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A NOTE ON THE GENERALIZED REFLEXION OF GUITART AND LAIR

by G. M. KELLY *

By a weak reflexion of a locally-small category $\mathcal{A}$ onto a full subcategory $\mathcal{B}$ we mean the assigning to each $A \in \mathcal{A}$ of a small projective cone $\pi_A$, with vertex $A$ and with base in $\mathcal{B}$, such that $\mathcal{A}(\pi_A, B)$ is a colimit-cone in $\textbf{Set}$ for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$. When each $\pi_A$ has its base indexed by a discrete category, $\pi$ is a multi-reflexion in the sense of Diers [1]; it is an actual reflexion if moreover each of these discrete categories is 1.

For example, let $\mathcal{A}$ be the category of commutative rings. When $\mathcal{B}$ consists of local rings, a weak reflexion is given by taking for $\pi_A$ the cone of localizations $A \rightarrow A_p$ of $A$; its base is indexed by the ordered set of prime ideals $p$ of $A$. When $\mathcal{B}$ consists of the fields, a multi-reflexion is given by the discrete cone $A \rightarrow A/m$ where $m$ runs through the maximal ideals of $A$. When $\mathcal{B}$ consists of the rings $A$ with $2A = 0$, an actual reflexion is given by $A \rightarrow A/2A$.

Guitart and Lair study in [4] the existence of weak reflexions when $\mathcal{B}$ is given as follows. We have a set $\Theta = \{ \theta_B \}$ of projective cones

$$\theta_B : \Delta N_B \rightarrow T_B : \mathcal{L}_B \rightarrow \mathcal{A}$$

in $\mathcal{A}$, where $\Delta N_B$ denotes the functor constant at $N_B$; and $\mathcal{B}$ consists of those $A \in \mathcal{A}$ for which each $\mathcal{A}(\theta_B, A)$ is a colimit-cone in $\textbf{Set}$. They further restrict themselves to the special case in which each generator of each cone $\theta_B$ is an epimorphism in $\mathcal{A}$.

Each of the examples above is of this kind. For local rings there are two cones $\theta_1$ and $\theta_2$ in $\mathcal{A}$; $\theta_1$ is the pushout diagram of the two (epimorphic) maps

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while $\theta_2$ is the cone of vertex 0 over the empty diagram. For fields there are again two cones: $\theta_2$ as above and $\theta_3$ the discrete cone

$$\mathbb{Z} \to \mathbb{Z}[x] \to \mathbb{Z}(x).$$

For rings with $2A = 0$, there is a single cone $\theta_4$ whose base is indexed by 1, namely $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.

We suppose henceforth that $\mathbb{B}$ is given as above. We recall that, for a regular cardinal $\alpha$, an object $A \in \mathcal{A}$ is called $\alpha$-presentable if $\mathcal{A}(A, \cdot) : \mathcal{A} \to \text{Set}$ preserves $\alpha$-filtered colimits. Guitart and Lair sketch in [4] a rather complicated proof by transfinite induction of the following: There is a weak reflexion $\pi$ of $\mathcal{A}$ onto $\mathbb{B}$ if $\mathcal{A}$ is cocomplete, if each $L_B$ is small, and if there is a regular cardinal $\alpha$ such that each $N_B$ and each $T_B L$ is $\alpha$-presentable. Moreover $\pi$ can be taken to be a multi-reflexion if each $L_B$ is discrete.

The $\alpha$-presentability hypothesis is a strong one; hardly any objects are $\alpha$-presentable in the category of topological spaces or in the dual of an algebraic category. By analogy with the case where each $L_B$ is 1 - the «orthogonal subcategory problem» of [2] - this hypothesis should not be needed when the generators of the cones $\theta_B$ are epimorphic: at least if $\mathcal{A}$ is cowellpowered, which is not a grave restriction. By the same analogy, there should be a simple and direct proof in this case. We now verify that this is so, and that moreover the base of each cone $\pi_A$ may then be taken to be an ordered set.

