

# CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES

MARIA CRISTINA PEDICCHIO

## **On the category of topological topologies**

*Cahiers de topologie et géométrie différentielle catégoriques*, tome 25, n° 1 (1984), p. 3-13

[http://www.numdam.org/item?id=CTGDC\\_1984\\_\\_25\\_1\\_3\\_0](http://www.numdam.org/item?id=CTGDC_1984__25_1_3_0)

© Andrée C. Ehresmann et les auteurs, 1984, tous droits réservés.

L'accès aux archives de la revue « Cahiers de topologie et géométrie différentielle catégoriques » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## ON THE CATEGORY OF TOPOLOGICAL TOPOLOGIES

by Maria Cristina PEDICCHIO

### SUMMARY.

We study the main properties of the category  $\mathcal{J}$  of all pairs  $(Y, Y^*)$  with  $Y$  a topological space and  $Y^*$  a topological topology on the lattice of open sets of  $Y$ . By  $\mathcal{J}$ , it is possible to classify monoidal closed and monoidal biclosed structures on  $Top$ .

### INTRODUCTION.

Isbell in [8] gave the following classification for adjoint endofunctors of the category  $Top$  of topological spaces. A pair  $F \dashv G: Top \rightarrow Top$  is completely determined by a pair  $(Y, Y^*)$  of topological spaces where

$$Y = F(\{*\}) \text{ and } Y^* = G(2),$$

with  $2$  a Sierpinski space; furthermore the set underlying  $Y^*$  is, up to isomorphism, just the lattice of open sets of  $Y$ , and the topology of  $Y^*$  is a *topological topology*, that is, it makes finite intersection and arbitrary union continuous operations.

It is natural then to define a category  $\mathcal{J}$  whose objects are all pairs  $(Y, Y^*)$ , and whose morphisms are all continuous maps  $f: X \rightarrow Y$ , such that  $f^{-1}: Y^* \rightarrow X^*$  is continuous too.

Our aim is to study the main properties of such a category  $\mathcal{J}$ .

Section 1 is completely devoted to recall definitions and properties about topological functors and initially complete categories drawing mainly on Herrlich [6, 7].

In Section 2, first we prove that  $\mathcal{J}$  is a small-fibred, initially complete category on  $Top$ , in order to apply the previous results; then we construct a monoidal closed (also biclosed) structure on  $\mathcal{J}$ .

Section 3 is devoted to the problem of classification of monoidal closed structures on the category of topological spaces [2, 3]). As a generalization of our results about structures induced by adjoining systems of filters [10], we give a complete description both of all monoidal closed structures and of all monoidal biclosed structures on  $Top$ , relating them to opportune functors from  $\mathcal{J}$  to  $Top$ . The author is greatly indebted to G. M. Kelly for many helpful conversations and advices.

1. For a concrete category over a base category  $X$ , we mean a pair  $(A, U)$ , where  $U: A \rightarrow X$  is a forgetful functor, that is, a faithful, amnesic and transportable functor.

A  $X$ -morphism  $c: X \rightarrow Y$  is said to be a constant morphism provided that, for each  $Z$  in  $X$  and for all  $u, v \in X(Z, X)$ ,  $c \cdot u = c \cdot v$ . If  $X$  is  $Set$  or  $Top$  this definition is just the classical one.

DEFINITION 1.1. A concrete category  $(A, U)$  over  $X$  is called a *topological category* iff the following hold:

- (i)  $U: A \rightarrow X$  is a topological functor in the sense of [6];
- (ii)  $A$  is small-fibred;
- (iii) every constant morphism  $U(A) \rightarrow U(B)$  underlies some  $A$ -morphism  $A \rightarrow B$ .

If the base category  $X$  is  $Set$  or  $Top$ , then condition (iii) is equivalent to the following

- (iv) For any  $X$  with cardinality one, the fibre  $U^{-1}(X)$  is a singleton.

We shall say that a concrete category is a *small-fibred, initially complete category* iff it verifies (i) and (ii) of Definition 1.1 (see [1, 11]).

