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# FROM FIBRE BUNDLES TO CATEGORIES

*by Andrée CHARLES EHRESMANN*

**ABSTRACT.**

The following paper has been written in 1980. It is a brief summary of Charles Ehresmann's main works, with a special emphasis on the way he proceeded from Algebraic Topology and Differential Geometry to Category Theory. It may be looked at as an introduction to the seven volumes of «Charles Ehresmann : Œuvres complètes et commentées» [0].

**INTRODUCTION.**

«Mathematics is very akin to Art; a mathematical theory not only must be rigourous, but it must also satisfy our mind in quest of simplicity, of harmony, of beauty... For the Platonists among the mathematicians, the motivation of their work lies in this search for the true structure in a given situation and in the study of such an abstract structure for itself... Mathematics is a never finished creation, which has not to justify its existence by the importance and the expanding number of its applications... It is the key for the understanding of the whole Universe».

This citation from [0 III : 94] * (written during our six months stay in Kansas, in 1966) reveals Charles' love for Mathematics, which explains that all his life has been devoted to them. Charles knew how to communicate his enthousiasm in (not very formal) lectures and long discussions. But his papers are often difficult to read, because of their concision and abstraction (motivations are scarcely given) and, for those on Category Theory, their non-standard notations. So it may be of some interest to point out the reasons which attracted Charles to categories and to show how naturally our ideas evolved from 1958 to 1979.

*) Such a reference denotes the paper numbered 94 in [0], Part III.

## 1. ALGEBRAIC TOPOLOGY AND DIFFERENTIAL GEOMETRY.

Charles' Thesis (in 1934), supervised by Elie Cartan for whom he had a great admiration, was written during a two years stay in Princeton. It is devoted to the topology of some homogeneous spaces, and it still remains a reference on *Grassmann manifolds*. Their homology and Poincaré groups are determined thanks to a powerful and then original method using cellular decompositions similar to those considered later on in the theory of CW-complexes [01:4]. The same method is also applied to more general manifolds in [01:8].

In [01:6], a *locally homogeneous space* $L$ is defined by «gluing together» spaces on which a local continuous group (= germ of group) acts. Conditions are obtained for $L$ to be locally equivalent to a homogeneous space. Among the numerous examples figure the locally projective spaces.

Charles introduced *fibre bundles* in 1941 (apart from Steenrod while the war had broken communications between France and the United States), and he defined locally trivial principal bundles and their associated bundles [01:15]. He gave Lifting of Homotopy Theorems and the exact sequence associated to a fibration ([01:14], written with his first student Feldbau, who was killed during the war; and [01:18, 23, 30]), special cases of which are already contained in his Thesis. He presented the general problem of restricting the structure group [01:16] and tackled it in several instances, in particular for the tangent bundle on a differentiable manifold [01:20, 17]; as a by-product, he proved that, if the Universe of Relativity is compact, its Euler-Poincaré characteristic must be zero (this result [01:17] brought him his first invitation to Rio de Janeiro in 1952).

This problem led him to the theory of *almost complex* (resp. *quaternionian*, resp. *hermitian) manifolds*, which is studied in [01:20, 29] and in the Theses of P. Libermann [39] and Wu Wen Tsun [55].

This same problem motivated the consideration of manifolds equipped with a completely integrable field of contact elements [01:17] whence the study of *foliated manifolds* ([01:19], written with Reeb whose Thesis [46] is a reference in this domain; [01:20, 23, 30]). Later on, Charles defined more general foliations and adapted the notion of holonomy and the stability theorems to locally simple foliations [OI: 45]. These results are still refined in a substantial paper [OII: 54], unfortunately ignored by most specialists, written during our stay in Montreal in 1961; it contains fine results on different notions of holonomy and unspreadings of a foliation, on stability, on complementary foliations and on foliations admitting a transversal riemannian structure. Foliations are also the main objects of several Theses supervised by Charles, e. g. Haefliger's [26].

