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AUTOMATA AND CATEGORIES

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Throughout this paper $\mathcal{V}$ denotes a monoidal closed category with tensor product $\otimes$ and internal-hom $[\ ,\ ]$. We assume that $\mathcal{V}$ has countable colimits and kernel pairs.

I. Given a sequence $\Omega = (\Omega_n)$ of objects of $\mathcal{V}$, an *Ω-algebra* is simply a pair (A, a_n) with $A \in \mathcal{V}$ and $a_n : \Omega_n \otimes A^n \to A$. ($\Omega_n$ is the object of symbols of n-ary operations of the algebra.) Homomorphisms are defined classically: they are morphisms which respect the Ω-algebra structure:

$$\begin{array}{ccc} \Omega_n \otimes A^n & \xrightarrow{\quad a_n \quad} & A \\ {\scriptstyle \Omega_n \otimes f^n}\downarrow & & \downarrow{\scriptstyle f} \\ \Omega_n \otimes B^n & \xrightarrow[\quad b_n \quad]{} & B \end{array} .$$

We can now construct the free Ω-algebra generated by an object X; its carrier object is the coproduct $\Omega(X) = \coprod_{n \geq 1} W_n$, where the objects W_n (trees of length n) are inductively defined as follows:

$$W_1 = X \amalg \Omega_0$$

....

$$W_{n+1} = \coprod_{\lambda_1 + \dots + \lambda_k = n} \Omega_k \otimes W_{\lambda_1} \otimes \dots \otimes W_{\lambda_k} .$$

II. One thing of central importance in this paper is the «free approximation» of the translation-monoid of an Ω-algebra (A, a_n).

Consider the object

$$A^* = \coprod_{1 \leq i \leq n < \omega} \Omega_n \otimes A^{i-1} \otimes \hat{A} \otimes A^{n-i}$$

where « ^ » means omission of the underlying factor, and ω denotes the first infinite ordinal.

We have a morphism $a^*: A^* \to [A, A]$ such that the adjoint transform of $a^* \circ in_{i,n}$ is a_n followed by a symmetry morphism.

Call tA the free monoid generated by A^*, i.e., $tA = \coprod_{n \geq 0} (A^*)^n$.

Then a^* is uniquely extended to a monoid homomorphism

$$a^+ : tA \to [A, A].$$

This says that tA acts on A by $s_A : tA \otimes A \to A$ obtained from a^+ via $\otimes \dashv [\,,\,]$.

III. In the framework described above we examine the realization problem for tree or algebra automata; precisely an *Ω-algebra automaton in* $\mathfrak{U}$ is a 5-tuple $\mathfrak{A} = (\mathrm{Q}, X, \tau, Y, \beta)$, where $\mathrm{Q} = (Q, \delta_n)$ is an Ω-algebra (Q the *state object*, δ_n the *transition* arrows) and $\tau : X \to Q$, $\beta : Q \to Y$ are morphisms of $\mathfrak{U}$ called *initial state* and *output*.

The *reachability map of* $\mathfrak{A}$ is the Ω-algebra homomorphism induced by τ, $\bar{\tau} : \Omega(X) \to Q$

$$\begin{array}{ccc} X & \xrightarrow{\;j_X\;} & \Omega(X) \\ & {\scriptstyle\tau}\searrow & \downarrow{\scriptstyle\bar{\tau}} \\ & & Q \end{array}$$

where j_X is the morphism «insertion of generators».

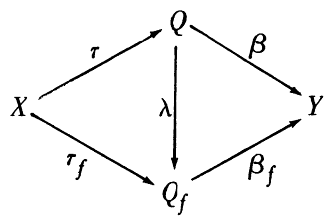
Call $\mathfrak{A}$ *$\mathfrak{E}$-reachable* if $\bar{\tau}$ belongs to a certain class $\mathfrak{E}$ of $\mathfrak{U}$-epimorphisms.

Finally the arrow

$$\Omega(X) \xrightarrow{\;\bar{\tau}\;} Q \xrightarrow{\;\beta\;} Y$$

is the *response of* $\mathfrak{A}$. Now given a morphism $f: \Omega(X) \to Y$, we say that $\mathfrak{A}$ *realizes* f iff its response is just f.