If  $(A, U)$  and  $(B, V)$  are concrete categories over  $X$ , for a functor  $F$  over  $X$ ,  $F: (A, U) \rightarrow (B, V)$ , we mean any functor  $F: A \rightarrow B$  with  $V \cdot F \simeq U$ .

DEFINITION 1.2. A functor  $F$  over  $X$ ,  $F: (A, U) \rightarrow (B, V)$ , is called an *extension* (of  $(A, U)$ ) if  $F$  is full and faithful.

DEFINITION 1.3. An extension  $E: (A, U) \rightarrow (B, V)$  is called a *small-*

*fibred, initial completion (of  $(A, U)$ ) if  $(B, V)$  is small-fibred and initially complete.*

The following list exhibits some properties of small-fibred, initially complete categories over the category *Top* of topological spaces; the same results are well known for topological categories over *Set* [7].

PROPOSITION 1.4. *If  $(A, U)$  is a small-fibred, initially complete category over *Top*, then the following hold:*

- (1) *A is finally complete;*
- (2) *A is complete and cocomplete and  $U: A \rightarrow \text{Top}$  preserves limits and colimits;*
- (3) *An A-morphism is a monomorphism (epimorphism) iff it is injective (surjective);*
- (4) *A is wellpowered and cowellpowered;*
- (5)  *$U: A \rightarrow \text{Top}$  has a full and faithful left adjoint  $D$  (= discrete structure);*
- (6)  *$U: A \rightarrow \text{Top}$  has a full and faithful right adjoint  $K$  (= indiscrete structure);*
- (7) *For any A-morphism  $f$ ,  $f$  is an embedding iff  $f$  is an extremal monomorphism iff  $f$  is a regular monomorphism;  $f$  is a quotient map iff  $f$  is an extremal epimorphism iff  $f$  is a regular epimorphism;*
- (8) *For any topological space  $X$ , the fibre  $U^{-1}(X)$  of  $X$  is a small complete lattice;*
- (9) *Any A-object  $A$ , with  $UA \neq \emptyset$  and  $A = DUA$ , is a separator;*
- (10) *An A-object  $A$  is a coseparator iff there exists an embedding of an indiscrete object with two points into  $A$ ;*
- (11) *An A-object  $A$  is projective iff  $A = DUA$ ;*
- (12) *An A-object  $A$  is injective iff  $UA \neq \emptyset$  and  $A = KUA$ .*

PROOF. (1), (2) and (3) follow from [6], Sections 5 and 6.

(4) is true for *Top* is well- and cowellpowered.

(5) and (6) follow by Theorem 7.1 of [6].

(7) follows easily by the definition of regular and extremal morphism.

(8) For any pair  $A^x, \bar{A}^x$  of objects in  $U^{-1}(X)$ , let us put

$A^x \leq \bar{A}^x$  iff there is a  $A$ -morphism  $g: A^x \rightarrow \bar{A}^x$  with  $U(g) = I_X$  ;

with this order relation,  $U^{-1}(X)$  forms a small complete lattice and, for any family  $(A_i^x)_{i \in I}$  in the fibre,  $inf(A_i^x)$  is the  $U$ -initial lifting of the source  $(X^i, I_X: X \rightarrow U(A_i^x)_{i \in I})$ .

(9), (10), (11) and (12) follow by properties of discrete and indiscrete structures.  $\square$

2. Let us recall the following definition from [8].

A topology  $T$  on the lattice  $OY$  of the open sets of a topological space  $Y$  is a *topological topology* iff it makes finite intersection and arbitrary union continuous maps (we use the symbol  $Y^*$  for the set  $OY$  topologized by  $T$ ). Isbell [8, 12] has proved that any pair of adjoint endofunctors  $F \dashv G$  of  $Top$  is determined (up to functorial isomorphism) by a pair  $(Y, Y^*)$ , where  $Y = F(\{*\})$  and  $Y^* = G(2)$ , with  $2$  a Sierpinski space; furthermore  $G(2)$  has, as underlying set, just  $OY$  (up to bijection) and  $Y^*$  is a topological topology on this lattice.