Bundle theory has important applications in *Differential Geometry*; in [OI: 22] Charles defined the differentiable bundles, which are an appropriate setting for *infinitesimal connections* as it is shown in [OI: 22, 28] (cf. also Ver Eecke's Thesis [53]). But he was unsatisfied with notations for differentials, and this prompted him to introduce jets. Thanks to the *bundles of jets* [OI : 32, 34] (now folklore), he was able to give modern fundations to Differential Geometry, and he developed a beautiful theory of prolongations of differentiable manifolds, both in the holonomic case and in the non-holonomic and semi-holonomic case [OI: 32, 34-38, 40, 41, 43]. Infinitesimal structures (which generalize geometrical objects) and their covariants, G-structures and their associated Lie pseudogroups are introduced in [OI: 36, 38, 42], where he posed the local equivalence problem for G-structures, studied in P. Libermann's Thesis [39].

Holonomic and semi-holonomic jets (not yet fully exploited) simplify several problems involving differential systems (cp. with sprays); for instance, it is proven in [OI: 48, 51] that the largest group of transformations included in a Lie pseudogroup of finite type is a Lie group.

## 2. TOPOLOGICAL AND DIFFERENTIABLE CATEGORIES.

As soon as 1950, Charles used the notion of a *groupoid* (= a category in which every morphism is invertible). Though these groupoids were introduced by Brandt in 1926 [9], they are often called Ehresmann's groupoids; this is somewhat confusing since so many important notions really due to Charles are attributed to others or looked at as common knowledge.

The first groupoid considered by Charles was the groupoid $HH^{-1}$

of isomorphisms from fibre to fibre of a fibre bundle E, used in [0I: 28]; a connection on E determines a functor (called a representation) from the groupoid of paths of the base B of E toward $HH^{-1}$. Later on [0I: 44] he showed that $HH^{-1}$ is equipped with a topology making the maps domain $\alpha$, codomain $\beta$ and composition $\gamma$ continuous, and satisfying:

For any $x \in B$ there exists a continuous local section $\sigma$ of $\beta$ on a neighborhood V of $x$ in B such that $\sigma(y): x \to y$ for any $y \in V$.

The theory of groupoids equipped with such a topology (called *locally trivial groupoids*) is equivalent to that of principal bundles; moreover the spaces on which these groupoids act (continuously) correspond to the associated fibre bundles; analogous results are obtained in the differentiable case [0I: 44, 50] and are applied to groupoids of jets. In this setting prolongations of manifolds and of bundles are described very nicely since they reduce to the spaces on which acts a groupoid of jets [0I:40, 43, 44]. Higher order connection elements on the bundle E appear as the jets of local sections of $\beta$ giving a constant jet by composition with $\alpha$; the prolongation, curvature and torsion of a connection are then described in terms of semi-holonomic jets [0I: 46].

In fact, the composition of jets gives the first example of a (general) category used by Charles (in [0I: 34, 40, 41], though not very explicitely). The differentiable bundles $J^k(M, M')$ of $k$-jets from M to M', equipped with their «source» and «target» maps, where M and M' are $(r+k)$-manifolds) form a polyad (in Bénabou's sense [5]) on the category $Diff^r$ of $C^r$-manifolds *. More precisely they may be glued into a large $C^r$-manifold, formed by all the jets between germs of $C^{r+k}$-manifolds, so that the domain, codomain and composition of the category $J^k$ *of k-jets* become $r$-differentiable. This result, only published in [0II: 53] was indicated by Charles in his lectures already in 1955; it led him to the formal definition of topological and of $r$-differentiable categories (i.e., internal categories in the category *Top* of topological spaces, and in $Diff^r$), and of their actions [0I: 50].

* This polyad is considered in the Appendix of [0 III- 2].

The theory of *prolongations of differentiable categories* and of their actions (generalizing that of manifolds and bundles) is developed in Charles's later concise categorical papers on Differential Geometry [OI:78, 103, 105, 116]. The main tool is the *k-jet functor* $J^k(M,-): Diff^{r+k} \to Diff^r$ which preserves existing limits, so that it may be applied to a sketched structure (cf. Section 5) in $Diff^{r+k}$ to give rise to its prolongations.

A more abstract setting for prolongations is to be found in Pradines Thesis [45]. Jets are also used by Kock in his work on *Synthetic Differential Geometry*, where he proves that a jet may be defined as a «map» in a topos [33]. Most results on prolongations extend in this setting, as Charles had hoped when he heard Dubuc's lecture on Synthetic Differential Geometry in Amiens in 1978 *.

