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**ON THE GROTHENDIECK TOPOLOGIES IN THE TOPOSES
 OF PRESHEAVES**
 by V. PATRYSHEV

Résumé. On discute les conditions sous lesquelles les topologies de Grothendieck dans un topos de préfaisceaux sont déterminées par les ensembles d'objets de la catégorie de base. Le livre de P. T. Johnstone "Topos Theory" contient toutes les définitions et résultats nécessaires.

0. Introduction.

When our program to build the Grothendieck topologies over a category was ready, we first obtained all the topologies for the categories as simple as **2** and **4**. They were 4 and 16. It became at once clear that for any 'n' the result should be 2^n . And for the finite trees it is still valid. The question now arises : what are the conditions for such a relation to hold?

1. The basic relation.

Let us consider a bounded geometric morphism f between two toposes, $f : E \rightarrow F$. Each Grothendieck topology j in E ($j \in GT(E)$) gives a topology f_*j in F : $sh_{f_*j}(F)$ is the image of $sh_j(E) \rightarrow E \rightarrow F$.

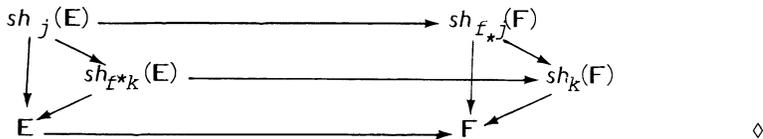
Conversely, any topology $k \in GT(F)$ gives a topology f^*k in E , $sh_{f^*k}(E)$ being the pullback of $sh_k(F)$ against k .

These f_* and f^* are lattice morphisms between $GT(E)$ and $GT(F)$.

1.1. Proposition. *The morphisms f_* and f^* constructed above are adjoint,*

$$f^* \dashv f_* : GT(E) \rightarrow GT(F).$$

This is easily seen from the following diagram :



Let C be a category in E . For the geometric morphism

$$c : E/C_0 \rightarrow E^{C^{OP}},$$

take

$$tops(D_0 \gg C_0) = c_*(j_{D_0}^o) \in GT(E^{C^{OP}}),$$

where $j_{D_0}^o$ is the open topology in E/C_0 corresponding to D_0 , and it gives the lattice morphism

$$tops : Sub(C_0)^{OP} \rightarrow GT(E^{C^{OP}}).$$

On the other hand, for a topology $j \in GT(E^{C^{OP}})$, $c^*(j)$ is the smallest topology k in E/C_0 such that $dns_o : 1_{C_0} \gg J_o$ is k -dense. Take $spot(j)$ to be the largest subobject of C_0 , $D_0 \gg C_0$, for which

$$\begin{array}{ccc} D_0 & \xrightarrow{\quad} & J_o \\ \downarrow & & \downarrow \rho_o \\ D_0 & \xrightarrow{\quad} & C_0 \end{array}$$

is a pullback. Or, in other words,

$$spot(j) = \{c \mid (\omega \in J_o \wedge \rho_o \omega = c) \Rightarrow \omega = true(c)\}.$$

The topology $tops(D_0)$ may also be expressed as

$$tops(D_0)_o = \{\omega \mid (\rho_o \omega \in D_0) \Rightarrow (\omega = true(\rho_o))\}.$$

2. When $(Sub C_0)^{OP}$ is a reflexive sublattice of $GT(E^{C^{OP}})$.

For a subobject $D_0 \gg C_0$ the construction

$$D_o^k = \{c \mid \exists \langle f, g \rangle ((m(f, g) = e(c) \wedge (d_1 f \in D_o))\} \gg C_0$$

is known as its *Karoubian closure*.

2.1. Proposition. *For any topology in $E^{C^{OP}}$ its spot is Karoubian-closed.*

Proof. Let

$$m(f, g) = e(c), \text{ and } d_1 f = d_o g \in spot(j),$$

and let ω of type J_o be such that $\rho_o(\omega) = c$, then

$$\rho_o(g(\omega)) = d_o g \in spot(j), \text{ and } g(\omega) = true(\rho_o g(\omega)).$$

So

$$\omega = fg(\omega) = f(true(\rho_o g(\omega))) = true(\rho_o fg(\omega)) = true(c),$$

that is $c \in spot(c)$. ◊

2.2. Proposition. *The lattice of Karoubian-closed subobjects of C_0 is a*

reflexive sublattice of $GT(\mathbf{E}^{\mathbf{C}^{\text{op}}})$.

