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ANDRÉE CHARLES EHRESMANN

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PARTIAL COMPLETIONS OF CONCRETE FUNCTORS

by *Andrée CHARLES EHRESMANN*

INTRODUCTION.

If f and g are differentiable maps between manifolds M and M' , the equation $f(m) = g(m)$ may not define a submanifold of M ; two differentiable maps toward M may not have a pullback unless they are transversal. Such difficulties have hindered a categorical study of Differential Geometry; e. g., differentiable categories /50/ are only those internal categories in the category $\mathcal{D}^n$ of manifolds whose domain and codomain maps are submersions. Is it not possible to embed $\mathcal{D}^n$ into an «adequate» category? Charles was mainly motivated by this question and inspired by his many works on completions of posets and of local categories /47, 55, 76, 85, 86/ when he wrote his paper /107/ in the mid-sixties: here he constructs «optimal» extensions of a concrete functor $P: H \rightarrow E$ into a concrete functor which initially lifts a given class of singleton sources («spreading» functors) or of limit cones («completions») in E . For example, the smallest spreading extension of $\mathcal{D}^n \rightarrow \text{Ens}$ equips each subset of a manifold M with a structure which has been independently worked out (without categorical aims!) by Ngo Van Qué [6], Aronszajn and Marshall [5].

Later on, several authors tackled analogous problems, often with a view to embedding the category of topological spaces in an initially complete cartesian closed category (Antoine, Chartrelle, Day, Wyler,...); generalizing results of Banaschewski and Bums on completions of posets, Herrlich describes in [2] the smallest - or Mac Neille - and the largest or universal (preserving initial lifts) initial completions of P ; for the bibliography, we refer to [3] where most papers on initial completions (Adamek, Herrlich, Strecker; Börger; Hoffmann; Tholen...) are summarized. Recently

universal completions of P have been constructed by Adamek-Koubek [1] and, without transfinite induction, by Herrlich [4].

Here all these results are unified: Given a class Γ of cones in E , and a class Δ of initial cones in H , the concrete functor $P: H \rightarrow E$ is extended into a concrete functor with initial lifts of cones of Γ , and for which the cones of Δ remain initial; two «optimal» solutions, the Mac Neille Γ -completion and the universal (Δ, Γ) -completion of P , are constructed by methods making the most out of the ideas of Charles [55, 107] and Herrlich [2, 4]. If Γ is «not too large» these solutions live in the same universe as P .

HYPOTHESES. There is given a category E (the «base category») and a class Γ of cones in E ; let $Ind \Gamma$ be the class formed by the indexing categories of the cones $\gamma \in \Gamma$. Cone always means projective cone.

We denote by $P: H \rightarrow E$ a concrete (i.e., faithful and amnesic) functor, by $S, S', \dots$ the objects of H , by $E, E', \dots$ the objects of E . These notations come from the primitive case where P is the forgetful (Projection) functor from the category of Homomorphisms between Structures of some kind to Ens .

$\mathcal{U}_0$ and $\mathcal{U}$ are two universes such that $\mathcal{U}_0 \in \mathcal{U}$; the elements of $\mathcal{U}$ are large sets, or classes, those of $\mathcal{U}_0$ are small sets. We suppose P lives in $\mathcal{U}$ (as does for instance the forgetful functor $\mathcal{D}^a \rightarrow Ens$).

1. PARTIAL COMPLETIONS.

In this section, we recall definitions, and state some results on cones and initial lifts which are used in the sequel.

A. Commutative hull of Γ .

Let γ be a cone in E with vertex E and basis $\phi: I \rightarrow E$, abbreviated in $\gamma: E \Rightarrow \phi$. If the indexing category I is discrete, γ is called a *source*, and also denoted by $(\gamma(I) \mid I \in I)$; for instance E defines the the singleton source $E^* = (Id_E)$.

To the cone γ is associated the source $(\gamma(I) \mid I \in I_0)$ indexed by the class I_0 of objects of I ; this source is written $\gamma_0: E \Rightarrow \phi_0$. We

denote by Γ_0 the class of all sources γ_0 , for $\gamma \in \Gamma$.