Our scope is to construct an $\mathfrak{E}$-reachable automaton ($\mathfrak{E}$ = coequalizers of reflexive pairs) $\mathfrak{A}_f = (\mathrm{Q}_f, X, \tau_f, Y, \beta_f)$ which realizes f in the most economical way, i.e. for every $\mathfrak{E}$-reachable automaton $\mathfrak{A}$ which also realizes f, there exists a unique algebra-homomorphism $\lambda : Q \to Q_f$ making



commutative.

THEOREM. *Any morphism $f : \Omega(X) \to Y$ has a reflexive coequalizer-reachable minimal realization.*

PROOF. Let

$$\tilde{s}_{\Omega(X)} : \Omega(X) \to [t\Omega(X), \Omega(X)]$$

be the adjoint transform of $s_{Q(X)}$,

$$E_f \underset{\gamma}{\overset{a}{\rightrightarrows}} \Omega(X)$$

the kernel pair of the morphism

$$\Omega(X) \xrightarrow[\tilde{s}_{\Omega(X)}]{} [t\Omega(X), \Omega(X)] \xrightarrow[{[1,f]}]{} [t\Omega(X), Y]$$

and $\tau_f : \Omega(X) \to Q_f$ the coequalizer of (a, γ).

We use Theorem 2.6 [A.A.M.]. Postulates i and iv of this theorem are evident. To prove Postulate iii, we need the following lemma:

LEMMA. *The monoid $t\Omega(X)$ acts on the object Q_f in such a way making τ_f an action homomorphism.*

PROOF. We have

$$f \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes s_{\Omega(X)}) \circ (t\Omega(X) \otimes t\Omega(X) \otimes a) =$$
$$= f \circ s_{\Omega(X)} \circ (m \otimes \Omega(X)) \circ (t\Omega(X) \otimes t\Omega(X) \otimes a) =$$
$$= f \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes a) \circ (m \otimes E_f),$$

m the multiplication of the monoid $t\Omega(X)$. Similarly

$$f \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes s_{\Omega(X)}) \circ (t\Omega(X) \otimes t\Omega(X) \otimes \gamma) =$$
$$= f \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes \gamma) \circ (m \otimes E_f).$$

Using the identity

$$f \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes a) = f \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes \gamma)$$

and passing to adjoint transforms we obtain

$$[1, f] \circ \bar{s}_{\Omega(X)} \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes a) =$$
$$= [1, f] \circ \bar{s}_{\Omega(X)} \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes \gamma).$$

Because (a, γ) is the kernel pair of $[1, f] \circ \bar{s}_{\Omega(X)}$ the above equality implies the existence of a morphism $\psi : t\Omega(X) \otimes E_f \rightarrow E_f$ such that

$$a \circ \psi = s_{\Omega(X)} \circ (t\Omega(X) \otimes a), \quad \gamma \circ \psi = s_{\Omega(X)} \circ (t\Omega(X) \otimes \gamma),$$

and consequently

$$r_f \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes a) = r_f \circ a \circ \psi = r_f \circ \gamma \circ \psi =$$
$$= r_f \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes \gamma).$$

But then the fact that

$$t\Omega(X) \otimes r_f = coeq(t\Omega(X) \otimes a, \ t\Omega(X) \otimes \gamma)$$

gives us the desired action $s : t\Omega(X) \otimes Q_f \rightarrow Q_f$. ■

Return to Postulate iii. We must prove that Q_f admits an Ω-algebra structure making r_f an algebra homomorphism. If we define the morphism

$$h_n^i : \Omega_n \otimes \Omega(X)^{i-1} \otimes Q_f \otimes \Omega(X)^{n-i} \rightarrow Q_f$$

to be the composite

$$\Omega_n \otimes \Omega(X)^{i-1} \otimes Q_f \otimes \Omega(X)^{n-i} \approx \Omega_n \otimes \Omega(X)^{n-1} \otimes Q_f \longrightarrow$$
$$\xrightarrow{in_n^i} \Omega(X)^* \otimes Q_f \xrightarrow{in_1 \otimes Q_f} t\Omega(X) \otimes Q_f \xrightarrow{s} Q_f,$$

then from the previous lemma we get

$$r_f \circ \mu_n = h_n^i \circ (\Omega_n \otimes \Omega(X)^{i-1} \otimes r_f \otimes \Omega(X)^{n-i})$$

(where $\mu_n : \Omega_n \otimes \Omega(X)^n \rightarrow \Omega(X)$ are the structural arrows of the free Ω-algebra $\Omega(X)$); therefore

$$\left.\begin{array}{l} h_n^i \circ (\Omega_n \otimes \Omega(X)^{i-1} \otimes r_f \otimes \Omega(X)^{n-i}) = \\ = h_n^j \circ (\Omega_n \otimes \Omega(X)^{j-1} \otimes r_f \otimes \Omega(X)^{n-j}) = r_f \circ \mu_n \end{array}\right] \quad (\Delta_n^{i,j})$$

for any $n \in \mathrm{N}$ and $1 \leq i, j \leq n$.