Proof. It is enough to show that for any $D_0 \twoheadrightarrow C_0$, $\text{spot}(\text{tops}(D_0))$ is its Karoubian closure. Let $c \in \text{spot}(\text{tops}(D_0))$. Then for $\omega \in \text{tops}(D_0)$,

$$\rho_\omega(\omega) = c \quad \text{iff} \quad \omega = \text{true}(c).$$

Take the free presheaf $R(c)$. $Z \twoheadrightarrow R(c)$ is $\text{tops}(D_0)$ -dense iff Z_0 equals to $R(c)_0$ over D_0 . But then the subpresheaf

$$Z = \{ f \mid d_1 f = c \wedge \exists \langle g, h \rangle (f = m(h, g) \wedge d_0 h \in D_0) \}$$

(which is the image of

$$C_1 \times_{C_0} D_0 \times_{C_0} C_1 \times_{C_0} U \twoheadrightarrow C_1 \times_{C_0} C_1 \times_{C_0} U \xrightarrow{m \times U} C_1 \times_{C_0} U)$$

is dense in $R(c)$. It follows that for ω classifying Z in $R(c)$, $\omega(c) = \text{true}(c)$, and $Z = R(c)$ at c , $e(c) \in Z_0$, and we have

$$E \models (\exists \langle g, h \rangle (e(c) = m(h, g) \wedge d_0 h \in D_0)). \quad \diamond$$

2.3. Definition. A category C in E is *pseudoantisymmetric* if

$$E \models ((d_0 f = d_1 g \wedge d_1 f = d_0 g) \Rightarrow (d_0 f = d_1 f \wedge h = f^{-1})),$$

or, in the diagram

$$\begin{array}{ccc} \text{Aut}(C) & \twoheadrightarrow & C \\ \downarrow Y & & \downarrow (d_0, d_1) \\ C & \xrightarrow{(d_0, d_1)} & C_0 \times C_0 \end{array}$$

is a pullback.

2.4. Proposition. For a pseudoantisymmetric C each $D_0 \twoheadrightarrow C_0$ is Karoubian-closed, and the lattice $(\text{Sub } C_0)^{\text{op}}$ is a reflexive sublattice in $GT(\mathbf{E}^{\mathbf{C}^{\text{op}}})$. \(\diamond\)

3. When the topologies in $E^{\mathbf{C}^{\text{op}}}$ are determined by subobjects of C_0 , or spot becomes iso.

For this property to hold, we obviously need still another restriction to be put upon the category C . Note first that any finite segment of \mathbf{N}^{op} possesses the property, while the whole \mathbf{N}^{op} does not. The obstacle is its infiniteness: take the topology $\top\top$ in $\mathbf{E}^{\mathbf{N}}$ - its spot is void. So we need some sort of boundedness, to be more precise, left-boundedness. And the topos E/C_0 should be Boolean.

3.1. Definition. A pseudoantisymmetric category C in a topos E is *fairly-ordered* if for any $D_0 \twoheadrightarrow C_0$ there is some $\rho: W \twoheadrightarrow D_0$ such that

$$E \models ((d_0f \in D_0 \wedge d_1f \in W) \Rightarrow d_0f = d_1f)$$

and $\pi_2: W \times D_0 \rightarrow D_0$ is epi.

Expressing this in Kripke-Joyal, we have

$$E \models (\exists \rho(\rho \in q) \Rightarrow (\exists \rho(\rho \in q \wedge \forall \rho'((\rho' \in q \wedge \rho' \leq \rho) \Rightarrow \rho' = \rho))))$$

for q of the type Ω^C , where $\rho' \leq \rho$ means

$$\exists f(d_0f = \rho' \wedge d_1f = \rho).$$

Any well-ordered poset is fairly-ordered. Finite antisymmetric categories in \underline{Ens} are also fairly-ordered.