If $\gamma_I: \phi(I) \Rightarrow \phi_I$ is a cone indexed by I_I for each $I \in I_0$, the source

$$(\gamma_I(J) \cdot \gamma(I): E \rightarrow \phi_I(J) \mid (J, I) \in \sum_{I \in I_0} I_{I_0})$$

is called the *composite source* of $((\gamma_I)_{I_0}, \gamma)$, denoted $(\gamma_I)_{I_0} \circ \gamma$. For instance: $(\phi(I)^*)_{I_0} \circ \gamma = \gamma_0$; if $I = 1$ and $\gamma(I) = f$, then $(\gamma_I)_1 \circ \gamma$ is the source $\gamma_I \circ f = (\gamma_I(J) \cdot f \mid J \in I_{I_0})$.

DEFINITION. A class Σ of sources in E is said *commutative* if

- 1° $E^* \in \Sigma$ for each object E of E ,
- 2° $\gamma \in \Sigma$ and $\gamma_I \in \Sigma$ for each $I \in I = I_0$ imply $(\gamma_I)_I \circ \gamma \in \Sigma$.

PROPOSITION 1. Let Σ be a class of sources in E ; the *smallest commutative class of sources* Σ° containing Σ is constructed by *transfinite induction*. If $\text{Ind } \Sigma$ and each $I \in \text{Ind } \Sigma$ belong to the universe $\mathcal{U}$, so does $\text{Ind } \Sigma^\circ$.

Δ . Σ° is the union of the transfinite increasing sequence $(\Sigma_\lambda)_\lambda$ defined by induction as follows:

$$\begin{aligned} \Sigma_0 &= \{E^* \mid E \in E_0\} \cup \Sigma; \quad \Sigma_\alpha = \bigcup_{\lambda < \alpha} \Sigma_\lambda \quad \text{for each limit ordinal } \alpha, \\ \Sigma_{\lambda+1} &= \{(\gamma_I)_I \circ \gamma \mid \gamma \in \Sigma_\lambda, \gamma_I \in \Sigma_\lambda \quad \forall I \in I\}. \end{aligned}$$

The construction stops at the limit ordinal larger than the ordinal of I , for each $I \in \text{Ind } \Sigma$. ∇

DEFINITION. The smallest commutative class of sources containing Γ_0 is called the *commutative hull* of Γ , denoted by Γ° .

EXAMPLES. The class *Sour* E of all sources in E is commutative and it is the commutative hull of the class *Cone* E of all cones in E . If A is a class of morphisms of E , the class $/A/$ of singleton sources (a) , $a \in A$ has for its commutative hull the class $/A'/$ corresponding to the sub-category of E generated by $A \cup E_0$.

B. Initial lifts.

Let $P: H \rightarrow E$ be a concrete functor. A morphism h from S to S' in H is written $g: S \rightarrow S'$, where $g = P(h)$. If θ is a cone in H with vertex S and basis Φ and if $\gamma = P\theta$, we also denote θ by $\gamma: S \Rightarrow \Phi$.

A P -cone indexed by I is a pair (Φ, γ) , where $\Phi: I \rightarrow H$ is a functor and $\gamma: E \Rightarrow P\Phi$ is a cone. An *initial lift* of (Φ, γ) is an object S of H such that:

1° $\gamma(I): S \rightarrow \Phi(I)$ is in H for each $I \in I_0$,

2° If $f: E' \rightarrow E$ in E and if $\gamma \circ f: S' \Rightarrow \Phi$ is a cone in H , then f lifts into $f: S' \rightarrow S$ in H .

As P is concrete, such an S (if it exists) is unique; it is then denoted by $il(\Phi, \gamma)$; so $\gamma: S \Rightarrow \Phi$ is an initial cone for P .

A P -source is a P -cone (Φ, γ) indexed by a discrete category I_0 ; it is often identified to the family of P -morphisms (singleton P -sources)

$$(\Phi(I), \gamma(I) \mid I \in I_0).$$

The dual notion is a P -sink.

To the P -cone (Φ, γ) indexed by I is associated the P -source (Φ_0, γ_0) , where $\Phi_0: I_0 \rightarrow H$ is the restriction of Φ to the objects of I .

