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ON THE STRUCTURE OF LOCALLY FINITE PURE SEMISIMPLE GROTHENDIECK CATEGORIES

by Daniel SIMSON

INTRODUCTION

Let $\mathcal{A}$ be a locally finitely presented Grothendieck category and denote by $fp(\mathcal{A})$ the full subcategory of $\mathcal{A}$ consisting of all finitely presented objects. We recall from [8] that $\mathcal{A}$ is pure semisimple if each of its objects is a direct sum of finitely presented objects. In [1] it is proved that if R is an Artin algebra then the category R-*Mod* of all left R-modules is pure semisimple iff R is of finite representation type, i.e. the category R-*mod* of all finitely generated left R-modules has only a finite number of isomorphism classes of indecomposable modules. Furthermore we know from [9] that if $\mathcal{A}$ is a pure semisimple Grothendieck category with the property:

(EA) *The endomorphism ring of any noetherian injective object in* $\mathcal{A}$ *is an Artin algebra*

and M is an object in $fp(\mathcal{A})$ then there exist an Artin algebra R of finite representation type and a pair of additive functors

$$(R\text{-}mod)^{op} \xrightarrow{T} fp(\mathcal{A}) \xrightarrow{F} (R\text{-}mod)^{op}$$

such that $FT = id$ and $M = TF(M)$. This shows that $fp(\mathcal{A})$ can be locally approximated by categories of the form $(R\text{-}mod)^{op}$ where R is an Artin algebra of finite representation type.

In the paper we establish a stronger result on the structure of the category $fp(\mathcal{A})$ under the assumption that $\mathcal{A}$ is locally finite Grothendieck and pure semisimple category with the property (EA). We show that there exist an inverse system of ring surjections $\{R_\alpha, f_{\alpha\beta}\}_{\alpha,\beta \in T}$ and full and exact embeddings $(R_\alpha\text{-}mod)^{op} \hookrightarrow fp(\mathcal{A})$, $\alpha \in T$, such that:

(i) R_α is an Artin algebra of finite representation type for all $\alpha \in T$

and the pseudocompact ring of the category $\mathfrak{A}$ (cf. [3]) is the limit of the system $\{R_\alpha, f_{\alpha\beta}\}_{\alpha,\beta\in T}$,

(ii) $fp(\mathfrak{A})$ is the union of all categories $(R_\alpha\text{-}mod)^{op}$, $\alpha\in T$,

(iii) For each $\alpha\in T$ there exists a $\beta\in T$, $\beta\geq\alpha$, such that for any $\gamma\geq\beta$ we have a factorization $f_{\alpha\beta}=f_{\alpha\gamma}g_{\gamma\beta}$ where $g_{\gamma\beta}: R_\beta\to R_\gamma$ is a homomorphism of left R_γ-modules.

The result was announced in [11].

1. NOTATION AND PRELIMINARIES.

If R is a ring we denote by R-*Mod* and *Mod*-R the categories of all left and all right R-modules, respectively. R-*mod* and *mod*-R will denote categories of finitely presented left R-modules and right R-modules.

We recall that a ring R is an Artin algebra if the center C of R is an Artinian ring and R is a finitely generated C-module.

Pure semisimple Grothendieck categories are investigated in [7-10, 2 and 4], where the reader is referred for details. The following result is taken from [9] (see also [1], Theorem A).

THEOREM 1.1. *A Grothendieck category* $\mathfrak{A}$ *is pure semisimple iff* $\mathfrak{A}$ *is locally Noetherian and for any sequence*

$$M_1 \xrightarrow{f_1} M_2 \longrightarrow \cdots \longrightarrow M_n \xrightarrow{f_n} M_{n+1} \longrightarrow \cdots$$

of monomorphisms between indecomposable Noetherian objects in $\mathfrak{A}$ *there exists an integer* n *such that* f_i *is an isomorphism for all* $i>n$.

In the next section we will need the following theorem proved in [8 and 9].

THEOREM 1.2. *Let* $\mathfrak{A}$ *be a pure semisimple Grothendieck category and suppose that* $\mathfrak{A}$ *has only a finite number of isomorphism classes of simple objects. If the endomorphism ring* S *of the minimal injective cogenerator in* $\mathfrak{A}$ *is an Artin algebra then* $\mathfrak{A}$ *is equivalent with the category Mod-*S *and* S *is of finite representation type.*

We also will need the following simple lemma.

