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MICHAEL BARR
RADU DIACONESCU

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ON LOCALLY SIMPLY CONNECTED TOPOSES AND THEIR FUNDAMENTAL GROUPS *)

by *Michael BARR and Radu DIACONESCU*

In SGA 1 Grothendieck introduced the notion of the fundamental group of a scheme. In terms of a topos Grothendieck's fundamental group classifies finite coverings of the terminal object. There is some evidence that in the context of schemes (in particular that of fields) all (connected) coverings may be finite. However in a general topos – in particular in the sheaves on a locally simply connected topological space – there are generally infinite coverings and a workable theory of the fundamental group should account for them.

In this paper we give a preliminary report on our investigation of the fundamental group of a topos. We define the notion of a locally simply connected topos and describe the group in that case.

If X is a topological space, the category $Sh(X)$ is locally simply connected in our sense iff X is locally connected and has a universal covering space.

We are working exclusively in the context of a molecular or locally connected topos $\underline{E}$. This means that every object E of $\underline{E}$ can be written $E = \Sigma M_i$ where M_i is a *molecule*, meaning that M_i cannot be written as a sum of two proper subobjects. The M_i are called the *molecules* or *connected components of* E. Let ΛE be the set of all molecules of E. Then $\Lambda: \underline{E} \to Sets$ is a functor since under a map $E \to E'$ the image of a molecule of E cannot be spread across two or more molecules of E', hence we get $\Lambda E \to \Lambda E'$. Assume $\Gamma = Hom(1, -): \underline{E} \to \underline{S} = Sets$ has a left adjoint

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Δ, given by $\Delta n = \sum_n 1$; then Λ is left adjoint to Δ. This is most easily seen for molecules and extended by additivity of Λ. See [Barr-Paré] for details.

Given that $\underline{E}$ is locally connected, it is not very restrictive to suppose that $\underline{E}$ is connected (i.e. $\Lambda 1 = 1$). The reason is that corresponding to $1 = \Sigma C_i$ we get a decomposition of $\underline{E} = \Sigma \underline{E}/C_i$ (the sum in the category of geometric morphisms) and every phenomenon of $\underline{E}$ can be studied on the individual components.

Throughout we will suppose $\underline{E}$ is a complete, connected molecular topos.

DEFINITION 1. Let U and E be objects of $\underline{E}$ such that U has global support (i.e. $U \to 1$ is epi). We say that E *is a locally constant object split by* U if there is an $n \in \underline{S}$ such that $E \times U \approx \Delta n \times U$ in $\underline{E}/U$. (We write $E \approx_U \Delta n$ to describe the above.) We let $Spl(U)$ denote the full subcategory consisting of the locally constant objects split by U.

For convenience we will sometimes say that E is split by U if $E \approx_U \Delta n$ even if U does not have global support (in which case E is not necessarily locally constant).

LEMMA 1. *Let U be a cover of 1 in $\underline{E}$. Then for any $V, W \in \Lambda(U)$ (i.e., V and W are connected components of U) there is a set*

$$V = V_0, V_1, \ldots, V_m = W \in \Lambda(U)$$

such that for $i = 1, \ldots, m$, $V_{i-1} \times V_i \neq \emptyset$.

PROOF. Let $U = \Sigma\{V_i \mid V_i \in \Lambda(U)\}$. Partition $\Lambda(U)$ into two sets I and J where I contains all indices i such that V_i can be «chained» to V in the above manner and J contains all the rest. Then $I \neq \emptyset$ and if also $J \neq \emptyset$ we have

$$V_i \times V_j = \emptyset \quad \text{for all} \quad i \in I,\ j \in J,$$

which implies that the images in 1 of $\sum_{i \in I} V_i$ and $\sum_{j \in J} V_j$ are disjoint and since their sum is 1 this contradicts the connectedness of 1.

LEMMA 2. *If* $V \to U$ *is a morphism in which* V *has global support, then* $Spl(U) \subset Spl(V)$. *The number of «leaves»* n *is the same for* U *and* V.