A general study of topological categories is done in [OII:92]. In particular *microtransitive categories* (which encompass locally trivial** categories) are well characterized; they possess a «quasi-uniform» structure (a notion deduced by localization from that of a uniformity, and englobing both topologies and uniformities). Local prolongations of topological categories are also dealt with.

Some results on actions of locally trivial categories are generalized in [OI:111] to actions of more general topological categories, and in [16] to partial actions of germs of categories (called there kernels of species of structures) which provide a good frame for some optimization problems.

## 3. LOCAL STRUCTURES AND THEIR OFFSPRINGS.

Topological, differentiable or analytic, fibred or foliated manifolds, locally homogeneous spaces,... are structures obtained by gluing together more elementary ones transported via charts, the change of charts pertaining to a given pseudogroup: the groupoid of homeomorphisms, of diffeomorphisms, of analytic isomorphisms, of isomorphisms between products equipped with the action of a group preserving or not the fibres, of affine

* Cf. also Kock [59].

** Microtransitive categories are locally trivial if their topology is nice [57].

isomorphisms. So is the philosophy which directed Charles toward «local structures», introduced in 1953 in order to unify all these examples.

In his former works on this subject [0I: 36, 39], a species of structures S (in Bourbaki's sense) is called local if it is equipped with an order (induction law) and a monotone map into the class $Top_0$ of topologies ordered by «is an open sub-space», satisfying the *sheaf axiom* :

There exists a join for any family $(s_i)_{i \in I}$ of structures such that the family of images $(p(s_i))$ admits a join and that

$$p(s_i \wedge s_j) = p(s_i) \wedge p(s_j) \quad \text{for any } i, j \in I.$$

Then the local automorphisms of S form a pseudogroup of transformations. Conversely any pseudogroup of transformations G gives rise to a local species, the structures of which are defined by complete ( = maximal) atlases having their changes of charts in G .

Charles soon realized (and said in his lectures from 1954 on) that the underlying topologies may be replaced by *paratopologies* ( = complete distributive lattices), since only their open sets are used. This idea of studying «topologies without points» (also considered by Nöbeling [43]) was developed by several of his students (Bénabou [4], Coppey [11], Tanré [51]). It is always actual, since paratopologies are now used, under the name of *locales*, in connection with Topos theory (Isbell [30], Johnstone [31], Joyal, ...).

In 1957, Charles published the essential paper [0II:47], from which many of his ten following years papers are directly issued; it improves his former results in several ways :

- species of structures over sets are axiomatized into species of structures over a category, for which extension theorems are given,

- pseudogroup of transformations are replaced by local categories,

- then local species of structures over a local category are defined and a general gluing together process is constructed.

We are going to emphasize these points and to show how these ideas reveal themselves fruitful and pioneering.

1. A *species of structures* S over a category C is given by an action of C on the set S; a Set-valued functor F (or presheaf over $C^{op}$) is

associated to it, as well as a *discrete opfibration* $p: \hat{S} \to C$ (called a hypermorphism functor in [0II:55] and often credited to Grothendieck). The equivalence of these three notions is proved in [0II:47].

Looking «upside-down» a Set-functor as a discrete opfibration has important applications (cf. Mac Lane's review [41]); for instance Set-functors may thus be composed (species of superstructures); the colimit of F may be computed as the set of components of $\hat{S}$, ... This paved the way to the theory of internal diagrams (defined as internal discrete opfibrations) or presheaves, and internal limits and colimits, to-day standard in Topos Theory (cf. e.g. Johnstone's book [32]).

2. If C is a subgroupoid of a category C', the *Extension Theorem* embeds S into a species S' over C'. The domain of the discrete opfibration associated to S' is the quotient of the comma category $C' \downarrow p$ by the «same component» equivalence.

The Set-functor associated to S' is the Kan-extension of F along the insertion $C' \hookrightarrow C$. But the Kan-extension, done at about the same time in a more general setting, takes the problem «upside-down». The way used by Charles led him to other kinds of generalization, $p$ being replaced by any functor. He so obtained different extension of functors theorems, which solve universal problems (cf. [0III:72, 77, 79] and the - unhappily almost hermetical - fifth chapter of his book «Catégories et structures»). In particular, one of these theorems gives back the *category of fractions* C/B, where B is a proper subcategory of C (= B admits a calculus of fractions, in a more recent terminology); nobody seems to know that C/B was first introduced in 1960, in the Appendix of [0II:55]).