LEMMA 1.3. *Let $\mathcal{C}$ be an abelian category such that each of its objects has finite length. Given a set $\mathcal{F}$ of simple objects in $\mathcal{C}$ we denote by $\mathcal{C}(\mathcal{F})$ the full subcategory of $\mathcal{C}$ consisting of all objects having composition series with factors from $\mathcal{F}$. Then:*

(a) $\mathcal{C}(\mathcal{F})$ is abelian and $\mathcal{F}$ is the set of representatives of isomorphism classes of simple objects in $\mathcal{C}(\mathcal{F})$.

(b) The embedding $\mathcal{C}(\mathcal{F}) \hookrightarrow \mathcal{C}$ is exact.

(c) Every object of $\mathcal{C}$ has a unique maximal subobject which belongs to $\mathcal{C}(\mathcal{F})$.

PROOF. [6], Theorem 1.2.

Throughout this paper we follow the terminology and notation of [5, 7 and 8]. In particular given an abelian category $\mathcal{C}$ we denote by $Lex\,\mathcal{C}$ the category of all left exact additive functors from $\mathcal{C}$ to the category of abelian groups. If X and Y are objects of $\mathcal{C}$ we denote by (X, Y) the abelian group of all morphisms from X into Y. $\mathcal{C}^{op}$ denotes the category opposite to $\mathcal{C}$.

2. THE MAIN RESULT.

We recall that a Grothendieck category $\mathcal{A}$ is locally finite if $\mathcal{A}$ has a set of generators of finite length.

Let $\mathcal{A}$ be a locally finite Grothendieck category and denote by Q the direct sum of representatives of a fixed family of isomorphism classes of indecomposable injective objects in $\mathcal{A}$. Fix a directed family of subobjects $L_t \subset Q$ of finite length such that $Q = \bigcup_t L_t$. The pseudocompact ring of the category $\mathcal{A}$ is the ring $R = End\,Q$ equipped with the linear topology defined by the ideals $(Q/L_t, Q)$ of R (see [3]). We also know from [3] that the functor

$$D: \mathcal{A} \to R\text{-}Mod \quad \text{given by} \quad D(-) = (-, Q)$$

defines a duality of $\mathcal{A}$ and the full subcategory R-PC of R-Mod consisting of all pseudocompact modules and continuous R-homomorphisms. The category of all left discrete R-modules will be denotde by R-Dis.

We are now able to prove the main result of the paper.

THEOREM 2.1. *Let* $\mathfrak{A}$ *be a locally finite pure semisimple Grothendieck category and suppose that the endomorphism ring of any injective object of finite length in* $\mathfrak{A}$ *is an Artin algebra. Let* R *be the pseudocompact ring of the category* $\mathfrak{A}$ *and* R' *the pseudocompact ring of the category* R*-Dis. Then there exist a directed set* T, *a directed family of full and exact abelian subcategories* $\mathcal{C}_a$ *of* $fp(\mathfrak{A})$, $a \in T$, *and inverse systems of rings* $\{R_a, f_{a\beta}\}_{a,\beta \in T}$ *and* $\{R'_a, f'_{a\beta}\}_{a,\beta \in T}$ *such that the following assertions hold:*

(a) For each $a \in T$ *the rings* R_a *and* R'_a *are Artin algebras of finite representation type and for each* $\beta \geq a$ *the ring homomorphisms*

$$f_{a\beta}: R_\beta \to R_a \quad \text{and} \quad f'_{a\beta}: R'_\beta \to R'_a$$

are surjective.

(b) $R = \varprojlim_{a \in T} R_a$ *and* $R' = \varprojlim_{a \in T} R'_a$. *The canonical maps* $f_a: R \to R_a$, $f'_a: R' \to R'_a$ *are surjective and the families* $\operatorname{Ker} f_a$, $a \in T$, *and* $\operatorname{Ker} f'_a$, $a \in T$, *form bases of neighborhoods of zero in* R *and* R' *respectively.*

(c) For each $a \in T$ *there exist natural equivalences*

$$\mathcal{C}_a \xrightarrow{D_a} (R_a\text{-mod})^{op} \xrightarrow{D'_a} R'_a\text{-mod}$$

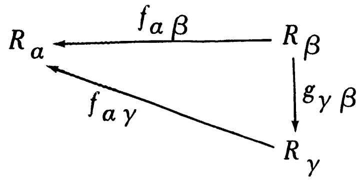
such that for any pair $\beta \geq a$ *the diagram*

$$\begin{array}{ccccc} \mathcal{C}_a & \xrightarrow{D_a} & (R_a\text{-mod})^{op} & \xrightarrow{D'_a} & R'_a\text{-mod} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}_\beta & \xrightarrow{D_\beta} & (R_\beta\text{-mod})^{op} & \xrightarrow{D'_\beta} & R'_\beta\text{-mod} \end{array}$$

is commutative where the right hand inclusions are induced by the surjections $f_{a\beta}$ *and* $f'_{a\beta}$ *respectively.*

(d) The category $fp(\mathfrak{A})$ *is the directed union of categories* $\mathcal{C}_a$, $a \in T$.