3.2. Proposition. Let C be a fairly-ordered category in a topos E , and let E/C_0 be Boolean. Then $(\text{tops}, \text{spot})$ is an isomorphism between

$$(\text{Sub } C_0)^{\text{op}} \text{ and } \text{GT}(E^{C^{\text{op}}}).$$

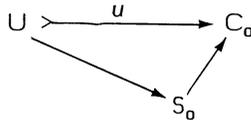
Proof.

3.2.1. In any case we have $\text{tops}(\text{spot}(j)) \geq j$, and for our C ,

$$\text{spot} \circ \text{tops} = 1.$$

Let j be a topology in $E^{C^{\text{op}}}$, and let j' denote $\text{tops}(\text{spot}(j))$. The corresponding subobjects of $\Omega_{E^{C^{\text{op}}}}$ (further denoted by Ω) are J and J' . The inclusion $J' \leq J$ is to be shown.

3.2.2. Let $\rho_0: R_0 \twoheadrightarrow C_0$ (corresponding to the full subcategory $\rho: R \twoheadrightarrow C$) be the largest subobject of C_0 such that $\rho * J' = \rho * J$ in $E^{R^{\text{op}}}$. Take $\sigma_0: S_0 \twoheadrightarrow C_0$ (corresponding to $\sigma: S \twoheadrightarrow C$) to be a complement of R_0 in C_0 . By 3.1 there is



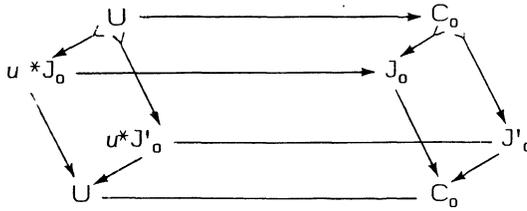
such that

$$((d_0f \in S_0 \wedge d_1f \in U) \Rightarrow d_0f = d_1f)$$

and $\pi_2: U \times S_0 \rightarrow S_0$ is epi. Let us prove now that $u * J'_0 \leq u * J_0$.

3.2.3. Look at the commutative diagram (a) hereafter. Suppose for some $t: T \rightarrow U$, $t * u * j_0 = 1_T$. Then

$$T \leq \text{spot}(j), \quad J' \leq \text{tops}(T),$$



and

$$t^*u^*J'_0 = t^*u^*tops(T) = 1_T = t^*u^*J_0.$$

So $T \leq R_0 \cap S_0 = 0$.

Now for $D_0 =$ complement of

$$u^*(C_0 \xrightarrow{dns_0} J_0)$$

in u^*J_0 the composition

$$D_0 \longrightarrow u^*J_0 \longrightarrow U$$

is epi. The composition

$$D_0 \longrightarrow u^*J_0 \longrightarrow U$$

gives a J -dense subpresheaf $Z \gg R(D_0)$ with

$$Z_0 \cap (D_0 \xrightarrow{(e, 1)} C_1 \times_{C_0} D_0) = 0.$$

3.2.4. Here we will see that

$$Z_0 \xrightarrow{\quad} C_1 \times_{C_0} D_0 \xrightarrow{d_0 \pi_1} C_0$$

factors through $R_0 \gg C_0$. Really, take $T = \sigma_0^*(Z_0)$, then

$$T \longrightarrow S_0 \times Z_0 \longrightarrow S_0 \times D_0 \longrightarrow S_0 \times U \longrightarrow C_0 \times C_0$$

factors through C_1 , it will be denoted $t: T \rightarrow C_1$. Since $d_0 t \in S_0$ and $d_1 t \in U$, we have $d_0 t = d_1 t$, and $T \rightarrow C_0 \times C_0$ factors through the diagonal, making the square

$$\begin{array}{ccc} T & \longrightarrow & D_0 \\ \downarrow & & \downarrow \\ Z_0 & \longrightarrow & C_1 \times_{C_0} D_0 \end{array}$$

commute. Then

$$T = T \cap (D_0 \longrightarrow C_1 \times_{C_0} D_0) = 0.$$

3.2.5. For $V_0 = D_0 \times u^*J'_0$ we have (as well as for D_0) a j -dense subpresheaf $Z \gg R(V_0)$ (classified by

$$C_1 \times_{C_0} V_0 \longrightarrow D_0 \longrightarrow u^*J_0 \longrightarrow \Omega_0$$

and Z_0 is also over R_0 .

