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Y. L. WONG

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CHAIN HOMOTOPY PULLBACKS AND PUSHOUTS

by Y. L. WONG

0. INTRODUCTION.

Homotopy pullbacks and pushouts in the category of topological spaces are introduced in [2] to characterize most of the standard constructions in topology and to deal with problems in homotopy theory; see for example [4] and [5]. Mather's Cube Theorems [2] so to speak are of fundamental importance within homotopy theory. In this paper, I prove an analogous result for chain complexes. Because of the exact dual character of chain fibration and chain cofibration, the two dual cube theorems also hold in this case. Chain homotopy pullbacks and pushouts are introduced in Section 1. In Section 2, the concept of fibration and cofibration, in this case I shall call chain fibration and chain cofibration, are recalled. Owing to a theorem of K.H. Kamps [1], one is able to glue chain fibrations or chain cofibrations together so that their properties actually resemble that of Dold's fibration and cofibration in topology. In Section 3, the Cube Theorems are established by a similar argument to that of Mather [2].

1. CHAIN HOMOTOPY PULLBACKS AND PUSHOUTS.

Let $f, g: X \to Y$ be chain maps and S, T be chain homotopies, from f to g. We shall say S *is equivalent to* T, in symbol $S \sim T$, if for each n, there is a group homomorphism $\theta_n: X_n \to Y_{n+2}$ such that

$$S_n - T_n = \partial_{n+2}\theta_n - \theta_{n-1}\partial_n .$$

One verifies that $\sim$ is an equivalence relation. We use $[T]$ to denote the equivalence class of chain homotopies with representative T. If T and S are chain homotopies from the chain map f to the chain map g, then addition of homotopies is given by $[T]+[S] = [T+S]$. It can be verified that this is well-defined. Moreover, by direct calculation, we have the following

Interchange Law: Let f, $g: X \to Y$ and p, $q: Y \to Z$ be chain maps. If T is a chain homotopy from f to g and S is a chain homotopy from p to q, then $[Sf]+[qT]=[pT]+[Sg]$.

A square

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "q"'] & B \arrow[d, "f"] \\ C \arrow[r, "p"'] \arrow[ur, phantom, "T"] & D \end{tikzcd}$$

of chain maps with a chain homotopy T from fg to pq is called *a chain homotopy commutative square*. In view of the interchange law above, such squares with a class of chain homotopy filling constitute a Special double category with connection CC introduced in [3]. To make the notation easier, we shall use the small case letter of the chain homotopy to denote such a square.

Let

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "q"'] & B \arrow[d, "f"] \\ C \arrow[r, "p"'] \arrow[ur, phantom, "T"] & D \end{tikzcd}$$

be a chain commutative square. It is called a *chain homotopy pullback* if given any chain maps $u: X \to B$, $v: X \to C$ and chain homotopy H from fu to pv, there exists a chain map $\theta: X \to A$ and chain homotopies U from u to $g\theta$, V from $q\theta$ to v such that

$$[pV]+[T\theta]+[fU]=[H].$$

Moreover if θ' is another such chain map and U', V' are such chain homotopies, then there exists a chain homotopy S from θ to θ' such that

$$[gS]+[U]=[U'] \quad \text{and} \quad [V']+[qS]=[V].$$

Chain homotopy pushout is defined dually.

Next, we shall describe the standard construction of chain homotopoy pullback and pushout.

The standard chain homotopy pullback of the chain maps

$$Y \xrightarrow{g} Z \xleftarrow{f} X$$

is the following chain homotopy commutative square

$$\begin{tikzcd} B \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] \arrow[ur, phantom, "T"] & Z \end{tikzcd}$$

where $B_n = X_n \oplus Z_{n+1} \oplus Z_n \oplus Z_{n+1} \oplus Y_n$ with boundary homomorphism given by

$$\partial(x, z_1, z, z_2, y) = (\partial x, -f(x) - \partial z_1 + z, \partial z, -g(x) - \partial z_2 + z, \partial y).$$

p_1 and p_2 are the respective projection chain maps. The chain homotopy T is given by

$$T(x, z_1, z, z_2, y) = z_2 - z_1.$$

Dually the standard chain homotopy pushout of the chain maps

$$Y \xleftarrow{g} Z \xrightarrow{f} X$$

is the chain homotopy commutative square

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i_1"] \\ Y \arrow[r, "i_2"'] \arrow[ur, phantom, "T"] & D \end{tikzcd}$$

where $D_n = X_n \oplus Z_{n-1} \oplus Z_n \oplus Z_{n-1} \oplus Y_n$ with boundary homomorphism given by

$$\partial(x, z_1, z, z_2, y) = (\partial x + f(z_1), -\partial z_1, \partial z - z_1 - z_2, -\partial z_2, \partial y + g(z_2)).$$

i_1 and i_2 are the respective inclusion chain maps. The chain homotopy T is given by $T(z) = (0, z, 0, -z, 0)$. We omit the verification of the definitions, which will be seen in later discussion.