(e) For each $a \in T$ *there exists a* $\beta \in T$, $\beta \geq a$, *such that for any* $\gamma \geq \beta$ *we have a factorization as indicated below where* $g_{\gamma\beta}$ *is a homomorphism of left* R*-modules:*



PROOF. Let S_i, $i \in I$, be a complete list of representatives of isomorphism classes of simple objects in $\mathfrak{A}$ and let T be the directed set of all finite subsets of I. Let

$$Q = \oplus_{i \in I} \hat{S}_i \quad \text{and} \quad Q_\alpha = \oplus_{i \in \alpha} \hat{S}_i, \ \alpha \in T,$$

where $\hat{S}_i$ is the injective envelope of S_i in $\mathfrak{A}$. Note that each $\hat{S}_i$ has finite length since $\mathfrak{A}$ is pure semisimple. Now applying the construction in the Lemma 1.3 to

$$\mathcal{C} = fp(\mathfrak{A}) \quad \text{and} \quad \mathcal{F} = \mathcal{F}_\alpha = \{ S_i, \ i \in \alpha \}, \ \alpha \in T,$$

we define an abelian exact subcategory $\mathcal{C}_\alpha$ of $fp(\mathfrak{A})$ putting

$$\mathcal{C}_\alpha = fp(\mathfrak{A})(\mathcal{F}_\alpha), \quad \alpha \in T.$$

It is clear that $\mathcal{C}_\alpha \subset \mathcal{C}_\beta$ whenever $\alpha \leq \beta$, and each Q_α is an object of a certain $\mathcal{C}_\beta$. We know from Lemma 1.3 that for every pair $\alpha, \beta \in T$ there exists a unique maximal subobject Q_α^β of Q_α which is an object in $\mathcal{C}_\beta$. It is easy to check that Q_α^α is an injective cogenerator in $\mathcal{C}_\alpha$ for any $\alpha \in T$. Now given $\alpha \in T$ we put $R_\alpha = EndQ_\alpha^\alpha$. It is easy to see that the natural inclusions $Q_\alpha^\alpha \subset Q_\beta^\beta \subset Q$, $\alpha \geq \beta$, induce isomorphisms

$$t_\alpha : (Q_\alpha^\alpha, Q_\alpha^\alpha) \to (Q_\alpha^\alpha, Q) \quad \text{and} \quad t_{\alpha\beta} : (Q_\alpha^\alpha, Q_\alpha^\alpha) \to (Q_\alpha^\alpha, Q_\beta^\beta).$$

We denote by $f_\alpha : R \to R_\alpha$ and $f_{\alpha\beta} : R_\beta \to R_\alpha$, $\beta \geq \alpha$, the composed maps

$$(Q, Q) \longrightarrow (Q_\alpha^\alpha, Q) \xrightarrow{t_\alpha^{-1}} (Q_\alpha^\alpha, Q_\alpha^\alpha),$$

$$(Q_\beta^\beta, Q_\beta^\beta) \longrightarrow (Q_\alpha^\alpha, Q_\beta^\beta) \xrightarrow{t_{\alpha\beta}^{-1}} (Q_\alpha^\alpha, Q_\alpha^\alpha).$$

It is obvious that f_α and $f_{\alpha\beta}$ are surjective ring homomorphisms such that

$$f_\alpha f_{\alpha\beta} = f_\beta \quad \text{and} \quad f_{\alpha\beta} f_{\beta\gamma} = f_{\alpha\gamma} \quad \text{provided} \quad \gamma \geq \beta \geq \alpha.$$

We observe that for each $\alpha \in T$ there exists $\beta \geq \alpha$, $\beta \in T$, such that

$Q_\alpha \subset Q_\beta^\beta \subset Q_\beta$. Then we have

$$Q = \bigcup_{\alpha \in T} Q_\alpha = \bigcup_{\beta \in T} Q_\beta^\beta$$

and therefore we get

$$R = (Q,Q) = \varprojlim_{\beta \in T} (Q_\beta^\beta, Q) = \varprojlim_{\beta \in T} (Q_\beta^\beta, Q_\beta^\beta) = \varprojlim_{\beta \in T} R_\beta .$$

Then we have proved the part of (b) related to the ring R.