PROPOSITION 2. *Let (Φ, γ) be a P -cone. It has an initial lift iff the P -source (Φ_0, γ_0) has one; in this case, $il(\Phi, \gamma) = il(\Phi_0, \gamma_0)$. ∇*

The following proposition (whose proof is straightforward) is important for the sequel. Let (Φ, γ) be a P -cone indexed by I and (Φ_I, γ_I) be a P -cone indexed by I_I such that $\gamma_I: \Phi(I) \Rightarrow \Phi_I$ is a cone in H , for I in I_0 . We denote by $(\Phi_I, \gamma_I)_{I_0} \circ (\Phi, \gamma)$ the P -source

$$(\Psi, (\gamma_I)_{I_0} \circ \gamma) \text{ where } \Psi: \sum_{I \in I_0} I_I \rightarrow H: (J, I) \mapsto \Phi_I(J).$$

PROPOSITION 3 (*Commutativity of initial lifts*). *If $\Phi(I) = il(\Phi_I, \gamma_I)$ for for each object I of I , we have*

$$il(\Phi, \gamma) = il((\Phi_I, \gamma_I)_{I_0} \circ (\Phi, \gamma))$$

as soon as one of these terms is defined. ∇

DEFINITION. P is called Γ -complete if each P -cone (Φ, γ) with $\gamma \in \Gamma$ admits an initial lift.

From Propositions 1, 2, 3, it follows by transfinite induction:

COROLLARY. *If P is Γ_0 -complete, then it is Γ -complete and Γ° -complete, where Γ° is the commutative hull of Γ .*

We consider the category of concrete functors over E , whose objects are the concrete functors $Q: K \rightarrow E$ and whose morphisms $F: Q \rightarrow Q'$, where $Q': K' \rightarrow E$, are the functors $F: K \rightarrow K'$ such that $Q'F = Q$. It has a non-full subcategory formed by the Γ -morphisms, which are the morphisms $F: Q \rightarrow Q'$ such that $F(il(\Phi, \gamma)) = il(F\Phi, \gamma)$ whenever (Φ, γ) is a Q -cone with $\gamma \in \Gamma$ which has an initial lift.

COROLLARY. *If $F: Q \rightarrow Q'$ is a Γ_0 -morphism, it is a Γ -morphism and if Q is Γ_0 -complete, a Γ^0 -morphism.*

C. Γ -density et Γ -generation.

Here $Q: K \rightarrow E$ is a concrete functor, H a full subcategory of K and $P: H \rightarrow E$ is the restriction of Q .

DEFINITION. H is called Γ -dense for Q if each object K of K is the initial lift of a P -cone (Ψ, γ) with $\gamma \in \Gamma$ and Ψ valued in H . If H is $Sour E$ -dense, it is said *initially dense*.

PROPOSITION 4. *The following conditions are equivalent:*

- 1° H is initially dense for Q .
- 2° For each $K \in K_0$ the source $(k: K \rightarrow S \mid S \in H_0)$ is initial for Q .
- 3° Let K, K' be objects of K and $g: Q(K) \rightarrow Q(K')$ a E -morphism; then we have $g: K \rightarrow K'$ in K iff

$$f: K' \rightarrow S \text{ in } K \text{ and } S \in H_0 \text{ imply } f \cdot g: K \rightarrow S \text{ in } K.$$

If they are satisfied, the insertion $H \hookrightarrow K$ preserves final lifts. ∇

The dual notion is «finally dense». It will be used in Section 2 through the third characterization above (introduced in /107/ under the name « Q is P -generated»).

DEFINITION. We call Γ -hull (resp. strict Γ -hull) of H for Q the smallest full subcategory (resp. smallest subcategory) H' of K containing H and $il(\Psi, \gamma)$ for each Q -cone (Ψ, γ) with $\gamma \in \Gamma$ and Ψ valued in H' . If $H' = K$, we say that Q is Γ -generated (resp. strictly Γ -generated) by H .

If Q is Γ -complete, so is its restriction to H' . If H is Γ -dense for Q , then Q is Γ -generated by H (but not conversely).