We define a category  $\mathcal{J}$  (= Isbell category) as follows :

Objects of  $\mathcal{J}$  are all pairs  $(X, X^*)$ , with  $X \in Top$  and  $X^*$  a topological topology on  $OX$ . An  $\mathcal{J}$ -morphism  $f: (X, X^*) \rightarrow (Y, Y^*)$  is a continuous map  $f: X \rightarrow Y$  such that  $f^{-1}: OY \rightarrow OX$  is continuous from  $Y^*$  to  $X^*$  (we shall write  $f^*$  for  $f^{-1}$ ). We call  $V$  the forgetful functor  $V: \mathcal{J} \rightarrow Top^0$  defined by

$$V(X, X^*) = X^* \quad \text{and} \quad V(f) = f^*,$$

and we call  $U$  the forgetful functor  $U: \mathcal{J} \rightarrow Top$  defined by

$$U(X, X^*) = X \quad \text{and} \quad U(f) = f.$$

Then, if  $Adj$  denotes the category whose objects are the pairs  $F \dashv G$  of adjoint endofunctors of  $Top$ , and whose morphisms are the pairs of natural transformations

$$(\alpha, \beta): (F, G) \rightarrow (F', G'), \quad \text{with} \quad \alpha: F \rightarrow F' \quad \text{and} \quad \beta: G' \rightarrow G$$

Isbell's result implies the following theorem :

**THEOREM 2.1.** *The categories  $Adj$  and  $\mathcal{J}$  are equivalent.*

PROPOSITION 2.2.  $(\mathcal{G}, U)$  is a small-fibred, initially complete category over  $Top$ .

PROOF. If  $(X_i, X_i^*)_{i \in I}$  is a family of  $\mathcal{G}$ -objects and

$$(X, f_i: (X_i, X_i^*) \rightarrow X)_{i \in I}$$

is a sink in  $Top$ , we denote by  $X^*$  the set  $OX$  with the initial topology with respect to  $(f_i^*)_{i \in I}$ . To show that  $X^*$  is a topological topology, it suffices to consider the following commutative diagram

$$\begin{array}{ccc} X^* \times X^* & \xrightarrow{\cap} & X^* \\ f_i^* \times f_i^* \downarrow & & \downarrow f_i^* \\ X_i^* \times X_i^* & \xrightarrow{\cap_i} & X_i^* \end{array}$$

and to apply properties of the initial topology to the continuous composite  $\cap_i \cdot (f_i^* \times f_i^*)$  (the same applies to the union map). Since, for any

$$b: U(X, X^*) = X \rightarrow Y = U(Y, Y^*),$$

$b^*: Y^* \rightarrow X^*$  is continuous iff  $f_i^* \cdot b^*$  is continuous for any  $i$ , then  $b$  is an  $\mathcal{G}$ -morphism iff  $b \cdot f_i$  is an  $\mathcal{G}$ -morphism (for any  $i$ ). This shows that  $(X, X^*)$  is the  $\mathcal{G}$ -final structure on  $X$  with respect to  $(f_i)_{i \in I}$ .

Any finally complete category is initially complete; if

$$(X, g_i: X \rightarrow U(X_i, X_i^*))_{i \in I}$$

is a source in  $Top$  and  $\Phi$  denotes the following set:

$$\Phi = \{ X^* \mid X^* \text{ is a topological topology on } OX, \text{ and } g_i: (X, X^*) \rightarrow (X_i, X_i^*) \text{ is an } \mathcal{G}\text{-morphism for any } i \}$$

then the  $\mathcal{G}$ -initial structure on  $X$  with respect to  $(g_i)_{i \in I}$  is the  $\mathcal{G}$ -final structure on  $X$  with respect to all the functions  $l: (X, X^*) \rightarrow X, X^* \in \Phi$ .