Now we are going to prove that each R_α, $\alpha \in T$, is an Artin algebra of finite representation type. For this purpose we fix $\alpha \in T$ and consider the category $\mathfrak{A}_\alpha = Lex\,\mathcal{C}_\alpha^{op}$. We know from [3] that $\mathfrak{A}_\alpha$ is a locally finite Grothendieck category. $fp(\mathfrak{A}_\alpha) \approx \mathcal{C}_\alpha$ and the natural embedding $\mathcal{C}_\alpha \to \mathfrak{A}_\alpha$ is exact. It follows that the finite set S_i, $i \in \alpha$, is a set of representatives of isomorphism classes of simple objects in $\mathfrak{A}_\alpha$. Since $\mathfrak{A}$ is pure semisimple then by Theorem 1.1 the category $\mathfrak{A}_\alpha$ is pure semisimple, too. Hence the injective cogenerator Q_α^α in $\mathcal{C}_\alpha$ is an injective cogenerator in $\mathfrak{A}_\alpha$. Furthermore the inclusion $Q_\alpha^\alpha \subset Q_\alpha$ induces the ring surjection

$$(Q_\alpha, Q_\alpha) \longrightarrow (Q_\alpha^\alpha, Q_\alpha^\alpha) = R_\alpha .$$

Since by our assumption (Q_α, Q_α) is an Artin algebra then so is R_α. Then by Theorem 1.2 the category $\mathfrak{A}_\alpha$ is equivalent with $Mod\text{-}R_\alpha$ and R_α is of finite representation type, as required.

Now we consider the duality

$$D: \mathfrak{A} \to R\text{-}PC \quad \text{given by} \quad D(-) = (-,Q).$$

Since Q_α^α is an injective cogenerator in $\mathcal{C}_\alpha$, $\alpha \in T$, and for any object N in $\mathcal{C}_\alpha$ we have $(N, Q_\alpha^\alpha) = (N,Q)$ then the restriction of the functor D to the category $\mathcal{C}_\alpha$ defines an equivalence $D_\alpha : \mathcal{C}_\alpha \to (R_\alpha\text{-}mod)^{op}$ (see [9], Proposition 2.3).

In order to define the inverse system $\{R'_\alpha, f'_{\alpha\beta}\}_{\alpha,\beta \in T}$ we consider the subcatgeory $R\text{-}Dis$ of $R\text{-}PC$. We know from [3] that $R\text{-}Dis$ is a locally finite Grothendieck category, the objects $S'_i = D(S_i)$, $i \in I$, form a complete list of representatives of isomorphism classes of simple objects in $R\text{-}Dis$ and

$$D(fp(\mathfrak{A})) = fp(R\text{-}Dis) = \bigcup_{\alpha \in T} R_\alpha\text{-}mod.$$

We consider the following injective objects in R-Dis :

$$E = \oplus_{i \in I} \hat{S}_i' \quad \text{and} \quad E_\alpha = \oplus_{i \in \alpha} \hat{S}_i', \; \alpha \in T,$$

and for each pair $\alpha, \beta \in T$ we define a left module E_α^β over the ring $R_\beta = R/Ker f_\beta$ by formula

$$E_\alpha^\beta = \{ x \in E_\alpha, (Ker f_\beta) x = 0 \}.$$

Since R_β is of finite representation type then E_α^β is a direct sum of modules of finite length. Then E_α^β has finite length because the socle of E_α^β is finite. Furthermore, we observe that for each $\alpha \in T$ the module E_α^α is an injective cogenerator in R_α-mod. Given $\alpha \in T$ we put $R_\alpha' = End E_\alpha^\alpha$. Similarly as in the first part of the proof we define ring surjections

$$f_\alpha' : R' \to R_\alpha' \quad \text{and} \quad f_{\alpha\beta}' : R_\beta' \to R_\alpha', \; \beta \geq \alpha, \quad \text{where } R' = End E.$$

Furthermore the duality

$$D' : R\text{-}Dis \to R'\text{-}PC \quad \text{given by} \quad D'(-) = (-, E)$$

yields an equivalence $D_\alpha' : (R_\alpha\text{-}mod)^{op} \to R_\alpha'\text{-}mod$ for each $\alpha \in T$. Now it is easy to verify the statements (a)-(d).