PROPOSITION 5. *The Γ -hull C and the strict Γ -hull B of H for Q are constructed by transfinite induction; they are in the same universe $\mathcal{U}$ as H if so are $\text{Ind } \Gamma$ and each $I \in \text{Ind } \Gamma$.*

Δ . C and B are respectively the union of the transfinite increasing sequence $(C_\lambda)_\lambda$ and $(B_\lambda)_\lambda$ defined as follows:

$$C_0 = H = B_0,$$

$$C_\alpha = \bigcup_{\lambda < \alpha} C_\lambda \quad \text{and} \quad B_\alpha = \bigcup_{\lambda < \alpha} B_\lambda \quad \text{for each limit ordinal } \alpha,$$

$C_{\lambda+1}$ is the full subcategory of K with objects $il(\Psi, \gamma)$ where (Ψ, γ) is a Q -cone, $\gamma \in \Gamma$ and Ψ valued in C_λ ,

$B_{\lambda+1}$ is the subcategory of K generated by all the morphisms

$$\gamma(I): il(\Psi, \gamma) \rightarrow \Psi(I) \quad \text{for each } I \in I_0,$$

$$g: K_\lambda \rightarrow il(\Psi, \gamma) \quad \text{whenever } \gamma \circ g: K_\lambda \Rightarrow \Psi_0 \text{ is a cone in } K,$$

where (Ψ, γ) is any Q -cone with $\gamma \in \Gamma$ and $\Psi: I \rightarrow K$ valued in B_λ .

The construction stops at the first limit ordinal greater than the ordinals of I for each $I \in \text{Ind } \Gamma$. ∇

COROLLARY. *If Q is Γ -generated by H , then H is Γ° -dense for Q .*

Proof by induction on C_λ using the commutativity of initial lifts.

D. Γ -completions.

DEFINITION. A Γ -completion of $P: H \rightarrow E$ is defined as a concrete Γ -complete functor $Q: K \rightarrow E$ of which P is a full restriction. The Γ -completion is Γ -dense (resp. *initially dense*) if so is H for Q ; it is (strictly) Γ -generated if K is the (strict) Γ -hull of H for Q .

An order is defined on the Γ -completions of P as follows:

$$Q \leq_\Gamma Q' \quad (\text{say } Q \text{ is } \Gamma\text{-smaller than } Q') \quad \text{iff there exists one unique } \Gamma\text{-morphism } F: Q \rightarrow Q' \text{ extending the identity on } H.$$

We are going to construct completions which are optimal for this order.

EXAMPLES. 1. *Sour* E -completions have been considered by several authors, e. g. Herrlich [2, 4] under the name: *initial completions*.

2. If A is a class of morphisms of E , the $/A/$ -completions of P are called *A-completions*; they are the A -spreading functors extending P which are dealt with in /107/.

3. If E is a complete category and $\text{Lim } E$ is the class of all its small limit-cones, $\text{Lim } E$ -completions, just called *completions of P* , are constructed in Adamek-Koubek [1] and Herrlich [3]. More generally, if μ is a partial (multiple) choice of limit-cones on E and Γ the class of limit-cones distinguished by μ , we find the μ -completions studied in /107/.

2. MAC NEILLE COMPLETIONS.

In this section, we construct a Γ -completion of $P: H \rightarrow E$ which is both finally dense and Γ -generated; such a Γ -completion is called a *Mac Neille Γ -completion of P* (by analogy with Herrlich's Mac Neille initial completions, named after the Mac Neille completions of posets).

In /107/, Charles constructs the Mac Neille A -completion of P for A a subcategory of E (Theorem 2, 3) and, using it, the Mac Neille μ -completion of P (Theorem 5, 6), which he calls «smallest prolongations of P ». His method, which generalizes for any class Γ of cones, may be sketched as follows: To H he adds «formal initial lifts» of cones of Γ and as many morphisms as possible for getting a faithful (non-amnestic) functor in which these formal initial lifts become initial lifts; so H is Γ -dense and finally dense for the associated concrete functor $Q: K \rightarrow E$ (this condition entirely characterizes Q). In the case $\Gamma = /A/$, this functor is the Mac Neille A -completion of P . For μ -completions or more generally, the construction has to be transfinitely reiterated, because Q is not Γ -complete. (In fact, Charles gets Q as a Mac Neille $\bar{A}$ -completion of a certain extension of P .) Now we remark that Q is Γ -complete whenever Γ is equal to its commutative hull Γ° thanks to the commutativity of initial lifts; in this case, an object of K , which is an equivalence class of P -sources, may be identified to the union of these P -sources, hence to a closed source in Herrlich's sense [3]; so, for $\Gamma = \text{Sour } E$, Q is exactly the Mac Neille initial completion P_4 as constructed by Herrlich in [2].