3. A *local class* is a $\Lambda$-lattice with distributive joins of bounded families (cp. definition of a locale). A *local category* C is an internal category in the category of local classes, the morphisms preserving finite meets and arbitrary joins of bounded families; some regularity axioms are added. Transformation pseudogroups are examples of local groupoids.

A *local species of structures over* C is a species of structures whose associated opfibration $p: S \to C$ is a local functor. It is complete if $p$ satisfies the sheaf axiom (above). The *Local completion Theorem* universally embeds a local species into a complete one, over the same local category.

This theorem reduces to the Associated Sheaf Theorem for presheaves :

- over a local class, if C is a discrete category (only objects),

- over the Grothendieck topology in which the covers of E are the sets of $E_a < E$ admitting E for join if C is completely regular (otherwise this is not a Grothendieck topology).

But in 1957 sheaves had been mainly considered over topological spaces. In fact Barr's Theorem [3] proves that this result is general enough, since it asserts that any Grothendieck topos is a quotient of a topos of sheaves over a locale.

4. Finally the *Local complete Extension Theorem* consists in applying successively the Extension Theorem and the Local completion Theorem to a local species of structures S over a local groupoid C ; the resulting structures may also be defined as atlasses compatible with the local groupoid $\hat{S}$ (or yet as colimits of the ind-objects corresponding to these atlasses).

For instance, if $\hat{S}$ is the groupoid of $r$-diffeomorphisms between open subsets of a Banach space, we so obtain the groupoid of diffeomorphisms between $C^r$-manifolds modelled on a Banach space. The category $Diff^r$ is constructed by the same process, a morphism being looked at as a structure (via the Hom functor).

These results on local categories are generalized in a series of papers on *ordered categories* (summing 700 pages which are collected in [O : II ]) : less particular orders are considered on categories or on species of structures, to englobe the ordered category of categories, of topological spaces, of topological vector spaces [15], ... Local categories of local jets are associated to local functors, thus defining germs of structures. The complete extension Theorem is generalized to preinductive opfibrations and to local functors ; the hypotheses are choosen so that the theorem be of use in some Analysis problems (cf. Section 6). Order-completion theorems are given for ordered categories or groupoids ; an application is the construction of the complete holonomy groupoid of a foliation; the main tool is the notion of an *atlas in a category* (and its generalizations : the rockets and super-rockets).

## 4. INTERNAL CATEGORIES AND QUOTIENTS.

Charles came to categories from groupoids, and to groupoids from groups (via the theory of fibre bundles). So he «felt» a category as a (small) set equipped with a partially defined composition (as it is done in MacLane [40]) rather than as a (large) class of sets Hom(E, E') (as it is more usual). Hence it seemed natural to equip the set of *all* morphisms with some kind of structure, compatible with the domain, codomain and composition maps, $\alpha$, $\beta$, $\gamma$. The usefulness of topological, differentiable, local or ordered categories, suggested that it was fruitful to develop this idea.

In 1963, this idea led to the definition of a *P-structured category*, where $P : H \to \mathrm{Set}$ is a forgetful functor: it is a category C and an object S of H such that $P(S) = C$ and that:

(i) $\alpha$ and $\beta$ lift into H-morphisms from S to a substructure of S,

(ii) $\gamma$ lifts into a H-morphism from the pullback of $(\alpha, \beta)$ to S.

In modern terms, it is an internal category in H (the faithfulness of P implies that the unitarity and associativity axioms are satisfied).

Besides the preceding examples, Charles had already introduced two special cases of double categories (= internal categories in Cat):

- the *2-category of natural transformations* Nat, which he had defined much earlier ([0IV: 52], written during our stay in Buenos-Aires in 1959), to give an axiomatic definition of Bourbaki's scale of sets, the sets being replaced by endofunctors of Set equipped with a natural transformation from the identity (naturalized functors); this paper is one of the sources of Guitart's Thesis [24];

- the double category of commutative squares of a category, used in his construction of categories of fractions [0II: 55].