□

REMARK 2.3. If  $S$  is an open set of a topological topology  $X^*$  and  $r, s$ , are objects of  $X^*$  such that  $r \in S$  and  $s \supseteq r$ , then  $s \in S$ . Consider the continuous map  $\bigcup_{n \in \mathbb{N}} \prod (X^*)_n \rightarrow X^*$ ; since  $U^{-1}(S)$  is open in the topological product  $\prod_{n \in \mathbb{N}} (X^*)_n$  and  $(r_n)_{n \in \mathbb{N}} = (r, r, \dots, r) \in U^{-1}(S)$ , then there ex-

ists a neighborhood  $U_r \subseteq \cup^{-1}(S)$  of  $(r_n)$  with

$$U_r = \prod_{n \in N} T_n = T_{l_1} \times T_{l_2} \times \dots \times T_{l_b} \times \mathbf{O}X \times \dots \times \mathbf{O}X.$$

and  $T_{l_j}$ ,  $1 \leq j \leq b$  open sets of  $X^*$  containing  $r$ . The sequence

$$(t_n)_{n \in N}, \text{ with } t_n = r \text{ if } n = l_1, l_2, \dots, l_b \text{ and } t_n = s \text{ otherwise}$$

is in  $U_r$ , so  $s$  is in  $S$ .

Note that a construction of  $\mathcal{G}$ -initial structures by *Top*-final structures generally fails.

As an example, consider a constant map  $k: 2 \rightarrow U(X, X^*)$ , with  $(X, X^*) \in \mathcal{G}$ , and denote by  $\beta = \{0, 1, 2\}$  the ordered set  $\mathbf{O}2$ . Since the set  $\{0, 2\}$  is open in the final topology  $2_f^*$  on  $\beta$  with respect to the map  $k^*: X^* \rightarrow \beta$ , it follows, by Remark 2.3, that  $2_f^*$  is not a topological topology, so, if  $(2, 2^*)$  is the  $\mathcal{G}$ -initial structure with respect to  $k$ ,  $2^* \leq 2_f^*$ .

By Theorem 2.2, Proposition 1.4 can be applied, hence completeness and cocompleteness of  $\mathcal{G}$  follow, where limits and colimits are obtained from limits and colimits in *Top* by  $\mathcal{G}$ -initial and  $\mathcal{G}$ -final structures respectively, with respect to the maps of the limit (or colimit) cone of *Top*. Again by Proposition 1.4,  $U: \mathcal{G} \rightarrow \text{Top}$  has a left adjoint  $D$  and a right adjoint  $K$  with the following properties:

$$\mathcal{G}(D(X), (Y, Y^*)) \simeq \text{Top}(X, U(Y, Y^*) = Y),$$

$$\text{Top}(U(Y, Y^*) = Y, X) \simeq \mathcal{G}((Y, Y^*), K(X))$$

for any  $X \in \text{Top}$  and  $(Y, Y^*) \in \mathcal{G}$ . If we denote by  $(X, X_D^*)$  the discrete structure  $D(X)$  and by  $(X, X_K^*)$  the indiscrete structure  $K(X)$ , then  $X_D^*$  and  $X_K^*$  determine, resp., the coarsest and the finest topological topology definable on  $\mathbf{O}X$ . For  $X = \{*\}$ , the fibre

$$U^{-1}(\{*\}) = D(\{*\}) = (\{*\}, 2), \quad K(\{*\}) = (\{*\}, 2).$$

(where  $2$  denotes an indiscrete space with two objects), hence  $\mathcal{G}$  is not topological. Since *Top* is topological over *Set* with the forgetful functor  $\bar{U}: \text{Top} \rightarrow \text{Set}$ , then  $\mathcal{G}$  is small-fibred, initially complete over *Set* with the forgetful functor  $W = \bar{U} \cdot U$ . We write  $\bar{D}$  and  $\bar{K}$  for the left and right adjoints to  $\bar{U}$  and  $\bar{D}$  and  $\tilde{K}$  for the composite functors  $D \cdot \bar{D}$  and  $K \cdot \bar{K}$ .

