. We show that

$$\nabla (\lambda \odot (t, s)) = \lambda \odot \nabla (t, s)$$

as in Proposition 4.

(4) ∇ is +-linear

$$\nabla (t, s_1 \oplus s_2)(d) = \nabla (t, s_1)(d) \oplus \nabla (t, s_2)(d)$$

Let

$$l_{\tilde{d}}(d_1, d_2) = e_\sigma (\tilde{d} \odot t \oplus d_1 \odot s_1 \oplus d_2 \odot s_2).$$

It suffices then to proceed as before. Same for

$$\nabla (\lambda.(t, s)) = \lambda . \nabla(t, s).$$

S (or σ) is the geodesic spray for :

$$\nabla(t, t)(d_1, d_2) = e_\sigma ((d_1 + d_2) \odot t) = \hat{\sigma} (d_1 + d_2, t)$$

by the assumed property ; this is another way of expressing that S (or σ) is the geodesic spray for ∇ since

$$S(t)(d_1)(d_2) = \sigma (t)(d_1 + d_2).$$

(6) The proof that ∇ is torsion-free is as in proposition 4. #

Remarks. In [1] is shown that if M is cut out of R^n by $f \equiv 0$, then, there is a unique $e : D_{(1)}(M) \to M$ such that

$$e(d \odot t) = t(d) \quad \text{for every} \quad d \in D .$$

By $D_{(k)}(M)$ we mean

$$. \Sigma\, D_{(k)}\left(\begin{array}{c} M^D \\ \downarrow \pi_M \\ M \end{array} \right)$$

where

$$D_{(k)}(V) = [\![v \in V \mid \varphi^{k+1}(v) = 0 \quad \forall \varphi : V \to R \text{ homogeneous}]\!]$$

(in [1], $D_{(2)}(M)$ is defined only for $k = 1$).

Examining the argument in [1] and using similar assumptions on E and R we find that although

$$D_{(2)}(M) = (M \times D_2(n)) \cap M^D$$

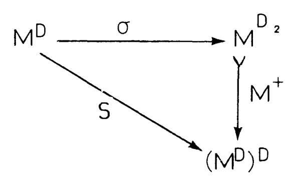
if we let $v \in D_{(2)}(M)$ be given by $v = (\underline{u}, \underline{v})$ in $M \times D_2(n)$, then "defining" $e(v) = \underline{u} + \underline{v}$ does not necessarily land in M ! Indeed, calculating

$$f(\underline{u} + \underline{v}) = f(\underline{u}) + df_{\underline{u}}(\underline{v}) + d^2 f_{\underline{u}}(\underline{v}) + h(\underline{u})(\underline{v})$$

where $h(\underline{u})(\underline{v})$ is of degree ≥ 3, we find $f(\underline{u}) = 0$ since M is defined by $f = 0$, $df_{\underline{u}}(\underline{v}) = 0$ since $(\underline{u}, \underline{v}) \in M^D$, $h(\underline{u})(\underline{v}) = 0$ since $\underline{v} \in D_2(n)$ but there seems to be no guarantee for $d^2 f_{\underline{u}}(\underline{v})$ to be zero. Hence, the need for the property of existence of exponential maps seems clear, not only on the local but also on the infinitesimal level, i.e., not only when dealing with Lawvere sprays but also when dealing with Joyal sprays. We can now establish a relationship between the two kinds of assumptions.

Proposition 7. *Let* M *be infinitesimally linear. If* M *satisfies the local property of existence of exponential maps, then* M *also satisfies the infinitesimal property of the existence of exponential maps.*

Proof. Let σ be a Joyal spray and let S be the Lawvere spray arising from σ as the composite