On the other hand, given a chain homotopy commutative square

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] \arrow[ur, phantom, "T'"] & Z \end{tikzcd}$$

it is a chain homotopy pullback iff B' is chain homotopy equivalent to B. Similarly, this is true for the dual case.

2. CHAIN FIBRATION AND COFIBRATION.

We just follow the usual definition in topology. A chain map $f: X \to Y$ is called a *Chain fibration* if for any chain maps $g_0: Z \to X$, $h_0, h_1: Z \to Y$ and chain homotopy T from h_0 to h_1 such that $f g_0 = h_0$, there exist chain map $g_1: Z \to X$ and chain homotopy S from g_0 to g_1 such that $fS = T$ and $fg_1 = h_1$. Dually, one defines *chain cofibration.* These definitions are equivalent to that given by Kamps in [1].

We recall that in [1], $f: X \to Y$ is a fibration (cofibration) iff, for each n, $f_n: X_n \to Y_n$ is a retraction (section). Here we state a further result. Its proof is actually categorical.

PROPOSITION 2.1. *Let*

$$\begin{tikzcd} P \arrow[r, dash, "p"] \arrow[d, dash, "g"'] & X \arrow[d, dash, "f"] \\ Q \arrow[r, dash, "q"'] & Y \end{tikzcd}$$

be a chain pullback (pushout) of chain complexes. If f is a chain fibration (g is a chain cofibration), then this square with the zero chain homotopy is a chain homotopy pullback (pushout).

Now, suppose $f: X \to Y$ is a chain map. Consider the chain homotopy pullback

$$\begin{tikzcd} P_f \arrow[r, "p"] \arrow[d, shift left, "t"] & Y \arrow[d, equal] \\ X \arrow[u, shift left, "r"] \arrow[r, "f"'] \arrow[ur, phantom, "T"] & Y \end{tikzcd}$$

where $(P_f)_n = X_n \oplus Y_{n+1} \oplus Y_n$ with boundary homomorphism given by

$$\partial(x, y_1, y) = (\partial x, -f(x) - \partial y_1 + y, \partial y).$$

p and t are the respective projection chain maps. The chain homotopy T is given by $T(x, y_1, y) = y_1$. It is clear that p is a chain fibration. Let $r: X \to P_f$ be the chain map given by $r(x) = (x, f(x), 0)$. One then verifies t is a chain homotopy equivalence with chain homotopy inverse r. Moreover, $f = pr$. Therefore, we are able to factor a chain map into a composite of a chain fibration and a chain homotopy equivalence.

Dually, consider the chain homotopy pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "T"] & Y \arrow[d, shift right, "i"'] \\ X \arrow[r, "j"'] & Z_f \arrow[u, shift right, "\bar{r}"'] \end{tikzcd}$$

where $(Z_f)_n = Y_n \oplus X_{n-1} \oplus X_n$ with boundary homomorphism given by

$$\partial(y, x_1, x) = (\partial y + f(x), -\partial x_1, \partial x - x_1).$$

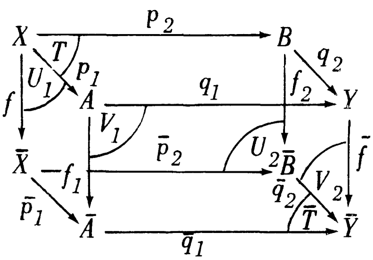
i and j are the respective inclusion chain maps. The chain homotopy T is given by $T(x) = (0, x, 0)$. Let $\bar{r}: Z \to Y$ be the chain map given by $\bar{r}(y, x_1, x) = y + f(x)$. In this case, j is a chain cofibration and i is chain homotopy equivalence with chain homotopy inverse $\bar{r}$. Also $f = \bar{r}j$. We state the results as follows:

PROPOSITION 2.2. *Any chain map f can be factored as pr or $\bar{r}j$ in which r, $\bar{r}$ are chain homotopy equivalences, p is a chain fibration and j is a chain cofibration.*

We remark also that chain fibration and chain cofibration are preserved by chain pullback and pushout respectively.