For  $X, Z \in Top$  and  $(Y, Y^*) \in \mathcal{J}$ , the symbol  $X \otimes Y$  denotes the set  $\bar{U}(X) \times \bar{U}(Y)$  with the topology  $O(X \otimes Y) = Top(X, Y^*)$ , and the symbol  $[Y, Z]$  denotes the set  $Top(Y, Z)$  with the initial topology with respect to the maps

$$\{\lambda_r : Top(Y, Z) \rightarrow Y^*\}_{r \in OZ} \text{ defined by } \lambda_r(f) = f^{-1}(r), f: Y \rightarrow Z$$

[8, 10]. Furthermore, from now on, we shall use letters  $A, B, C$  to denote  $\mathcal{J}$ -objects

$$A = (X, X^*), \quad B = (Y, Y^*), \quad C = (Z, Z^*) ;$$

$J$  will be the  $\mathcal{J}$ -object  $D(\{*\}) = (\{*\}, 2)$  and  $I$  the  $\mathcal{J}$ -object

$$K(\{*\}) = (\{*\}, 2).$$

**THEOREM 2.4.** *The category  $\mathcal{J}$  admits a structure of monoidal closed category [4]  $(\mathcal{J}, \circ, I, \{-, -\})$  with*

$$A \circ B = (X \otimes Y, (X \otimes Y)^* = [X, Y^*]), \quad \{B, C\} = ([Y, Z], [Y, Z]^*)$$

where  $[Y, Z^*]$  is the set  $O[Y, Z]$  with the final topological topology with respect to the maps  $\{p_S : Z^* \rightarrow O[Y, Z]\}_{S \in OY^*}$ , defined by

$$p_S(r) = \lambda_r^{-1}(S) \text{ with } r \in Z^* \text{ and } \lambda_r : [Y, Z] \rightarrow Y^*.$$

**PROOF.** Since the initial topology on  $Top(X, Y^*)$  with respect to all the maps  $\{\lambda_S\}_{S \in OY^*}$  is a topological topology (see Theorem 2.2), then the tensor product object is in  $\mathcal{J}$ . For  $\mathcal{J}$ -morphisms  $f: A \rightarrow A'$  and  $g: B \rightarrow B'$  we define

$$f \circ g : (X \otimes Y, [X, Y^*]) \rightarrow (X' \otimes Y', [X', Y'^*])$$

as the product  $f \times g$ . If  $r: X' \rightarrow Y'^*$  is an open set of  $X' \otimes Y'$ , then

$$(f \times g)^{-1}(r) = g^*.r.f: X \rightarrow Y^*,$$

hence  $f \times g: X \otimes Y \rightarrow X' \otimes Y'$  is continuous; to prove the continuity of  $(f \times g)^*: [X', Y'^*] \rightarrow [X, Y^*]$ , it suffices to apply properties of the initial topology.  $I = (\{*\}, 2)$ , so

$$A \circ I \simeq A \text{ and } I \circ B \simeq B, \text{ for } A, B \in \mathcal{J}.$$

The associativity of  $\circ$  follows from Proposition 2.3 of [10], for

$$(X \otimes Y)^* = [X, Y^*].$$



As for the internal hom we get the final topological topology  $[Y, Z]^*$  as in Theorem 2.3 and the  $\mathcal{J}$ -morphism  $\{f, g\}: \{B, C\} \rightarrow \{B', C'\}$  is  $hom(f, g)$ , for any  $f: B' \rightarrow B$  and  $g: C \rightarrow C'$ . If  $B = (Y, Y^*) \in \mathcal{J}$ , it is known by [8], that

$$Top(X \otimes Y, Z) \xrightarrow[\phi]{\cong} Top(X, [Y, Z]).$$

Let us denote by  $k: X \rightarrow [Y, Z]$  the map  $\phi(b)$ , for any  $b: X \otimes Y \rightarrow Z$ .  $b^*$  is continuous iff the following composite maps  $t_S$

$$\begin{array}{ccc} Z^* & \xrightarrow{t_S} & X^* \\ & \searrow b^* & \nearrow \lambda_S \\ & [X, Y^*] & \end{array}$$

are continuous, for any  $S \in \mathcal{O} Y^*$ , but, for the commutative diagram

$$\begin{array}{ccc} Z^* & \xrightarrow{t_S} & X^* \\ & \searrow p_S & \nearrow k^* \\ & [Y, Z]^* & \end{array}$$