The monomorphism $u^*J'_0 \twoheadrightarrow \Omega_0$ gives a j' -dense subobject $T \twoheadrightarrow R(u^*J'_0)$. Its j -closure is the pullback

$$\begin{array}{ccc} \bar{T} & \longrightarrow & J_0 \\ \downarrow \gamma & & \downarrow \gamma \\ C_1 \times_{C_0} u^*J'_0 & \longrightarrow & \Omega_0 \end{array}$$

In E/R_0 this looks like

$$\begin{array}{ccc} R_0 \times_{C_0} \bar{T}_0 & \longrightarrow & R_0 \times_{C_0} J_0 \\ \downarrow \gamma & & \downarrow \gamma \\ R_0 \times_{C_0} C_1 \times_{C_0} u^*J'_0 & \longrightarrow & R_0 \times_{C_0} \Omega_0 \end{array}$$

But we have

$$R_0 \times_{C_0} J_0 = R_0 \times_{C_0} J'_0, \quad \text{so} \quad R_0 \times_{C_0} \bar{T}_0 = \overline{R_0 \times_{C_0} T_0}$$

is also the j' -closure of T , that is, it equals to $R_0 \times_{C_0} C_1 \times_{C_0} u^*J'_0$. Now, since Z_0 is over R_0 ,

$$\begin{array}{ccc} Z_0 & \longrightarrow & \bar{T}_0 \\ \downarrow \gamma & & \downarrow \gamma \\ C_1 \times_{C_0} V & \longrightarrow & C_1 \times_{C_0} u^*J'_0 \end{array}$$

commutes, and we have

$$\begin{array}{ccc} Z & \longrightarrow & \bar{T} \\ \downarrow \gamma & & \downarrow \gamma \\ R(V) & \longrightarrow & R(u^*J'_0) \end{array}$$

with $Z \twoheadrightarrow R(V)$ j -dense and \bar{T} closed, thus T is also j -dense in $R(u^*J'_0)$. But therefore

$$u^*J' = u^*J, \quad \text{then} \quad U \leq S_0 \cap R_0 = 0, \quad \text{and} \quad S_0 = 0.$$

We have

$$R_0 = C_0, \quad \text{and} \quad \text{tops}(\text{spot}(j)) = j. \quad \diamond$$

The question naturally arises : what properties are necessary for a category C and $\text{topos } E/C_0$ to give the isomorphism between topologies

and subobjects? The properties are just the same as in 3.2. The propositions below prove this. The following lemma will be an example.

3.3. Lemma. *If a category C in a topos E is a monoid, and $(\text{tops}, \text{spot})$ is an isomorphism, then C is a group(oid).*

Proof. For $D_0 \gg C_0$ the full subcategory $D \rightarrow C$ is in fact a subobject of 1 in E^{COP} . But then any topology in E^{COP} is open, E^{COP} is Boolean, and is thus a group. \diamond

This shows how one could prove in an arbitrary category with *spot* iso all the endomorphisms are invertible.

But first of all we should prove the following

3.4. Lemma. *If for a category C in a topos E , $(\text{tops}, \text{spot})$ is an isomorphism, then the topos E/C_0 is Boolean.*

Proof. We have the reflection

$$\text{GT}(E/C_0) \longrightarrow \text{GT}(E^{\text{COP}}),$$

the isomorphism *tops* factors through it, thus any topology in E/C_0 will come from $\text{Sub}C_0$, and is consequently open. E/C_0 is Boolean. \diamond

Next we shall successively obtain the properties of C for

$$\text{spot} = \text{tops}^{-1}.$$

3.5. Lemma. *Let $(\text{tops}, \text{spot})$ be an iso for C in a topos E , and let $f : U \rightarrow C$ be such that $d_1 f = d_0 f$. Then f is invertible.*

Proof. Let \bigcup_{ni} be the complement of $f^*(\text{Iso}(C))$, D_0 be the image of

$$\bigcup_{ni} \xrightarrow{fni} C \xrightarrow{d_0} C_0,$$

and D the corresponding full subcategory in C . It is clear that *fni* is not invertible anywhere in D . The complement of the image of $D_1 \times \bigcup_{D_0} \rightarrow D_1$