In 1963, these two examples were mixed to define the double category Q of squares of Nat called quintets in [0III: 58, 63]). Afterwards Bénabou considered general 2-categories and their double categories of squares. In our last paper [0 IV : 121], we prove that every double category «is» a subcategory of such a double category (and theorefore * of Q).

*Multiple categories* are defined by induction [0 III : 63]; they provide a good setting for the study of lax (or pseudo-) transformations, as we have shown in [0 IV : 119 - 121]; there, existence theorems for lax limits are not only generalized to higher dimensions, but also proved by a «structural» method much easier to handle that the one previously devised by Gray [21], Bourn [8], Street [49] in the case of 2-categories.

The general theory of structured categories is developed in a series of papers (cf. [0 III and IV].). In the first ones the definition is more cumbersome. Indeed the correct notion of a P-*substructure* is only cleared up in [0 III : 66]; now it would be called an initial lifting. The dual notion is that of a *quotient* structure. Seeking examples of quotients, Charles found independently the notion of a reflective subcategory (when P is a functor toward 2); this complicated some of his papers, in which the study of adjoint functors is transformed into that of reflections (via the elaborate construction of a comma category). Some quotient internal categories are described in [0 III : 66, 91].

But quotients are scarce, e. g. in Cat [0 III : 61, 80, 91]; hence the idea of defining *quasi-quotients* [0 III : 82, 100] (which are now also called semi-final lifts). Fine constructions of quasi-quotients and of colimits (as quasi-quotients of coproducts) are done in Cat and in categories of P-structured categories [0 III : 65, 91, 100].

Since explicit constructions are not always available (even in Cat) existence theorems of free objects, quasi-quotients and colimits are obtained in [0 III : 100] and [0 IV : 102, 108] for a functor $P : H \to C$. Instead of restricting the problem via a solution set condition (as in Freyd theorems) Charles enlarges it: he considers the product of all the possible solutions; this product is «too large» for existing in H, but it exists in the «similar category» $\hat{H}$ relative to a larger universe (this has a precise meaning for usual concrete categories); the solution is then a substructure of

* Cf. Comment 105.1 in [0 III].

this product, which actually is small enough to lie in H.

This is used to prove the existence of quasi-quotients and colimits of P-structured categories and to internalize the extension of functors theorems [0III: 90, 95, 96, 100, 113]. However it is necessary that P satisfies «good» properties, e.g.:

- creation or preservation and commutation of limits or colimits of some kinds,

- existence of quasi-quotients,

- existence of generated substructures: for an object S of H and a subset A of P(S), there exists a smallest substructure S' of S such that P(S') contains A («subgenerating functors») or, more strictly, equals A («spreading functors»).

So, classes of functors are pointed out, for instance:

- functors of a topological type, which are the spreading functors creating products (afterwards considered by Antoine [2], Herrlich [27], Wischnewsky [54], ... and called initialstructure functors or, yet, topological functors, not to be confused with internal functors in Top);

- functors of an algebraic type, which are the subgenerating functors creating limits (for which there exist quasi-quotients, cp. with the duality theorems), which now are called semi-topological (and hence, according to Tholen-Wischnewsky [52], are reflective restrictions of topological functors). Here algebraic englobes partial compositions, and not only global operations like in monadic functors.

More general classes of functors are presently studied, e.g. by many German categorists. This seems to justify the prevision:

«Mathematics research, I believe, will be less concerned with the study of a given functor; instead its aim will be to define classes of functors» [0III: 94].

A natural problem, initially inspired by the forgetful functor from $Diff^r$ (which does not create equalizers) is to extend a given functor P from H to C into a better one with the same codomain; some results are given in [0IV: 107]:

- P is extended into a minimal K-spreading functor, where K is a given subset of C;

- P is extended into a minimal reflecting some kinds of limits functor (thanks to the preceding result applied transfinitely) and into a maximal one. Recently this problem has been tackled by several authors (e. g., Adamek-Herrlich-Strecker [1], Hoffmann [28], ...* ).