In order to prove the last statement we consider the inverse system $\{R_\alpha, f_{\alpha\beta}\}_{\alpha, \beta \in T}$ in the category R-PC and observe that under D it is the image of the direct system Q_α^α, $\alpha \in T$, in the category $fp(\mathfrak{A})$. Since $\mathfrak{A}$ is pure semisimple then by [7], Theorems 6.3, 5.4, 3.16, the system Q_α^α, $\alpha \in T$, is factorizable in the sense of [7], Definition 3.1. Hence (e) follows and the theorem is proved.

REMARK. We don't know if the statements (a)-(e) in Theorem 2.1 are sufficient for the pure semisimplicity of the category $\mathfrak{A}$.

Now we give a simple example to illustrate our main theorem.

EXAMPLE 2.2. Let K be a field and denote by $\mathfrak{R}$ the category of K-representations of the infinite quiver

$$1 \to 2 \to \dots \to n \to n+1 \to \dots$$

(see [2]). Let $\mathfrak{A}$ be the full subcategory of $\mathfrak{R}$ consisting of directed unions of objects of finite length. We know from [2] that $\mathfrak{A}$ is pure semisimple and every indecomposable object in $\mathfrak{A}$ has the form

$$I_{nm}: 0 \to \dots \to 0 \to \underset{n}{K} \xrightarrow{id} K \xrightarrow{id} \dots \xrightarrow{id} \underset{m}{K} \to 0 \to 0 \to \dots, \quad n \leq m < \infty.$$

Let $\mathcal{F}_n$ be the set of simple objects $I_{11}, \dots, I_{nn}$. Then $\mathcal{C}_n = fp(\mathfrak{A})(\mathcal{F}_n)$ consists of all representations of the form

$$V_1 \to V_2 \to \dots \to V_n \to 0 \to 0 \to \dots$$

with $\dim V_j < \infty$ for $j = 1, 2, \dots, n$. Observe that $fp(\mathfrak{A})$ is the union of all $\mathcal{C}_n$, the object

$$Q_n = I_{11} \oplus I_{12} \oplus \dots \oplus I_{1n}$$

is the minimal injective cogenerator in the category $\mathcal{C}_n$, the ring $R_n = EndQ_n$ is the ring of $n \times n$ lower triangular matrices with entries in the field K and we have $\mathcal{C}_n \approx (R_n\text{-}mod)^{op} \approx mod\text{-}R_n$.

We conclude this paper with some remarks concerning the pure semi simple property of comodule categories.

Let C be a coalgebra over a field K and denote by C-$Comod$ the category of left C-comodules (see [12]). Let C_j, $j \in J$, be a directed set of finite-dimensional subcoalgebras of C such that $C = \bigcup_{j \in J} C_j$. Then the dual K-algebra C^* with the linear topology defined by the two-sided ideals $Hom_K(C/C_j, K)$ in C^*, $j \in J$, is pseudocompact and $C^* = \varprojlim_{j \in J} C_j^*$ (see [13]). It is clear that C-$Comod$ is a locally finite Grothendieck category. Furthermore there exist natural equivalences

$$C\text{-}Comod \approx C^*\text{-}Rat \approx C^*\text{-}Dis$$

where C^*-Rat is the category of left rational C^*-modules (cf. [12] and [13]).

If R is the pseudocompact ring of the category C-$Comod$ and R' is the pseudocompact ring of the category R-Dis then we know from [3] that there is an equivalence $C\text{-}Comod \approx R'\text{-}Dis$. The continuous K-dual space C' to R' has a natural K-coalgebra structure such that $C'^* \approx R'$ and C' is a minimal injective cogenerator in $C'\text{-}Comod \approx C\text{-}Comod$. It is clear that these properties determine C' uniquely up to a coalgebra isomorphism and we call any such coalgebra C' basic.

Now suppose C is a basic coalgebra and C-*Comod* is pure semisimple. Then by Theorem 2.1 there exists a directed set of finite-dimensional subcoalgebras L_α of C, $\alpha \in T$, such that $C = \bigcup_{\alpha \in T} L_\alpha$, the dual K-algebra L_α^* is of finite representation type for all $\alpha \in T$ and

$$C^* = \varprojlim_{\alpha \in T} L_\alpha^* .$$

It would be interesting to have a characterization of the coalgebras C with C-*Comod* pure semisimple. In the cocommutative case such a characterization is given in [7], Theorem 7.1.

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