Combining the last relations and the fact

$$\Omega_n \otimes \Omega(X)^{i-1} \otimes r_f \otimes \Omega(X)^{n-i} =$$
$$= coeq(\Omega_n \otimes \Omega(X)^{i-1} \otimes \alpha \otimes \Omega(X)^{n-i}, \Omega_n \otimes \Omega(X)^{i-1} \otimes \gamma \otimes \Omega(X)^{n-i})$$

we deduce

$$\left.\begin{array}{l} h_n^i \circ (\Omega_n \otimes \Omega(X)^{i-1} \otimes r_f \otimes \Omega(X)^{n-i}) \circ \\ \quad \circ (\Omega_n \otimes \Omega(X)^{i-1} \otimes \alpha \otimes \Omega(X)^{n-i}) \circ (\Omega_n \otimes \Omega(X)^{i-1} \otimes E_f \otimes \gamma^{n-i}) = \\ h_n^i \circ (\Omega_n \otimes \Omega(X)^{i-1} \otimes r_f \otimes \Omega(X)^{n-i}) \circ \\ \quad \circ (\Omega_n \otimes \Omega(X)^{i-1} \otimes \gamma \otimes \Omega(X)^{n-i}) \circ (\Omega_n \otimes \Omega(X)^{i-1} \otimes E_f \otimes \gamma^{n-i}) \end{array}\right] (\Lambda_n^i)$$

for all $n \in$ N and $1 \leq i \leq n$.

Using successively $(\Delta_n^{i,j})$ and (Λ_n^i) we obtain the following string of isomorphisms:

$$h_n^n \circ (\Omega_n \otimes \Omega(X)^{n-1} \otimes r_f) \circ (\Omega_n \otimes \Omega(X)^{n-1} \otimes \alpha) \circ (\Omega_n \otimes \alpha^{n-1} \otimes E_f) =$$
$$h_n^n \circ (\Omega_n \otimes \Omega(X)^{n-1} \otimes r_f) \circ (\Omega_n \otimes \Omega(X)^{n-1} \otimes \gamma) \circ (\Omega_n \otimes \alpha^{n-1} \otimes E_f) =$$
$$h_n^n \circ (\Omega_n \otimes \Omega(X)^{n-1} \otimes r_f) \circ (\Omega_n \otimes \Omega(X)^{n-2} \otimes \alpha \otimes \Omega(X)) \circ (\Omega_n \otimes \alpha^{n-2} \otimes E_f \otimes \gamma) =$$
$$h_n^{n-1} \circ (\Omega_n \otimes \Omega(X)^{n-2} \otimes r_f \otimes \Omega(X)) \circ (\Omega_n \otimes \Omega(X)^{n-2} \otimes \alpha \otimes \Omega(X)) \circ$$
$$\circ (\Omega_n \otimes \alpha^{n-2} \otimes E_f \otimes \gamma) =$$

. . . .

$$h_n^2 \circ (\Omega_n \otimes \Omega(X) \otimes r_f \otimes \Omega(X)^{n-2}) \circ (\Omega_n \otimes \Omega(X) \otimes \gamma \otimes \Omega(X)^{n-2}) \circ$$
$$\circ (\Omega_n \otimes \alpha \otimes E_f \otimes \gamma^{n-2}) =$$
$$h_n^2 \circ (\Omega_n \otimes \Omega(X) \otimes r_f \otimes \Omega(X)^{n-2}) \circ (\Omega_n \otimes \alpha \otimes \Omega(X)^{n-1}) \circ (\Omega_n \otimes E_f \otimes \gamma^{n-1}) =$$
$$h_n^1 \circ (\Omega_n \otimes r_f \otimes \Omega(X)^{n-1}) \circ (\Omega_n \otimes \alpha \otimes \Omega(X)^{n-1}) \circ (\Omega_n \otimes E_f \otimes \gamma^{n-1}) =$$
$$h_n^1 \circ (\Omega_n \otimes r_f \otimes \Omega(X)^{n-1}) \circ (\Omega_n \otimes \gamma \otimes \Omega(X)^{n-1}) \circ (\Omega_n \otimes E_f \otimes \gamma^{n-1});$$

so (1)

$$h_n^n \circ (\Omega_n \otimes \Omega(X)^{n-1} \otimes r_f) \circ (\Omega_n \otimes \alpha^n) = h_n^n \circ (\Omega_n \otimes \Omega(X)^{n-1} \otimes r_f) \circ (\Omega_n \otimes \gamma^n)$$