Another method to define classes of functors is to give a general process of construction; such is one of the motivations for introducing:

## 5. SKETCHED STRUCTURES AND COMPLETIONS.

The idea of a category consists of the graph

$$\begin{tikzcd} 0 \arrow[r, "\iota"] & 1 \arrow[l, bend right, "\alpha"'] \arrow[l, bend left, "\beta"] & 2 \arrow[l, "\gamma"'] \end{tikzcd}$$

and of relations on the free category on it, in order to state the unitarity and associativity axioms. So if H is a (no more concrete) category, an *internal category* in H (called a generalized structured category in [0III: 93, 104]) is a morphism from this graph to H mapping *2* on the pullback of $(\alpha, \beta)$ and satisfying the given relations (which were always satisfied in the concrete case).

Whence the *sketch of categories* [0III: 93]: it is the full subcategory of the opposite of the simplicial category with objects *0, 1, 2, 3*, equipped with the two cones:

$$\begin{tikzcd} 0 & 1 \arrow[l, "\beta"'] \\ 1 \arrow[u, "\alpha"] & 2 \arrow[l, "\pi"] \arrow[u, "\pi'"'] \end{tikzcd} \qquad \begin{tikzcd} 1 & 2 \arrow[l, "\pi'"'] \\ 2 \arrow[u, "\pi"] & 3 \arrow[l] \arrow[u] \end{tikzcd}$$

which are to be mapped onto pullbacks.

Similarly, algebraic structures (in a wide sense) may be sketched by the data $\sigma$ of a neocategory $\Sigma$ (graph, equipped with a partial composition) and of cones and cocones on it. The corresponding internal structures (or models) in H are the functors $\Sigma \to H$ which map the cones to limits and the cocones to colimits. This is explained in [0III: 93], and in [0IV: 98, 106, 115, 117], where general theorems about such *sketched*

*) Cf. also [56], where these results are generalized.

structures are proved (mainly for sketches without cocones), and where examples are given (algebraic structures in Lawvere and Bénabou's sense, actions of a category and fibrations, n-categories, ...). More elaborate examples figure in a series of Theses in the seventies, e. g. topologies and categories with choices of limits in A. Burroni [10]. Sketches and their models are extensively studied in Lair's Thesis [36].

Charles introduced sketched structures in 1966; several more or less equivalent notions have been considered since this time: Gabriel-Ulmer's locally presentable categories [19] are categories of sketched structures, as are more general Diers' localizable categories [13] (which give interesting examples, like metric spaces and Banach spaces, thanks to a powerful method using «local colimits» * ). Models of lim-theories (M. Coste [12], who works in a more logical setting) are sketched structures, as are those of a theory on the corpus lim in Bénabou's sense [6]; but theories on other corpus (e. g. geometric theories in topoi, Joyal-Reyes), might be more general. In [18] Freyd-Kelly replace cones by cylinders.

The interest in taking a neocategory with cones for the sketch instead of a category with limits is to get a small enough presentation of the structures; in fact, one problem is to find minimal presentations (idea of a sketch [0III: 93] and Lair [35]). But for theoretical purposes, it is often easier to consider the associated prototype or type [0IV: 106, 114], which, in an up-to-date language, could be called the classifying category with limit-cones, and the classifying complete category.

This is one of the reasons which led to construct *completions* of a category C. In [0IV: 102], a completion $\hat{C}$ of C is obtained, with respect to some kinds L of limits, such that the functor $J : C \to \hat{C}$ map a given set of cones (void for the «free» completion) on limits and that $\hat{C}$ be *universal up to isomorphism* for some choice of limits; this solution is also universal up to *equivalence* in the category of L-continuous functors.

These results are refined in [0IV: 115], the choice of limits becoming a «relational choice»; then existence theorems cannot be applied,

* Sketched structures for sketches with both cones and cocones have been recently studied by Guitart and Lair [58].

and the construction is by transfinite induction (recently Street [50] has given a simpler construction.) In particular, the free completion with respect to filtered categories is the category of pro-objects of C (Duskin-Verdier [14]) very useful in Shape Theory.

As it is explained in [0IV: 107] these completion theorems and those of Isbell [29] and Lambek [37] are of a different nature, these later ones seeking a minimal dense embedding (while J is universal, not dense).