Whence the idea of the following proof: we first construct the Mac Neille Γ° -completion of P ; the Γ -hull of H in it then gives the Mac Neille Γ -completion of P (constructed by induction via Proposition 5).

THEOREM 1. *P admits a Mac Neille Γ -completion $P_\Gamma: H_\Gamma \rightarrow E$, which*

lives in the universe $\mathcal{U}$ if so do P , $\text{Ind } \Gamma$ and each $I \in \text{Ind } \Gamma$.

Δ . Construction of the Mac Neille Γ° -completion $V: M \rightarrow E$ of P :
If (Φ, γ) is a P -source, we denote by $(\Phi, \gamma)^*$ the opposite P -sink [3]:

$$(S, f: P(S) \rightarrow E \mid S \in H_0, \gamma \circ f: S \Rightarrow \Phi),$$

where E is the vertex of γ . Let M_0 be the class of the P -sinks of the form $(\Phi, \gamma)^*$ for some P -source with $\gamma \in \Gamma^\circ$; the vertex of γ is denoted by $V(M)$. If M and M' are in M_0 , there'll be a morphism $g: M \rightarrow M'$ in M mapped by V on g iff

$$(S, f) \text{ in } M \text{ implies } (S, g.f) \text{ in } M'.$$

This defines the concrete functor $V: M \rightarrow E$. We identify H to a full subcategory of M by identifying $S \in H_0$ to the P -sink

$$(S, id_{P(S)})^* = (S', h \mid h: S' \rightarrow S \text{ in } H).$$

So H becomes finally dense for Q ; it is also Γ° -dense, because the object M is the initial lift of each P -source (considered as a V -source!) (Φ, γ) such that $M = (\Phi, \gamma)^*$. Hence V is a Mac Neille Γ° -completion of P if it is Γ° -complete; this is true: let (Ψ, θ) be a V -source with $\theta \in \Gamma^\circ$, indexed by I ; for each I in I , we have

$$\Psi(I) = (\Phi_I, \gamma_I)^* = il(\Phi_I, \gamma_I) \quad \text{for some } \gamma_I \in \Gamma^\circ;$$

as Γ° is commutative $(\Phi_I, \gamma_I)_I \circ (\Psi, \theta)$ is a P -source σ whose dual P -sink σ^* is in M_0 , so that $\sigma^* = il(\sigma) = il(\Psi, \theta)$ (by Proposition 3).

- Let H_Γ be the Γ -hull of H for V , and $P_\Gamma: H_\Gamma \rightarrow E$ the restriction of V . It is Γ -complete (Corollary, 1), and H is still finally dense, H_Γ being a full subcategory of M . Hence P_Γ is a Mac Neille Γ -completion of P . If $\text{Ind } \Gamma$ and all $I \in \text{Ind } \Gamma$ are in the universe $\mathcal{U}$, then so does $\text{Ind } \Gamma^\circ$ (Proposition 1), which implies M and H_Γ are also in $\mathcal{U}$. ∇

REMARK. M is a full subcategory of the Mac Neille initial completion P_4 so that H_Γ may also be defined as the Γ -hull of H for P_4 . However M is in $\mathcal{U}$ while P_4 may not; conditions for it to be in $\mathcal{U}$ are given in [3].

The «optimality» of P_Γ will be deduced from the following proposition, which generalizes Theorems 3 and 6 of /107/ and has a similar proof.

THEOREM 2. Let $Q: K \rightarrow E$ be a Γ -generated Γ -completion of $P: H \rightarrow E$ and $Q': K' \rightarrow E$ be a finally dense Γ -completion of $P': H' \rightarrow E$. Let $F: P \rightarrow P'$ be a morphism satisfying the «lifting-cones» condition:

If $\Phi: I \rightarrow H$ is a functor with $I \in \text{Ind } \Gamma$, each cone in H' with basis $F\Phi$ is the image by F of a cone in H with basis Φ .