PROOF. Both squares

$$\begin{array}{ccc} E\times V & \longrightarrow & V \\ \downarrow & & \downarrow \\ E\times U & \longrightarrow & U \end{array} \qquad \begin{array}{ccc} \Delta n\times V & \longrightarrow & V \\ \downarrow & & \downarrow \\ \Delta n\times U & \longrightarrow & U \end{array}$$

are pullbacks.

DEFINITION 2. Let $V \in \Lambda(U)$. We have for $E \in \underline{E}$ an adjunction morphism $\eta = \eta(E\times V): E\times V \to \Delta\Lambda(E\times V)$ which gives

$$\tau_V E = (\eta, p): E\times V \to \Delta\Lambda(E\times V)\times V = T_V(E).$$

Adding this up over all $V \in \Lambda(U)$ we define a functor $T = \Sigma T_V$ and

$$\tau(E) = \Sigma\tau_V(E): E\times U \to \Sigma T_V(E).$$

THEOREM 1. *Let* $\underline{E}$ *be a complete, connected, molecular topos,* U *an object of* $\underline{E}$ *with global support and* E *an arbitrary object of* $\underline{E}$. *Then the following are equivalent:*

(i) $E \in Spl(U)$.

(ii) E *is split by every* $V \in \Lambda(U)$.

(iii) $\tau_V E$ *is an isomorphism for all* $V \in \Lambda(U)$.

(iv) τE *is an isomorphism.*

(v) *There is a morphism* $f: TE \to E$ *such that* $f.\tau E = p$ *(projection).*

(vi) *There is for each* $V \in \Lambda(U)$ *a morphism*

$$f_V: T_V E \to E \quad \textit{such that} \quad f_V.\tau_V E = p.$$

PROOF. We will show that

$$(i) \Leftrightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (ii).$$

(i) => (ii): This is an immediate consequence of Lemma 2.

(ii) => (i): For any $V \in \Lambda(U)$ we have an

$$n_V \in S \quad \text{such that} \quad E\times V \approx \Delta n_V \times V.$$

If also $W \in \Lambda(U)$ suppose that $V\times W \neq 0$ and $Y \in \Lambda(V\times W)$. Then by pulling back as in the proof of Lemma 2, we have $E\times Y \approx \Delta n_V \times Y$ as well as $E\times Y \approx \Delta n_W \times Y$, whence applying Λ, $n_V \approx n_W$. It follows from Lemma 1 $n_V = n$ does not depend on V.

(ii) => (iii): Let $f\colon E\times V \to \Delta n$ be the morphism such that

$$(f,p)\colon E\times V \to \Delta n \times V$$

is an isomorphism. Let $\bar{f}\colon \Lambda(E\times V) \to n$ be the map which corresponds under adjointness. Then

$$\begin{tikzcd}
E\times V \arrow[rr, "{(f,p)}"] \arrow[d, "{\tau E=(\eta,p)}"'] & & \Delta n\times V \\
\Delta\Lambda(E\times V)\times V \arrow[urr, "\Delta\bar{f}\times V"'] & &
\end{tikzcd}$$

commutes. Apply Λ and use the Frobenius isomorphism to obtain

$$\begin{tikzcd}
\Lambda(E\times V) \arrow[r, "\approx"] \arrow[d, "\approx"'] \arrow[dr, "\bar{f}"] & n\times\Lambda(V) \arrow[d, "\approx"] \\
\Lambda(E\times V) \arrow[r] & n
\end{tikzcd}$$

whence $\bar{f}$ is an isomorphism. Hence so is $\Delta\bar{f}\times V$ and so is τE.

(iii) => (iv): Just add up over all V.

(iv) => (v): Let $f = p.\tau E^{-1}$.

(v) => (vi): Compose f with $T_V E \to TE$.