Suppose that a local flow $(\Phi_S, D(S))$ exists for S. Recall that this defines

$$exp_S : D_S \to M \quad \text{by} \quad exp_S(t) = \pi_M \circ \Phi_S(1, t).$$

Define

$$e_\sigma = D(M) \xrightarrow{\zeta} D_S \xrightarrow{exp_S} M$$

ζ according to the Lemma preceding Proposition 4 of this section). We show that given $d_1, d_2 \in D$,

$$e_\sigma((d_1 + d_2) \odot t) = \sigma(d_1 + d_2, t).$$

Indeed :

$$\begin{aligned} e_\sigma((d_1 + d_2) \odot t) &= exp_S((d_1 + d_2) \odot t) \\ = \pi_M \circ \Phi_S(1, (d_1 + d_2) \odot t) &= \pi_M((d_1 + d_2). \Phi_S(d_1 + d_2, t)) \\ = \pi_M(\Phi_S(d_1 + d_2, t)) = \pi_M(\Phi_S(d_1, \Phi_S(d_2, t))) &= \pi_M(\Phi_S(d_1, \hat{S}(d_2, t))) \\ = \pi_M(\hat{S}(d_1, \hat{S}(d_2, t))) = (\pi_M^D \circ S)(\hat{S}(d_2, t))(d_1) &= \hat{S}(d_2, t)(d_1) \\ = S(t)(d_2)(d_1) &= \sigma(t)(d_1 + d_2) \end{aligned}$$

(again $(d_1 + d_2, t) \in D(S)$ since it is a Penon open containing $(0, t)$). #

Added in proofs :

One question not dealt with in this paper is that of the validity of either property of existence of exponential maps for the classically constructed objects in the models of SDG and its corresponding implications for the derivability of the classical counterparts of those synthetic results depending on them. Concerning the infinitesimal property of existence of exponential maps, this follows from recent work of Kock and Lavendhomme ("Strong infinitesimal linearity, with applications to strong difference and affine connections", Aarhus Preprint Series **47**, 1983/84), since they prove that the property of strong infinitesimal linearity (widely satisfied for familiar objects in any model) implies the "ray property" considered earlier by Kock ("Remarks on connections and sprays", in Category Theoretic Methods in Geometry, Aarhus Var. Pub. Ser. **35**, 1983) in an attempt to do without assumptions of the kind we employ in our paper. In fact, almost direct inspection shows that the ray property of order 2 implies our infinitesimal property (called "axiom" in the preliminary version of this paper) of existence of exponential maps, for $n \leqq 2$, if one assumes furthermore that $T_x M \simeq R^n$. (Notice that it iis only the case $n \leqq 2$ that is needed in the proof of our theorem.) It follows from this that our second synthetic proof of the Ambrose-Palais-Singer Theorem (§ 4, Prop. 6) does give the corresponding classical theorem.

Two further remarks are in order concerning the two notions of spray considered here, and as follow from remarks made by Kock and by Kock and Lavendhomme in the above mentioned articles. Firstly, two of the three assumptions which render the two notions equivalent are, in fact, also consequences of strong linearity (namely, the Symmetric Functions and the Itarated Tangent Bundle properties). Secondly, not only the Lawvere sprays are a synthetic version of an existing classical notion ; but according to Kock and Lavendhomme, the notion of a Joyal spray (which they just refer to as "spray") is the synthetic counterpart of a notion introduced by Smale and utilized in the work of Libermann.

References

1. E. DUBUC & A. KOCK, On 1-form classifiers, Aarhus Preprint Series 1981/82, **39.**
2. C. GODBILLON, Géométrie différentielle et mécanique analytique, Hermann, Paris 1969.
3. A. KOCK, Synthetic Differential Geometry, Cambridge Univ. Press, 1981.
4. A. KOCK & G. REYES, Connections in formal Differential Geometry, Aarhus Math. Inst. Var. Publ. Series **30**, 1979.
5. S. LANG, Introduction to differentiable manifolds, Interscience, 1967.
6. F.W. LAWVERE, Categorical Dynamics, in Aarhus Math. Inst. Var. Publ. Series **30**, 1979.
7. L. N. PATTERSON, Connections and prolongations, Can. J. Math. **XXVII**, 4 (1975), 766-791.
8. J. PENON, Topologie et intuitionnisme, in : Journées de Faisceaux et Logique, Mai 1981, Univ. Paris-Nord (pré-publications math.), 1982.
9. W.A. POOR, Differential geometric structures, McGraw Hill, 1981.
10. B. VEIT RICCIOLI, Structures on iterated tangent bundles (manuscript), 1981.

Department of Mathematics and Statistics
McGill University
805 Sherbrooke St. W.
MONTREAL, PQ, H3A 2K6. CANADA

and

Groupe Interuniversitaire en études catégoriques
MONTREAL