Then F extends in a unique Γ -morphism $F': Q \rightarrow Q'$.

Δ . $K = \bigcup_{\lambda} C_{\lambda}$ (Proposition 5) and F' is defined by induction on C_{λ} : If F' is defined on C_{λ} and if $K = \text{il}(\Phi, \gamma)$ with $\gamma \in \Gamma$ and Φ valued in C_{λ} , we take $F'(K) = \text{il}(F'\Phi, \gamma)$; since Q' is finally dense, $F'(K)$ does not depend on the choice of (Φ, γ) , and $g: K \rightarrow K'$ in $C_{\lambda+1}$ implies that $g: F'(K) \rightarrow F'(K')$ in K' . Whence F' is defined on $C_{\lambda+1}$. ∇

COROLLARY 1. The Mac Neille Γ -completion P_{Γ} of P is the Γ -smallest finally dense Γ -completion of P and the Γ -largest Γ -generated one; in particular, two Mac Neille completions are isomorphic.

COROLLARY 2. The Mac Neille Γ -completion P_{Γ} of P is fully embedded in any finally dense Γ -completion Q' of P .

Indeed, if $F = \text{id}_H$ and Q finally dense, F' above is a full embedding.

REMARK. Corollary 2 says that P_{Γ} is also the smallest finally dense Γ -completion of P for the preorder on completions:

$Q < Q'$ iff there exists a full embedding $Q \rightarrow Q'$ (not a Γ -morphism!) extending the identity on H .

For this preorder, Herrlich proves that P_4 is in fact the smallest initial completion of P ; this stronger result comes from the duality Theorem for initially complete functors, which has no analogon for a general Γ .

As in /107/, Theorem 2 is easily adapted to characterize the strict Γ -hull H_{Γ} of H for V (or for P_{Γ}). We say that a Γ -completion $Q: K \rightarrow E$ of P is weakly dense if $K = K'$ whenever:

for each $S \in H_0$, we have: $g: S \rightarrow K$ iff $g: S \rightarrow K'$ in K .

THEOREM 3. The restriction $P_{\Gamma}^{\dagger}: H_{\Gamma} \rightarrow E$ of P_{Γ} is the unique (up to isomorphism) Γ -completion of P which is both weakly dense and strictly Γ -generated; it is the Γ -smallest weakly dense Γ -completion and its Γ -

largest strictly Γ -generated one. ∇

3. UNIVERSAL PARTIAL COMPLETIONS.

In this section Δ denotes a given class of initial cones in H for $P: H \rightarrow E$ such that $P\delta \in \Gamma$ for each $\delta \in \Delta$. Let $\Delta_0 = \{ \delta_0 \mid \delta \in \Delta \}$; by Proposition 1, Δ_0 is a class of initial sources for P .

DEFINITION. A Γ -completion $Q: K \rightarrow E$ of P is called a (Δ, Γ) -completion of P if the insertion $H \hookrightarrow K$ sends (all δ in) Δ on initial cones. It is called a *universal (Δ, Γ) -completion* if it also satisfies:

Let $Q': K' \rightarrow E$ be a Γ -complete concrete functor and $F: P \rightarrow Q'$ be a morphism sending Δ on initial cones; then there exists one unique Γ -morphism $F': Q \rightarrow Q'$ extending F .

The universal (Δ, Γ) -completion is unique (up to isomorphism) if it exists, and it is strictly Γ -generated.

EXAMPLES. If $\Delta = \emptyset$, a universal (Δ, Γ) -completion is called a *free Γ -completion*. If Δ is the class of all initial cones δ in H with $P\delta \in \Gamma$, a universal (Δ, Γ) -completion is just called a *universal Γ -completion*. The free and universal Γ -completions are proved to exist in [107] for Γ associated to a subcategory A of E or to a partial choice μ of limits (Theorem 10), but no explicit construction is given in this last case. Herrlich describes the free initial completion P_2 and the universal initial completion P_3 in [1] and, in [4] the universal completion P^* (for $\Gamma = \text{Lim } E$), whose objects are the «complete sources». Adapting his method as in Section 2, we'll obtain the universal (Δ_0, Γ^0) -completion U of P , with objects the Δ -complete P -sources; the Γ -hull of H for U is the universal (Δ, Γ) -completion of P .