and, by properties of the final topological topology,  $t_S$  is continuous (for any  $S$ ) iff  $k^*$  is continuous; hence

$$\begin{aligned} \phi: \mathcal{J}(A \circ B, C) &= \mathcal{J}((X \otimes Y, [X, Y^*]), (Z, Z^*)) \xrightarrow{\cong} \\ &\rightarrow \mathcal{J}(A, \{B, C\}) = \mathcal{J}((X, X^*), ([Y, Z], [Y, Z]^*)) \end{aligned}$$

and  $- \circ B \dashv \{B, -\}$  for any  $B \in \mathcal{J}$ .  $\square$

We end this section with two further comments on Theorem 2.4. The tensor  $\circ$  is not symmetric, for there exist non symmetric monoidal closed structures over  $Top$  (for example Greve structures [5]), induced by  $(\mathcal{J}, - \circ -, I, \{-, -\})$  (see Section 3).  $A \circ -$  preserves coproducts and coequalizers for any  $A \in \mathcal{J}$ , then by Freyd's Special Adjoint Functor Theorem, it has a right adjoint, hence the  $\mathcal{J}$ -structure is biclosed.

3. Consider  $Top$  as a concrete category over  $Top$  itself:

$$(Top, I: Top \rightarrow Top).$$

PROPOSITION 3.1 (see also [10]). *A monoidal closed structure on  $Top$ ,*

$$(Top, -\square-, \{*\}, <-, ->)$$

is equivalent to a functor over  $Top$ ,  $F: (Top, I) \rightarrow (\mathcal{J}, U)$  such that

- (i)  $F(\{*\}) = I$ ,
- (ii)  $F(X \otimes U F Y) \approx F(X) \circ F(Y)$ .

PROOF. By [8] (or [12]),

$$F(Y) = \langle Y, 2 \rangle, \quad \langle Y, Z \rangle \approx [Y, V F(Z)], \quad X \square Y = X \otimes U F(Y)$$

for  $X, Y, Z \in Top$ . From Proposition 2.3 of [10], (i) and (ii) are equivalent to monoidality of  $(Top, -\square-, \{*\})$ .  $\square$

Then, for any monoidal closed structure

$$(Top, -\square-, \{*\}, <-, ->),$$

the associated functor  $F$  is strict monoidal and the pair  $(\mathcal{J}, F: Top \rightarrow \mathcal{J})$  is a small-fibred, initial completion of  $(Top, I)$ . Further the monoidal structure  $(\mathcal{J}, - \circ -, I)$  restricted to the full subcategory  $F(Top) \approx Top$ , is just  $(Top, -\square-, \{*\})$ .

EXAMPLES. The canonical symmetric monoidal closed structure on  $Top$  (separate continuity and pointwise convergence) is determined by  $F(Y) = (Y, Y^*)$  where the topology of  $Y^*$  is generated by the family of all principal ultrafilters of open sets. Similarly the  $\mathcal{Q}$ -structures of Booth and Tiltonson [2] and Greve [5] are obtained associating to any  $Y \in Top$  the topological topology generated by the family

$$\{\langle b(X) \rangle, b \in Top(X, Y), X \in \mathcal{Q}\}$$

where  $\langle b(X) \rangle$  denotes the set of all open sets of  $Y$  containing  $b(X)$ .

PROPOSITION 3.2. A monoidal biclosed structure over  $Top$  is equivalent to a pair of functors over  $Top$ ,  $(F, G): Top \rightarrow \mathcal{J}$ , such that  $F$  satisfies (i) and (ii) of Proposition 3.1 and  $V.F \dashv V \circ G \circ: Top \rightarrow Top^o$ .