There was another reason yet for considering completions. Indeed let $\sigma$ be a sketch (without cocones) and $Set^{\sigma}$ its category of models in Set. If F is a model of $\sigma$ in H, there exists a functor $G: H^{op} \to Set^{\sigma}$ such that $G(I) = Hom(I, F\text{-})$ for any object I of H. Conversely, a functor $G: H^{op} \to Set^{\sigma}$ is so associated to a model iff it is pointwise representable [0IV: 93, 115]. When $\sigma$ is the sketch of categories and H admits pullbacks (resp. equalizers) G is pointwise representable iff $G(\text{-})(1)$ is [0IV: 113, 117], i.e. iff G is a Grothendieck's object of categories [22]; otherwise a Grothendieck's object of categories only defines an internal category in the completion of C by pullbacks (resp. equalizers). So completions are useful to compare both notions.

### 6. ENRICHMENTS.

Differential Geometry led Charles to internal categories and their actions. Analysis aroused our interest in enriched categories and H(-enriched) species of structures ( = functors to the concrete category H, called dominated species of structures in [0III: 58, 64]).

Indeed, the problem considered in [15] was the unification of various concepts of «generalized functions» such as Schwartz's distributions [48], Mikusinski's operators [42], Sato's hyperfunctions [47], ... and the definition of infinite-dimensional distributions. This is managed by dissecting the local definition of distributions which may be briefed into the sheaf of distributions is the sheafification of the presheaf of formal derivatives of continuous functions. It is done as follows in the case of distributions on R, with values in a locally convex space E (cf. [16]):

(i) Let U be a bounded open subset of R. The space C(U) of continuous maps $f: U \to E$ is equipped with the compact-open topology and with the partial action of the additive monoid of integers N :

$$(n, f) \mapsto f^{(n)} \text{ iff } f \text{ has an } n\text{-th derivative.}$$

(ii) Extending this partial action into a global one, we get the space FD(U) of formal derivatives, formed by the cosets $/n, f/$ for the equivalence relation on $N \times C(U)$ :

$$(n, f) \sim (m, g) \text{ iff there exist integers } n', m' \text{ and a continuous map } h \text{ such that } n + n' = m + m',\ f = h^{(n')},\ g = h^{(m')};$$

it is equipped with the final topology for all maps $f \mapsto /n, f/$ from C(U) and with the action of N :

$$(m, /n, f/) \mapsto /m + n, f/.$$

(iii) This defines a presheaf FD with values in the category of locally convex spaces equipped with an action of N. Its sheafification (in the same category) is the sheaf of E-valued distributions; it is constructed in [16] via the Local completion Theorem (cf. Section 2).

If N is replaced by the monoid of words $(v_1, \ldots, v_n)$ on the vectors of a Banach space B and $f^{(n)}$ by the partial derivative $\dfrac{\partial^n f}{\partial v_1 \ldots \partial v_n}$, we get the sheaf of E-*valued distributions on* B (to be compared with the prodistributions of Krée [34]).

So decomposed, this problem suggested the introduction of several notions :

1. *Partial actions of a (neo)category* C on a set S, called systems of structures [0III: 87]; they correspond to those functors $p: \hat{S} \to C$ for which there is at most one morphism of $\hat{S}$ with fixed domain and image (well-faithful functors). The enriched notion is called a H-system of structures, where the fibres are equipped with H-objects compatible with the partial action.

A theorem of extension of a H-system of structures into a H-species of structures is given in [0III: 87]; it generalizes the Kan extension Theorem (and was motivated by (ii) above). An internal version of this theorem is given in [0III: 89, 90]; for instance, it associates the action of a topological category to a germ of action (cf. Section 3).

2. *Actions of a category* C *on a category* A (called category of categories in [15]) and the equivalent notions [0III: 58, 63]: species of morphisms over C (= functors from C to Cat) and opfibrations with a cleavage considered in [0III: 70] via the *crossed product* of A and C (in analogy with the case of groups).

Following the lead of Lawvere [38] and Bénabou [7], general fibrations (introduced by Grothendieck in [23]) are now widely used, e.g. by Topos theorists. Indeed, they provide a good setting for the study of families of categories (or «large» categories) relative to a category C; since an internal category (or «small» category) in C may be replaced by a fibration (its associated Grothendieck's object of category, cf. 5), they so unify the treatment of «large» and «small» categories relative to C.