DEFINITION. A P -source $\sigma = (S_I, f_I \mid I \in I)$ is said *Δ -complete* if it contains the P -morphisms

- (a) (S, h, f_I) for each $h: S_I \rightarrow S$ in H ,
- (b) (S', g) if there exists $(d_J: S' \rightarrow S'_J \mid J \in J)$ in Δ_0 with
 $(S'_J, d_J \cdot g)$ in σ for each $J \in J$.

(Intuitively, σ is closed under left composition by H and factors through

the initial sources of Δ_0 .)

PROPOSITION 6. *Each P -source $\sigma = (\Phi, \gamma)$ is included in a smallest Δ -complete P -source, denoted by $\Delta\sigma = (\Delta\Phi, \Delta\gamma)$, which is constructed by transfinite induction. We have $il\sigma = il\Delta\sigma$ as soon as one of them is defined.*

Δ . The P -source $\Delta\sigma$ is the union of the transfinite sequence $(\sigma_\lambda)_\lambda$ where $\sigma_0 = \sigma$, $\sigma_\alpha = \bigcup_{\lambda < \alpha} \sigma_\lambda$ for a limit ordinal α , and $\sigma_{\lambda+1}$ is deduced from σ_λ by adding elements of the form (a) and (b) above. For the last assertion, we prove by induction on σ_λ that, if the P -source $\sigma \circ g = (\Phi, \gamma \circ g)$ lifts into a source $\gamma \circ g: S \Rightarrow \Phi$ in H , then the P -source $\Delta\sigma \circ g$ lifts into a source with basis $\Delta\Phi$ in H . $\forall$

THEOREM 4. *P has a universal (Δ, Γ) -completion $U_\Gamma: L_\Gamma \rightarrow E$ which is Γ -generated, hence Γ° -dense. It lives in the universe $\mathcal{U}$ if so do $Ind\Gamma$ and each of its elements.*

Δ . 1. Construction of a (Δ_0, Γ°) -completion $U: L \rightarrow E$ of P . Let L_0 be the class formed by the Δ -complete P -sources L of the form (Proposition 6) $\Delta(\Phi, \gamma)$ for some $\gamma \in \Gamma^\circ$; let $U(L)$ be the vertex of γ . If L' is also in L_0 , then $g: L' \rightarrow L$ is a morphism in L mapped by U on g iff $(\Phi, \gamma \circ g)$ is included in L' (which implies $L \circ g \subset L'$). We identify H to a full subcategory of L by identifying the object S to

$$\Delta(S, id_{P(S)}) = (S', f \mid f: S \rightarrow S' \text{ in } H).$$

As we have $h: L \rightarrow S$ in L iff (S, h) is in L , it follows that L is the initial lift of (Φ, γ) considered as a U -source, and that $\delta_0 \in \Delta_0$ remains an initial source for U . The fact that U is Γ° -complete is proved as in Theorem 1, thanks to the commutativity of initial lifts and of Γ° .

2. Universality of U . Let $Q: K \rightarrow E$ be a Γ° -complete functor, and $F: P \rightarrow Q$ a morphism sending Δ_0 on initial sources. If there is a Γ° -morphism $F': U \rightarrow Q$ extending F , it maps L on $il(F\Phi, \gamma)$ and $g: L' \rightarrow L$ on $g: F'(L') \rightarrow F'(L)$. So we have just to prove that this F' is well-defined i.e., that

$$il(F\Phi', \gamma') = il(F\Phi, \gamma) \quad \text{if} \quad \Delta(\Phi', \gamma') = L = \Delta(\Phi, \gamma).$$

Indeed $(F\Phi, \gamma)$ generates a $F\Delta$ -complete Q -source σ and $il\sigma = il(F\Phi, \gamma)$ (Proposition 6). From the construction of $\Delta(\Phi, \gamma) = (\Delta\Phi, \Delta\gamma)$, we deduce by induction that $(F\Delta\Phi, \Delta\gamma) \supset (F\Phi', \gamma')$ is included in $\Delta(F\Phi, \gamma) = \sigma$. Therefore

$$\Delta(F\Phi', \gamma') = \sigma \quad \text{and} \quad il(F\Phi', \gamma') = il\sigma = il(F\Phi, \gamma).$$