(vi) => (ii): The composite

$$E\times V \xrightarrow{\tau V} \Delta\Lambda(E\times V)\times V \xrightarrow{(f_V,p)} E\times V$$

is the identity. If we pass to the connected topos E/V, this becomes $E \to \Delta n \to E$. Apply $\Delta\Lambda$ to get

$$\begin{tikzcd}
E \arrow[r, tail] \arrow[d, "\eta E"'] & \Delta n \arrow[r, two heads] \arrow[d, "\eta\Delta n"] & E \arrow[d, "\eta E"] \\
\Delta\Lambda E \arrow[r, tail] & \Delta\Lambda\Delta n \arrow[r] & \Delta\Lambda E
\end{tikzcd}$$

and with the middle map isomorphism we get successively that ηE is mono and epi.

We are now going to construct a left adjoint to $Spl(U) \subset \underline{E}$. Let a functor S and a natural transfomation σ be constructed so that

$$\begin{tikzcd} E \times U \arrow[r, "p"] \arrow[d, "\tau E"'] & E \arrow[d, "\sigma E"] \\ TE \arrow[r] & SE \end{tikzcd}$$

is a pushout. It follows easily from the equivalence of (i), (iv) and (v) above that $E \in Spl(U)$ iff σE is an isomorphism iff σE has a left inverse. We observe that since SE is constructed from pushout, $- \times U$, Δ and Λ, all of which commute with arbitrary colimits, so does S. Now let $L_U E = L\ E$ denote the colimit of the diagram

$$\begin{tikzcd} E \arrow[r, "\sigma E"] & SE \arrow[r, shift left, "S\sigma E"] \arrow[r, shift right, "\sigma SE"'] & S^2 E \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & S^3 E \ \ldots \end{tikzcd}$$

THEOREM 2. *For any* $E \in \underline{E}$, $L(E) \in Spl(U)$.

PROOF. We use the above mentioned observation that S commutes with colimit. Thus $L(E)$ and $SL(E)$ are the colimits respectively of the two sequences below.

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\sigma E"'] & SE \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\sigma SE"'] & S^2 E \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\sigma S^2 E"'] & S^3 E \ \ldots \arrow[d, "\sigma S^3 E"] \\ SE \arrow[r] \arrow[ur, dash] & S^2 E \arrow[r, shift left] \arrow[r, shift right] \arrow[ur, dash] & S^3 E \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[ur, dash] & S^4 E \ \ldots \end{tikzcd}$$

The map in one direction is induced by the vertical maps σ as shown. In the other we use the identity maps as shown. The duals of the simplicial face identities – which are evidently satisfied here – show that these both induce maps. Moreover, since the transition maps coequalize all these face maps, the same identities show that the composites induce the identity on $L(E)$ and $SL(E)$ respectively.

PROPOSITION 1. *The inclusion* $Spl(U) \subset \underline{E}$ *preserves equalizers.*

PROOF. Let

$$E \xrightarrow{f} E' \overset{g}{\underset{h}{\rightrightarrows}} E''$$

be an equalizer in $\underline{E}$ with E' and E'' in $Spl(U)$. For any $V \in \Lambda(U)$, let n be the equalizer of $\Lambda(g \times V)$ and $\Lambda(h \times V)$. Then from

$$\begin{tikzcd}
E \times V \arrow[r, "f \times V"] \arrow[d] & E' \times V \arrow[r, shift left, "g \times V"] \arrow[r, shift right, "h \times V"'] \arrow[d, "\approx"] & E'' \times V \arrow[d, "\approx"] \\
\Delta n \times V \arrow[r] & \Delta\Lambda(E' \times V) \times V \arrow[r, shift left, "\Delta\Lambda(g \times V) \times V"] \arrow[r, shift right, "\Delta\Lambda(h \times V) \times V"'] & \Delta\Lambda(E'' \times V) \times V
\end{tikzcd}$$

it follows that $E \times V \approx \Delta n \times V$ and the conclusion follows from theorem 1(ii).

PROPOSITION 2. *Let*

$$\begin{tikzcd}
E_1 \arrow[r, tail] \arrow[d] & F_1 \arrow[d] \\
E \arrow[r, tail] & F
\end{tikzcd}$$

be a pullback. Then E *has a complement in* F *iff* E_1 *has a complement in* F_1.