Our next step is to define inductively morphisms

$$k_\lambda : \Omega_n \otimes \Omega(X)^{n-\lambda} \otimes Q_f^\lambda \to Q_f \quad (1 \leq \lambda \leq n)$$

such that

$$k_\lambda \circ (\Omega_n \otimes \Omega(X)^{n-\lambda} \otimes r_f \otimes Q_f^{\lambda-1}) = k_{\lambda-1}.$$

We take $k_1 = h_n^n$. k_1 coequalizes

$$(\Omega_n \otimes \Omega(X)^{n-2} \otimes a \otimes Q_f,\ \Omega_n \otimes \Omega(X)^{n-2} \otimes \gamma \otimes Q_f).$$

(This comes if we put (1) together with

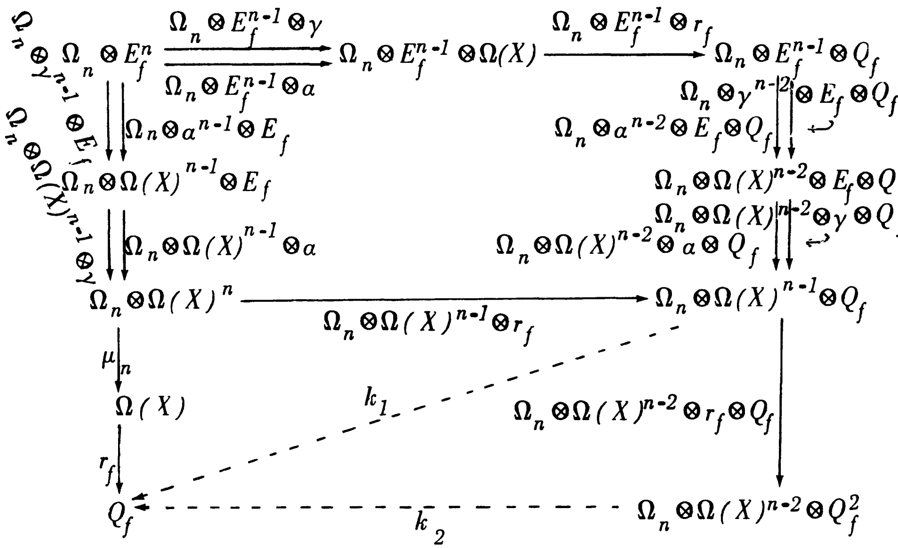
$$\Omega_n \otimes E_f^{n-1} \otimes r_f = coeq(\Omega_n \otimes E_f^{n-1} \otimes a, \Omega_n \otimes E_f^{n-1} \otimes \gamma)$$

and the fact that

$$(\Omega_n \otimes a^{n-2} \otimes E_f \otimes Q_f,\ \Omega_n \otimes \gamma^{n-2} \otimes E_f \otimes Q_f)$$

is reflexive.) Consequently there exists a unique

$$k_2 : \Omega_n \otimes \Omega(X)^{n-2} \otimes Q_f^2 \to Q_f \quad \text{with} \quad k_2 \circ (\Omega_n \otimes \Omega(X)^{n-2} \otimes r_f \otimes Q_f) = k_1.$$



k_3 results from k_2 by a similar as above argument, and so on. Finally, we determine $k_n : \Omega_n \otimes Q_f^n \to Q_f$ which is the n-th structural arrow of Q_f.