We refer to [5] for the notion of strong monomorphism, and for the fact that epimorphisms and strong monomorphisms constitute a factorization system (see [2]) on $\mathcal{A}$ if $\mathcal{A}$ admits finite limits and all intersections of strong monomorphisms, or if $\mathcal{A}$ admits finite colimits and all cointersections of epimorphisms; certainly, therefore, if $\mathcal{A}$ is complete and well-powered, or cocomplete and cowellpowered.

**THEOREM 1.** Let the full subcategory $\mathbb{B}$ of the locally-small category $\mathcal{A}$
be determined as above by a set $\Theta$ (not necessarily small) of cones $\theta_\beta$ (not necessarily small), where each generator of each $\theta_\beta$ is epimorphic in $\mathfrak{A}$. Let epimorphisms and strong monomorphisms constitute a factorization system on $\mathfrak{A}$, and let $\mathfrak{A}$ be cowellpowered.

For each $A \in \mathfrak{A}$ denote by $S_A$ the small category whose objects are (a set of representatives of) the epimorphisms $p : A \to C$ in $\mathfrak{A}$ with domain $A$ and codomain in $\mathfrak{B}$, and whose maps $p \to p'$ are the maps $q : C \to C'$ with $qp = p'$; clearly $S_A$ is an ordered set. Let $d_A : S_A \to \mathfrak{B} \subset \mathfrak{A}$ be the projection functor sending $p : A \to C$ to $C$, and let

$$\pi_A : \Delta A \to d_A : S_A \to \mathfrak{A}$$

be the cone whose $p$-th component is $p$ itself.

Then an object $B$ of $\mathfrak{A}$ lies in $\mathfrak{B}$ if and only if each $\mathfrak{A}(\pi_A, B)$ is a colimit-cone in $\text{Set}$.

**Proof.** The essential observation is that $\mathfrak{B}$ is closed in $\mathfrak{A}$ under strong subobjects. To see this it suffices to consider a single cone $\theta : \Delta N \to T$ of $\Theta$, with epimorphic generators $\theta_i : N \to T_i$. Let $j : D \to B$ be a strong monomorphism in $\mathfrak{A}$, with $B \in \mathfrak{B}$. By the diagonal-fill-in property for epimorphisms and strong monomorphisms, the diagram

$$\begin{array}{ccc}
\mathfrak{A}(T_i, D) & \xrightarrow{\mathfrak{A}(\theta_i, D)} & \mathfrak{A}(N, D) \\
\downarrow & & \downarrow \\
\mathfrak{A}(T_i, j) & \xrightarrow{\mathfrak{A}(\theta_i, j)} & \mathfrak{A}(N, j) \\
\downarrow & & \downarrow \\
\mathfrak{A}(T_i, B) & \xrightarrow{\mathfrak{A}(\theta_i, B)} & \mathfrak{A}(N, B)
\end{array}$$

is a pullback in $\text{Set}$. Since colimits are universal in $\text{Set}$, and since $\mathfrak{A}(\theta_i, B)$ is a colimit-cone in $\text{Set}$, so is $\mathfrak{A}(\theta_i, D)$; so that $D \in \mathfrak{B}$.

It is now easy to see that $\mathfrak{A}(\pi_A, B)$ is a colimit-cone for $B \in \mathfrak{B}$.

For let $f : A \to B$, and let $f$ factorize as an epimorphism $p : A \to C$ followed by a strong monomorphism $j : C \to B$. Since $C \in \mathfrak{B}$ by the above, $p$ is a generator of $\pi_A$ through which $f$ factorizes. If $f$ also factorizes as $g p'$ through another generator $p' : A \to C'$ of $\pi_A$, the diagonal-fill-in property applied to $g p' = j p$ gives a $q : C' \to C$ with $qp' = p$ and $jq = g$. Hence $\mathfrak{A}(\pi_A, B)$ is a colimit-cone.
Conversely, if $\mathfrak{C}(\pi_A, B)$ is a colimit-cone for each $A$, then $\mathfrak{C}(\pi_B, B)$ is a colimit-cone; so that $1 : B \to B$ factorizes as $1 = j p$ for some epimorphism $p : B \to C$ with $C \in \mathfrak{B}$. But then the epimorphism $p$, being a coretraction, is invertible; and $B \in \mathfrak{B}$. □

**Theorem 2.** Add to the hypotheses of Theorem 1 the completeness of $\mathfrak{A}$, and suppose each cone $\theta_B$ to have a discrete base $\mathfrak{L}_B$. Then the restriction of $\pi_A$ to a suitable full subcategory of $S_A$ gives a multi-reflexion of $\mathfrak{A}$ onto $\mathfrak{B}$.