(considered as the subpresheaf of $R(D_0)$ in E^{DOP}) is empty, so this image is $\neg\neg$ -dense. Then $\neg\neg$ is not true anywhere in E^{DOP} ,

$$\text{tops}(\text{spot}(\neg\neg)) = (\Omega \longrightarrow 1 \xrightarrow{\text{true}} \Omega)$$

in E^{DOP} , $\text{sh}_{\neg\neg}(E^{\text{DOP}})$ is degenerate, and E^{DOP} is also degenerate. $D_0 = 0$, and f is invertible in C . \diamond

3.6. Lemma. *Let $(\text{tops}, \text{spot})$ be an iso for C in a topos E . Let f be an invertible U -element of C_1 , $d_0 f = x$, $d_1 f = y$. Then $x = y$.*

Proof. Consider two topologies :

$$j_x = \text{tops}(x), \quad \text{and} \quad j_y = \text{tops}(y).$$

The monomorphism $u : A \rightarrow B$ in E^{COP} is

$$\begin{aligned} j_x\text{-dense} & \text{ iff } x^*A_0 = x^*B_0, \\ j_y\text{-dense} & \text{ iff } y^*A_0 = y^*B_0. \end{aligned}$$

The two conditions are equivalent because $f : x \rightarrow y$ is invertible. \diamond

The next proposition is merely the corollary of the two lemmas above.

3.7. Proposition. *Let $(\text{tops}, \text{spot})$ be an iso for C in a topos E . Then C is pseudoantisymmetric.* \diamond

3.8. Proposition. *Let $(\text{tops}, \text{spot})$ be an iso for C in a topos E . Then C is fairly-ordered.*

Proof. Let $I \rightarrow C_0$ be the *spot* of double negation ($\text{spot}(\neg\neg)$), and $f : U \rightarrow C_1$ be such that $d_1 f$ is in I . Take the presheaf $R(I)$ and its subpresheaf Z - the image of

$$C_1 \times_{C_0} U \xrightarrow{-C_1 x f} C_1 \times_{C_0} C_1 \times_{C_0} I \xrightarrow{m} C_1 \times_{C_0} I.$$

The complement of the image is empty, then Z is $\neg\neg$ -dense, which means equal to $R(I)$ over I , so f is an isomorphism.

Note that the same holds for an arbitrary full subcategory $D \rightarrow C$, and the property of the *tops* being iso is hereditary with respect to full subcategories. \diamond

3.9. Remark. The " I " used in 3.8 is actually "the set of initial points" of C . To be more precise,

$$I = \Pi_{d_1} [(d_0, d_1)^* \Delta_C].$$

Really, the full subcategory $I \rightarrow C$ is almost discrete - it has only automorphisms. It is the subobject of 1 in E^{COP} , and the corresponding *tops* (I) is open. $sh_{j_I} \rightarrow E^{\text{COP}}$ is logical, and $\neg\neg = 1$ in sh_{j_I} . So $sh_{j_I} \rightarrow E^{\text{COP}}$ factors through $sh_{\neg\neg} \rightarrow (E^{\text{COP}})$. \diamond

4. Conclusion.

The situation when any topology in E^{COP} is determined by the subobjects of C_0 seems to be most pleasant from the "materialistic" point

of view. That is, if we think of C as a generalized time (internal time for a physical body), the isomorphism

$$(\text{Sub } C_0) \longrightarrow \text{GT}(\mathbf{E}^{C^{\text{op}}})$$

means that any event, which develops in time, can be discerned by marking time, and ghosts cannot appear. The conditions upon our time are simple : it ought to be discrete, irreversible, and bounded at one side. But groups of automorphisms (space symmetry) are permitted at any moment.

Or, to be less philosophical, let C be a computational scheme of some process in a computer. It is non-linear in the case of multiprocessing. And the property of $\text{tops} = \text{spot}^{-1}$ gives the possibility to spot a deviation of the realization at the points of the scheme - the quality the structured programs are expected to possess.

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