It remains to show that r_f is an algebra homomorphism. We have

$$r_f \circ \mu_n = k_1 \circ (\Omega_n \otimes \Omega(X)^{n-1} \otimes r_f)$$
$$= k_2 \circ (\Omega_n \otimes \Omega(X)^{n-2} \otimes r_f \otimes Q_f) \circ (\Omega_n \otimes \Omega(X)^{n-1} \otimes r_f)$$
$$\cdots\cdots\cdots$$
$$= k_{n-1} \circ (\Omega_n \otimes \Omega(X) \otimes r_f \otimes Q_f^{n-2}) \circ \ldots \circ (\Omega_n \otimes \Omega(X)^{n-1} \otimes r_f)$$

$$= k_n \circ (\Omega_n \otimes r_f \otimes Q_f^{n-1}) \circ \dots \circ (\Omega_n \otimes \Omega(X)^{n-1} \otimes r_f)$$
$$= k_n \circ (\Omega_n \otimes r_f^n).$$

We now establish the postulate i. It suffices to prove that f co-equalizes the pair $(\bar{a}, \bar{\gamma})$ in a universal way, i.e. wherever $f \circ \bar{p} = f \circ \bar{q}$ also holds for a reflexive pair

$$R \underset{q}{\overset{p}{\rightrightarrows}} \Omega(X),$$

there exists a unique arrow $h : R \to E_f$ such that $a \circ h = p$, $\gamma \circ h = q$. ($\bar{a}$, $\bar{\gamma}$, $\bar{p}$, $\bar{q}$ are the unique algebra homomorphisms deduced by a, γ, p, q respectively.) The equality $f \circ \bar{a} = f \circ \bar{\gamma}$ comes easily if we prove that $f \circ a = = f \circ \gamma$. Let $\phi : Q_f \to [\,t\Omega(X), Y]$ be the unique arrow such that

$$\phi \circ r_f = [\,1, f] \circ \bar{s}_{\Omega(X)}.$$

If $u : I \to t\Omega(X)$ is the unit of the monoid $t\Omega(X)$, then we have

$$\psi \circ (u \otimes E_f) = Id_{E_f},$$

because ψ is an action. Therefore

$$f \circ a = f \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes a) \circ (u \otimes E_f)$$
$$= \tilde{\phi} \circ (t\Omega(X) \otimes r_f) \circ (t\Omega(X) \otimes a) \circ (u \otimes E_f)$$
$$= \tilde{\phi} \circ (t\Omega(X) \otimes r_f) \circ (t\Omega(X) \otimes \gamma) \circ (u \otimes E_f)$$
$$= f \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes \gamma) \circ (u \otimes E_f) = f \circ \gamma,$$

where $\tilde{\phi}$ is the transform of ϕ via $\otimes \dashv [\ ,\ ]$.

Now, let $\epsilon : \Omega(X) \to \Omega(R)$ be the common right inverse of $\bar{p}$ and $\bar{q}$: $\bar{p} \circ \epsilon = Id = \bar{q} \circ \epsilon$. We have

$$\bar{q} \circ s_{\Omega(R)} = s_{\Omega(X)} \circ (t\bar{q} \otimes \bar{q}), \quad \bar{p} \circ s_{\Omega(R)} = s_{\Omega(X)} \circ (t\bar{p} \otimes \bar{p}),$$

where $t\bar{p}$, $t\bar{q}$ are the free monoid homomorphisms induced by $\bar{p}, \bar{q}$ respectively. Hence

$$f \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes \bar{p}) = f \circ s_{\Omega(X)} \circ (t\bar{p} \otimes \bar{p}) \circ (t\epsilon \otimes \Omega(R)) =$$
$$= f \circ \bar{p} \circ s_{\Omega(R)} \circ (t\epsilon \otimes \Omega(R)) = f \circ \bar{q} \circ s_{\Omega(R)} \circ (t\epsilon \otimes \Omega(R)) =$$
$$= f \circ s_{\Omega(X)} \circ (t\bar{q} \otimes \bar{q}) \circ (t\epsilon \otimes \Omega(R)) = f \circ s_{\Omega(X)} \circ (t\Omega(X) \otimes \bar{q}),$$

or

$$[1, f] \circ \tilde{s}_{\Omega(X)} \circ \bar{p} = [1, f] \circ \tilde{s}_{\Omega(X)} \circ \bar{q}$$

by adjoint transformation. Therefore there exists a (unique) arrow

$$\delta : \Omega(R) \to E_f \text{ such that } \alpha \circ \delta = \bar{p}, \quad \gamma \circ \delta = \bar{q}.$$

It is now clear that the morphism

$$h = \delta \circ j_R : R \to E_f$$

has the desired property, and this concludes the proof of the theorem.

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