**Proof.** Since connected limits commute with discrete colimits in $\text{Set}$, we have $\mathfrak{B}$ closed in $\mathfrak{A}$ under connected limits. For each connected component $\delta$ of $S_A$, therefore, the limit of $d_A|\delta : \delta \to S_A \to \mathfrak{A}$ is an object $E_\delta$ of $\mathfrak{B}$; and the $p : A \to C$ of $S_A$ induce a map $r_\delta : A \to E_\delta$. Let this factorize as the epimorphism $s_\delta : A \to K_\delta$ followed by the strong monomorphism $k_\delta : K_\delta \to E_\delta$. Then $K_\delta \in \mathfrak{B}$, and $s_\delta$ is an object of $S_A$; clearly the greatest object of the ordered set $S_A$ which belongs to $\delta$. It is now evident that any $f : A \to B$ with $B \in \mathfrak{B}$ factorizes uniquely through some $s_\delta$, and through one only. □

We include for completeness the classical:

**Theorem 3.** If each $\mathfrak{L}_B = 1$ in Theorem 2, $\mathfrak{B}$ is closed under limits in $\mathfrak{A}$, and we get an actual reflexion $\rho_A$ of $\mathfrak{A}$ onto $\mathfrak{B}$, where $\rho_A$ is the epimorphic part of the factorization of $A \to \text{lim} d_A$ into an epimorphism followed by a strong monomorphism. □

We end by observing that the cowellpoweredness hypothesis of Theorem 1 does hold in the example to which Guitart and Lair give most prominence - that of the algebras for a mixed sketch $S$. By this is meant a small category $\mathfrak{S}$ in which are given a small set $\Phi = \{\phi_a\}$ of small projective cones and a small set $\Psi = \{\psi_\beta\}$ of small inductive cones; unlike Guitart and Lair, we do not ask the $\phi_a$ to be limit-cones nor the $\psi_\beta$ to be colimit-cones. The category $S\text{-}Alg$ of $S$-algebras is the full subcategory of $[\mathfrak{S}, \text{Set}]$ given by those $A : \mathfrak{S} \to \text{Set}$ for which each $A\phi_a$ is a limit-cone and each $A\psi_\beta$ is a colimit-cone. The sketch $S$ is projective when
the set $\Psi$ is empty; write $S_0$ for the projective sketch obtained from $S$ by discarding $\Psi$. It is classical that categories of the form $S_0$-$Alg$ are the locally presentable ones of Gabriel-Ulmer [3]; and that such a category is reflective in $[\hat{S}, \text{Set}]$, and is therefore complete and cocomplete.

Let $Z : S^{op} \to S_0$-$Alg$ be the composite of the Yoneda embedding $Y : S^{op} \to [\hat{S}, \text{Set}]$ and the reflexion $R : [\hat{S}, \text{Set}] \to S_0$-$Alg$. Clearly $B = S$-$Alg$ is the full subcategory of $\hat{A} = S_0$-$Alg$ consisting of those objects $A$ such that $\hat{A}(\cdot, A)$ sends the projective cone $\theta_\beta = Z \psi_\beta$ of $\hat{A}$ to a colimit-cone in $\text{Set}$ for each $\beta$. Note that each generator of $\theta_\beta$ is epimorphic if each generator of $\psi_\beta$ is monomorphic.

Finally, observe that $\hat{A}$ is cowellpowered by Satz 7.14 of [3], an account of which in English can be found in Section 8.6 of [6].

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