3. CUBE THEOREMS.

A cube



(FIG. 1)

with chain homotopy commutative faces t, $\bar{t}$, u_1, u_2, v_1, v_2 is said to be *chain homotopy commutative* if

$$[\bar{f}T] + [V_1 p_1] + [\bar{q}_1 U_1] = [\bar{T} f] + [V_2 p_2] + [\bar{q}_2 U_2].$$

THEOREM 1. *Refer to the chain homotopy commutative cube in Figure 1. If t, $\bar{t}$ are chain homotopy pushouts and u_1, u_2 are chain homotopy pull-*

backs, then v_1, v_2 *are chain homotopy pullbacks.*

PROOF. Following Mather's argument, it is just sufficient to prove the theorem in the case where f_1, f_2 are chain fibrations, u_2 is a chain pullback and $U_1 = 0$. Actually such reductions are valid in any special double category with connection [6]. Let X' be the chain pullback of $\bar{p}_1$ and f_1 so that we get a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "p_1"] \arrow[dr, "\theta"] \arrow[dd, "f"'] & & A \arrow[dd, "f_1"] \\
& X' \arrow[ur, "p'"'] \arrow[dl, "f'"] & \\
\bar{X} \arrow[rr, "\bar{p}_1"'] & & \bar{A}
\end{tikzcd}$$

in which θ is a chain homotopy equivalence. Let D' be the chain complex given by $D'_n = A_n \oplus X_{n-1} \oplus X'_n \oplus X_{n-1} \oplus B_n$ and

$$\partial(a, x_1, x', x_2, b) = (\partial a + p_1(x_1), -\partial x_1, \partial x' - \theta(x_1) - \theta(x_2), -\partial x_2, \partial b - p_2(x_2)).$$

If $\bar{D}$ is the standard chain homotopy pushout of $\bar{p}_1$ and $\bar{p}_2$, then we have the following homotopy commutative cube :

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, "S'"'] \arrow[dd] & & B \arrow[dr, "i_2"] \arrow[dd] & \\
& A \arrow[rr, "i_1"] \arrow[dd] & & D' \arrow[dd, "h"] \\
\bar{X} \arrow[rr] \arrow[dr] & & \bar{B} \arrow[dr, "\bar{i}_2"] \arrow[dr, phantom, "\bar{S}"'] & \\
& \bar{A} \arrow[rr, "\bar{i}_1"'] & & \bar{D}
\end{tikzcd}$$

in which i_1, i_2, $\bar{i}_1$, $\bar{i}_2$ are the respective inclusion chain maps and the chain map h is given by

$$h(a, x_1, x', x_2, b) = (f_1(a), f(x_1), f'(x'), f(x_2), f_2(b)).$$

Since all the f's are chain fibrations, Kamps's result implies that h is also a chain fibration. It is also clear the front and the right faces of this cube are chain pullbacks so that they are actually chain homotopy pullbacks. Now we shall show the top face s' is a chain homotopy pushout. At this point it is sufficient to show that D' is chain homotopy equivalent to the standard chain homotopy pushout D of p_1 and p_2. Here, we may assume the chain homotopy equivalence $\theta: X \to X'$ is a strong chain homotopy equivalence with inverse $\bar{\theta}$. This means there are chain homotopies K from $\bar{\theta}\theta$ to 1_X and K from $\theta\bar{\theta}$ to $1_{X'}$, such that

$$[\theta K] = [\bar{K}\theta] \quad \text{and} \quad [K\bar{\theta}] = [\bar{\theta}\bar{K}].$$

For this, see Proposition 2.3 of [3]. The chain maps $\phi: D \to D'$ and $\bar{\phi}$: $D' \to D$ are given by

$$\phi(a, x_1, x, x_2, b) = (a, x_1, \theta(x), x_2, b),$$

$$\bar{\phi}(a, x_1, x', x_2, b) = (a, x_1, \bar{\theta}(x') - K(x_1) - K(x_2), x_2, b).$$

The chain homotopies L from $\bar{\theta}\theta$ to 1_D and $\bar{L}$ from $\theta\bar{\theta}$ to $1_{D'}$, are respectively given by

$$L(a, x_1, x, x_2, b) = (0, 0, K(x), 0, 0),$$

$$\bar{L}(a, x_1, x', x_2, b) = (0, 0, \bar{K}(x') - M(x_1) - M(x_2), 0, 0),$$

where M is the homomorphism such that $\theta K - \bar{K}\theta = \partial M - M\partial$. The rest is categorical. See for example [6].

THEOREM 3.2. *Refer to Figure 1. If u_1, u_2, v_1, v_2 are chain homotopy pullbacks and $\bar{t}$ is a chain homotopy pushout, then t is a chain homotopy pushout.*