Directed by the cohomology of groups, Charles defined the crossed homomorphisms, whence the first class of cohomology of C relative to (its action on) A, and its first category of central cohomology [0III: 70] (to be compared with Giraud's results [20]). Then he constructed the groupoids of central cohomology of a complex K of species of morphisms over C with values in A; taking for K the simplicial resolution of C, he obtained the $n$-th cohomology of C to A. This was the first step toward a general study of non-abelian cohomology begun in [0III: 73, 74, 91], and [0IV: 102], though these almost forgotten papers might be much improved. In [0II: 75], the first cohomology of ordered species of structures is defined, with a view to applications to foliations.

3. *Categories of acting categories* [15], which are enriched species of structures in the category of discrete fibrations, or, more precisely, of discrete fibrations over the same category C (as in (iii) above). This last notion is equivalent to the notion of a pair of acting categories (cf. Chapter II of «Catégories et Structures»), which is the same as a distributor (or profunctor or, following Lawvere, a bimodule), a notion Bénabou considered to add adjoints to functors. When a distributor $D: A \dashrightarrow B$ is so looked at as «two actions» on S, it gives rise to an atlas in an appropriate category $\overline{D}$, containing $A^{op}$, B and S as the set of morphisms from A-ob-

jects to B-objects ; then the category associated to the composite $D' \otimes D$, is the pushout of the insertions $B \hookrightarrow \bar{D}$ and $B^{op} \hookrightarrow \bar{D}'$. Hence the compositions of distributors and of atlases are alike. Conversely the category of atlases of any category [0 II : 75] admits an embedding into the category of distributors *.

4. *Enriched categories.* Already in [0 II: 47] Charles used the fact that a category C is a species of structures over $C \times C^{op}$ (corresponding to the Set-functor Hom). To enrich this species in the concrete category H is to «naturally» equip the sets Hom(E, E') with H-objects; this is called a dominated category in [0 III : 77]. Examples in Analysis (categories enriched in the category of Banach spaces [15], the category of limit spaces [0 II: 92]) led to refine this notion and define a H-category, where H is a concrete cartesian category (strongly dominated category in [0 III: 104, 109]). But we did not study the non-concrete case.

When we discovered Eilenberg-Kelly's paper [17], we extensively used monoidal closed categories. For examples, we constructed *monoidal closed structures*: on categories of sketched structures, on categories of internal categories, on categories of topological ringoids, on the categories of all multiple categories and of $n$-fold categories [0 IV: 109, 115, 118-121].

In the Appendix of [0 IV : 120]**, it is proved that a V-category is the same as an internal category in V with a «discrete» object of objects, as soon as V is a cartesian category, admitting commuting coproducts (in Penon's sense [44]); it is another way to unify the treatment of «large» and «small» categories relative to a category.

## 7. CONCLUSION.

From his initial motivations in Differential Geometry which suggested the study of small categories, Charles was early led to the notions

* For a generalization of atlasses and their link with pro-objects, cf. Comment 199. 1 of [0 IV].

** A simpler (and more general) proof is given in Appendix of [0 III - 2].

of internal categories and internal presheaves (though initially in concrete categories) and more generally of sketched structures representing algebraic structures in a wide enough sense to include categories. He used enrichments only afterwards.

On the other hand, most of the categorists came to categories from Algebra or Topological Algebra, and they first thought of large categories and of universal algebra. Hence they initially studied Set-valued functors, enriched categories (e. g. abelian categories) and monadic functors; and only then the theory of Lawvere-Tierney led them toward internal functors.

Bénabou who was well aware of both developments, helped to bridge the gap between these tendancies; often we heard of fundamental notions from him. In fact, it is unfortunate that communications were not better developed during the sixties, due to several reasons: Charles was still esteemed as a Geometer more than a Categorist; and categorists soon took the bad habit of making confidential publications (that we did not receive). So, much time was wasted to discover anew well-known notions or to read or ignore paper with opposite notations. This difficulty only began to clear up in the early seventies, thanks to more personal contacts - for instance during the two Amiens Colloquiums [0IV: 142, 143] - and to the issuing of several Lecture Notes on categories.

I hope that the publication of the complete works of Charles, with numerous comments to link his papers with other and more recent developments, might help to assess his contributions to Category theory, and to point out several papers which should still be exploited and which concern up-to-date problems.

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