PROOF. If

$$(Top, -\square-, \{*\}, <-, ->, \langle\langle -, - \rangle\rangle)$$

is a monoidal biclosed structure,  $F$  and  $G$  are defined by

$$F(X) = (X, \langle X, 2 \rangle) \quad \text{and} \quad G(X) = (X, \langle\langle X, 2 \rangle\rangle).$$

Since, for  $X, Y, Z \in Top$ ,

$$Top^{\circ}(\langle X, Z \rangle, Y) \simeq Top(X, \langle \langle Y, Z \rangle \rangle),$$

then if  $Z = 2$ ,

$$Top^{\circ}(VF(X), Y) \simeq Top(X, V^{\circ}G^{\circ}(Y)),$$

so  $V.F \dashv V^{\circ}.G^{\circ}$ . Conversely,  $F$  determines a monoidal closed structure

$$(Top, -\square-, \{*\}, \langle -, - \rangle) \quad \text{with } \langle X, 2 \rangle = VF(X)$$

and  $G$  determines an adjunction  $-\bar{\square}Y \dashv \langle \langle Y, \cdot \rangle \rangle$  with

$$\langle \langle -, - \rangle \rangle \cdot Top^{\circ} \times Top \rightarrow Top, \quad \langle \langle Y, 2 \rangle \rangle = VG(Y).$$

$V.F \dashv V^{\circ}.G^{\circ}$  implies

$$\begin{aligned} Top(Y \square X, 2) &\simeq Top(Y, \langle X, 2 \rangle) \simeq Top(X, \langle \langle Y, 2 \rangle \rangle) \simeq \\ &\simeq Top(X \bar{\square} Y, 2). \end{aligned}$$

Since

$$Top(Y \square X, 2) \simeq Top(X \bar{\square} Y, 2)$$

and  $(2, 2)$  is a strong cogenerator of  $Top$ , then  $Y \bar{\square} X \simeq Y \square X$ , for any  $X$ , hence  $Y \bar{\square} - \dashv \langle \langle Y, \cdot \rangle \rangle$ , for any  $Y$ .  $\square$

Observe that a symmetric monoidal closed structure over  $Top$  is determined by a pair  $(F, G)$  with  $G = F$ .

## REFERENCES.

1. J. ADAMEK, H. HERRLICH & G. E. STRECKER, Least and largest initial completions, *Gen. Top. Appl.* 4 (1974), 125-142.
2. P. BOOTH & J. TILLOTSON, Monoidal closed, cartesian closed and convenient categories of topological spaces, *Pacific J. Math.* 88 (1980), 35-53.
3. J. ČINČURA, Tensor product in the category of topological spaces, *Comment. Math. Univ. Carol.* 20 (1979), 431-446.
4. S. EILENBERG & G. M. KELLY, Closed categories, *Proc. Conf. Cat. Algebra* (La Jolla), Springer, 1966, 421-562.
5. G. GREVE, How many monoidal closed structures are there in *Top*? *Arch. Math.* 34, Basel (1980), 538-539.
6. H. HERRLICH, Topological functors, *Gen. Top. Appl.* 4 (1974), 125-142.
7. H. HERRLICH, Cartesian closed topological categories, *Math. Colloq., Univ. Cape Town* 9 (1974), 1-16.
8. J. R. ISBELL, Function spaces and adjoints, *Math. Scand.* 36 (1975), 317-339.
9. G. M. KELLY, *Basic concepts of enriched category theory*, London Math. Soc., Lecture Notes 64, Cambridge Univ. Press, 1982.
10. M. C. PEDICCHIO, Closed structures on the category of topological spaces determined by systems of filters, *Bull. Austral. Math. Soc.* 28 (1983), 161-174.
11. H. E. PORST, Characterizations of MacNeille completions and topological functors, *Bull. Austral. Math. Soc.* 18 (1978), 201-210.
12. M. C. VIPERA, Endofuntori aggiunti di *Top*, *Quad. del gruppo di ricerca Alg. Geom.-Topologia*, Dip. Mat. Univ. Perugia, 1980.
13. O. WYLER, On the categories of general topology and topological algebra, *Arch. Math.* 22 (1971), 7-17.
14. O. WYLER, Top-categories and categorical topology, *Gen. Topology Appl.* 1 (1971), 17-28

Institute of Mathematics  
 Università degli Studi  
 I-34100 TRIESTE. ITALY