BARRY MITCHELL

Monoidal structures on graded categories

Cahiers de topologie et géométrie différentielle catégoriques, tome 23, n° 1 (1982), p. 43-45

<http://www.numdam.org/item?id=CTGDC_1982__23_1_43_0>
Two monoidal structures on a category $\mathcal{V}$ are equivalent if there is a bimonoidal (that is, strict monoidal) structure on the identity functor $1_{\mathcal{V}}$, using one monoidal structure in the domain and the other in the range. Also if $\mathcal{V}$ and $\hat{\mathcal{V}}$ are monoidal categories, then two bimonoidal structures on a functor $T: \mathcal{V} \to \hat{\mathcal{V}}$ are equivalent if there is a monoidal isomorphism $T = T$ using one bimonoidal structure in the domain and the other in the range.

Let $G$ be a group acting on a monoidal category $\mathcal{V}$. This means that for each $x \in G$, there is a bimonoidal equivalence $T_x: \mathcal{V} \to \mathcal{V}$ and monoidal isomorphisms making a couple of obvious diagrams commute. Such an action induces an action of $G$ on the abelian group $Z^*\mathcal{V}$ of automorphisms of $1_{\mathcal{V}}$. Let $G\mathcal{V}$ denote the category of $G$-graded objects of $\mathcal{V}$ (that is, the direct product of $G$ copies of $\mathcal{V}$). If $\mathcal{V}$ has coproducts, we can define a tensor product in $G\mathcal{V}$ by the rule

\begin{equation}
(A \otimes B)_z = \bigotimes_{x, y = z} A_x \otimes T_x B_y.
\end{equation}

Under the assumption that the tensor product of $\mathcal{V}$ preserve coproducts and epimorphisms and that the unit of this tensor product be a generator for $\mathcal{V}$, we show:

**Theorem 1.** The equivalence classes of monoidal structures on $G\mathcal{V}$ using the tensor product (2) are in 1-1 correspondence with the elements of $H^3(G, Z^*\mathcal{V})$. Moreover, the equivalence classes of bimonoidal structures on $I_{G\mathcal{V}}$, using any one of the above monoidal structures in both domain and

*) Work supported by NSF grant MCS-7703645.
range, are in 1-1 correspondence with the elements of $H^2(G, Z^*V)$.

Two symmetric monoidal structures on a category $V$ are equivalent if there is a symmetric bimonoidal structure on $I_V$ making the monoidal structures equivalent. Now for $G^V$ with the tensor product (2) to have a symmetric monoidal structure at all, one must assume that $V$ is symmetric monoidal and that $G$ is abelian and acts trivially on $V$ (that is, $T_x = 1_V$ for all $x \in G$ with the isomorphisms (1) identities). In this case the monoidal structure on $G^V$ using the tensor product (2) and the trivial 3-cycle will be referred to as the trivial monoidal structure. Then again with the above blanket assumptions on $V$, we show:

**Theorem 2.** If $G$ is abelian and $V$ is symmetric monoidal, then the equivalence classes of symmetric structures on the trivial monoidal structure on $G^V$ are in 1-1 correspondence with the equivalence classes of bilinear antisymmetric maps $f: G \times G \to Z^*V$, where two such maps $f, f'$ are equivalent if there is a 2-dimensional cocycle $h$ such that

$$f'(x, y) - f(x, y) = h(x, y) - h(y, x)$$

for all $x, y \in G$.

An immediate consequence of the above theorems, using the fact that the group of integers has cohomological dimension one, is that if $K$ is a commutative ring, then up to equivalence there is precisely one monoidal structure on the category of $Z$-graded $K$-modules (with the usual graded tensor product), and the symmetries for this structure are in 1-1 correspondence with the elements $k \in K$ such that $k^2 = 1$. In particular, if $K$ is a domain, we find that the only symmetries are given by

$$a_p \otimes b_q \mapsto b_q \otimes a_p \text{ and } a_p \otimes b_q \mapsto (-1)^{pq} b_q \otimes a_p.$$ 

Finally, if we start with an abelian group $K$ on which a group $G$ acts, then we can take $V$ to be $Sets^K$ (so that $Z^*V = K$ as $G$-modules), in which case Theorem 1 gives new interpretations of the cohomology groups $H^3(G, K)$ and $H^2(G, K)$.
Details of this work will be appearing in reference [5].

REFERENCES


4. S. MAC LANE, Natural associativity and commutativity, Rice University Studies 49 (1963), No 4, 28-46.


Department of Mathematics
Rutgers University
NEW BRUNSWICK, N. J. 08903
U. S. A.