3. Universal (Δ, Γ) -completion of P . Let $U_\Gamma: L_\Gamma \rightarrow E$ be the restriction of U to the Γ -hull of H for U ; it is a (Δ, Γ) -completion of P . To prove the universality, let $Q': K' \rightarrow E$ be a Γ -complete functor, and $G: P \rightarrow Q'$ a morphism sending Δ on initial cones. We consider the universal Γ° -completion $U': L' \rightarrow E$ of Q' ; as $Q' \hookrightarrow U'$ is a Γ° -morphism, hence a Γ -morphism, $G: P \rightarrow U'$ still sends Δ on initial cones and, by Part 2, it extends into a unique Γ° -morphism $G': U \rightarrow U'$. If G' maps L_Γ into the full subcategory K' of L' its restriction $G'': U_\Gamma \rightarrow Q'$ will be the unique Γ -morphism extending G . This is proved by induction on C_λ , where $L_\Gamma = \bigcup_\lambda C_\lambda$ (Proposition 5): suppose G' maps C_λ into K' ; if L is an object of $C_{\lambda+1}$, we have $L = il(\Psi, \gamma')$, where $\gamma' \in \Gamma$ and Ψ valued in C_λ ; as $G'\Psi$ is valued in K' and Q' is Γ -complete, the Q' -cone $(G'\Psi, \gamma')$ has an initial lift for Q' , which remains an initial lift for the universal Γ° -completion U' , hence is equal to $G'(L) = il(G'\Psi, \gamma')$. It follows that G' maps $C_{\lambda+1}$ into the full subcategory K' . ∇

REMARKS. 1. Suppose $\Gamma = \text{Lim } E$. Then the universal completion U_Γ of P is (Γ) -dense (not only Γ -generated); this has been proved by Adamek-Koubek [1] via a construction which is transfinite only for morphisms, and by Herrlich [4] thanks to his one-step construction. His proof rests on the two facts:

A functor is complete as soon as it lifts products and equalizers;

Let $\sigma = \Delta(\Phi, \gamma)$ with $\gamma \in \text{Lim } E$; any P -source included in σ is also included in a P -source $(\tilde{\Phi}, \tilde{\gamma}) \subset \sigma$ with $\tilde{\gamma}$ a limit-cone [4]; it is easily adapted, whatever be Δ , to prove that L_Γ reduces to the full subcategory C_I of L (Part 3 above), whence:

COROLLARY. *The universal $(\Delta, \text{Lim } E)$ -completion of P is $(\text{Lim } E)$ -dense.*

2. The free initial completion of P is the largest initial completion, while its universal one is the largest preserving initial lifts initial completion [2]. This maximality property is no more valid for a general class Γ ; we only prove as in Theorem 4 the

PROPOSITION 7. *Let $Q: K \rightarrow E$ and $Q': K' \rightarrow E$ be (Δ_0, Γ^0) -completions of P . If Q satisfies*

(1) *For each K in K_0 there exists a P -source (Φ, γ) with $\gamma \in \Gamma^0$ such that $\Delta(\Phi, \gamma) = (S', h \mid h: K \rightarrow S' \text{ in } K)$ and $K = il(\Phi, \gamma)$, then there exists a morphism $Q \rightarrow Q'$ extending the identity on H .*

COROLLARY. *The universal (Δ_0, Γ^0) -completion U of P is the largest (Δ_0, Γ^0) -completion which satisfies (1).*

Another maximality property of the universal (Δ, Γ) -completion is given in Comment 91-1 [0], Part IV-1.

3. The problem of «lifting singleton sources» may be translated in the world of internal functors in a category, leading to universal internal $(\Delta, /A/)$ -completions; cf. /95, 96/ and Synopsis n° 5 [0], Part III-2.

4. Many authors consider only concrete functors which are transportable. The preceding results are easily adapted to this case.

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U. E. R. de Mathématiques
Université de Picardie
80039 AMIENS CEDEX.