PROOF. Following Mather's reduction, it is equivalent to prove that in the commutative cube

$$\begin{tikzcd}
X' \arrow[rr, "j_2"] \arrow[dr, "j_1"] \arrow[dd, "p"'] & & B' \arrow[dr, "i_2"] \arrow[dd, "r_2"] & \\
& A' \arrow[rr, "i_1"] \arrow[dd, "r_1"] & & P_{\theta\bar{f}} \arrow[dd, "\bar{p}"] \\
X \arrow[rr, "\bar{j}_2"] \arrow[dr, "\bar{j}_1"'] & & Z_{\bar{p}_2} \arrow[dr, "\bar{i}_2"] & \\
& Z_{\bar{p}_1} \arrow[rr, "\bar{i}_1"'] & & D
\end{tikzcd}$$

the upper face is a chain homotopy pushout. In this case, all the vertical faces are chain pullbacks, the lower face is a flat chain homotopy pushout with D being the standard chain homotopy pushout of $\bar{p}_1$ and $\bar{p}_2$, and $\bar{p}$ is a chain fibration. Here θ is the chain homotopy equivalence between D and $\bar{Y}$. The chain maps $\bar{i}_1$, $\bar{i}_2$, $\bar{j}_1$ and $\bar{j}_2$ are the obvious inclusion chain maps. The chain complex X' is given by $X'_n = Y_n \oplus D_{n+1} \oplus \bar{X}_n$ with boundary homomorphism

$$\partial(y, d, \bar{x}) = (\partial y, -\bar{p}(y) - \partial d + \bar{i}_1 \bar{j}_1(\bar{x}), \partial \bar{x}).$$

The chain complex A' is given by $A'_n = Y_n \oplus D_{n+1} \oplus (Z_{\bar{p}_1})_n$ with boundary homomorphism

$$\partial(y, d, z_1) = (\partial y, -\bar{p}(y) - \partial d + \bar{i}_1(z_1), \partial z_1).$$

The structure of B' is similar. Here all the chain maps of the top face are the inclusion chain maps and all the vertical ones are projection chain maps. Now suppose $u : A' \to Q$ and $v : B' \to Q$ are chain maps such that $u j_1 = v j_2$. Then the unique chain map ϕ from P_f to Q is given by

$$\begin{aligned}\phi(y, d, (\bar{a}, \bar{x}_1, \bar{x}, \bar{x}_2, \bar{b})) &= u(y, d, (\bar{a}, \bar{x}_1, \bar{x})) + v(0, 0, (\bar{b}, \bar{x}_2, 0)) = \\ &= u(0, 0, (\bar{a}, \bar{x}_1, 0)) + v(y, d, (\bar{b}, \bar{x}_2, \bar{x})).\end{aligned}$$

Hence the top face is a chain pushout. Since j_1 or j_2 is a chain cofibration, it is a chain homotopy pushout.

We remark also that Theorem 3.2 follows from Theorem 3.1 if all are free chain complexes. Dualizing the proofs, we have

THEOREM 3.3. *Refer to Figure 1.*

(a) If t, $\bar{t}$ are chain homotopy pullbacks and v_1, v_2 are chain homotopy pushouts, then u_1 and u_2 are chain homotopy pushouts.

(b) If u_1, u_2, v_1, v_2 are chain homotopy pushouts and $\bar{t}$ is a chain homotopy pullback, then t is a chain homotopy pullback.

4. FLAT HOMOTOPY PUSHOUT.

Suppose the commutative square

$$\begin{array}{ccc} X & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & a & \downarrow{\scriptstyle q} \\ A & \xrightarrow[h]{} & Y \end{array}$$

is a homotopy pushout of topological spaces. We want to investigate the commutative square

$$\begin{tikzcd}
S(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "S(a)"] & S(B) \arrow[d] \\
S(A) \arrow[r] & S(Y)
\end{tikzcd}$$

where S is the singular homology functor.

Consider the commutative cube

$$\begin{tikzcd}
X \times I \arrow[rr] \arrow[dr] \arrow[dd] & & I_g \arrow[dr] \arrow[dd] & \\
& I_f \arrow[rr] \arrow[dd] & & M(f,g) \arrow[dd] \\
X \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& A \arrow[rr] & & Y
\end{tikzcd}$$

in which all the vertical maps are homotopy equivalences. Applying the singular homology functor S, we obtain the following chain commutative cube

$$\begin{tikzcd}
S(X \times I) \arrow[rr] \arrow[dr] \arrow[dd] & & S(I_g) \arrow[dr] \arrow[dd] & \\
& S(I_f) \arrow[rr] \arrow[dd] & & S(M(f,g)) \arrow[dd] \\
S(X) \arrow[rr] \arrow[dr] & & S(B) \arrow[dr] & \\
& S(A) \arrow[rr] & & S(Y)
\end{tikzcd}$$

In this case, all the vertical chain maps are chain homotopy equivalences. By the Mayer-Vietoris Theorem, the upper face is a chain homotopy pushout. Hence the bottom square is also a chain homotopy pushout. Moreover, by the Whitehead Theorem, the converse is also true for simply connected CW complexes. We thus have:

PROPOSITION 4.1. *Suppose all spaces are simply connected CW complexes. Then a is a homotopy pushout iff $S(a)$ is a chain homotopy pushout.*

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Department of Mathematics
University of Hong-Kong
HONG KONG