PROOF. Let E' be the pseudocomplement of E in F and E'_1 the complement of E_1 in F_1. Then it is evident that $E'_1 \supset E' \times_F E_1$. On the other hand,

$$E'_1 \times_F E \approx (E'_1 \times_{F_1} F_1) \times_F E \approx E'_1 \times_{F_1} (F_1 \times_F E) = E'_1 \times_{F_1} E_1 = \emptyset .$$

Thus if E'' is the image of E'_1 in F, $E'' \cap E = \emptyset$ and hence $E'' \subset E'$ from which

$$E' \times_F F_1 \supset E'' \times_F F_1 \supset E'_1$$

so that $E'_1 = E' \times_F F_1$. The inclusion $E + E' \to F$ thus pulls back to an isomorphism and with $F_1 \twoheadrightarrow F$, it had to be an isomorphism.

COROLLARY 1. *If* $U \twoheadrightarrow 1$ *and* $E \rightarrowtail F$ *are such that* $E \times U$ *has a complement in* $F \times U$, *then* E *has a complement in* F.

COROLLARY 2. *Let* $f: E \hookrightarrow F$ *with* E *and* F *in* $Spl(U)$. *Then* E *has a complement in* $Spl(U)$.

PROOF. For any $V \epsilon \Lambda(U)$,

$$\begin{tikzcd} E\times V \arrow[r, hook, "f\times V"] \arrow[d, "\tau_V E"', "\approx"] & F\times V \arrow[d, "\tau_V F", "\approx"'] \\ \Delta\Lambda(E\times V)\times V \arrow[r, "\Delta\Lambda(f\times V)\times V"] & \Delta\Lambda(F\times V)\times V \end{tikzcd}$$

commutes and $\Lambda(f\times V): \Lambda(E\times V) \to \Lambda(F\times V)$ is a mono because $f\times V$ is and has a complement being in $\underline{S}$. Thus $f\times V$ has a complement for each $V \epsilon \Lambda(U)$ so that $f\times U$ does and hence f does. It is clear that the complement is also split by each V.

THEOREM 3. *The functor $L: \underline{E} \to Spl(U)$ is left adjoint to the inclusion.*

PROOF. If $f: E \to F$ is given and $F \epsilon Spl(U)$ we get

$$SE \xrightarrow{Sf} SF \xleftarrow{\approx} F, \quad S^2E \xrightarrow{S^2f} S^2F \xleftarrow{\approx} F$$

etc. which gives $LE \to F$ whose restriction to E is f. To see the uniqueness, it is sufficient to consider the case that E is non-empty and connected. Then

$$\Lambda(\tau_V): \Lambda(E\times V) \to \Lambda(\Delta\Lambda(E\times V)\times V) \approx \Lambda(E\times V)\times\Lambda V \approx \Lambda(E\times V)$$

is an isomorphism whence so are $\Lambda(\tau)$ and $\Lambda(\sigma)$ so that

$$\Lambda S(E) \approx \Lambda(E) = 1.$$

Thus $S(E)$ and similarly each $S^m(E)$ is connected. Since Λ commutes with colimits, $\Lambda L(E) = 1$ so $L(E)$ is connected. Now if two distinct maps $L(E) \rightrightarrows F$ agree on E, their equalizer contains E and is thus non-empty while by Proposition 1 the equalizer lies in $Spl(U)$. By Corollary 2 above, that equalizer has a complement and is evidently non-empty, whence since $L(E)$ is connected, it is all of $L(E)$.

We remark that every object of $Spl(U)$ has global support since $E\times U \approx \Delta n \times V$ implies that the support of $E\times U$, hence of E, is 1.

PROPOSITION 3. *Let $V \epsilon \Lambda(U)$ and $A = L(V)$. Then A is connected and $Spl(V) = Spl(A)$.*

PROOF. From the Frobenius isomorphism, it is immediate that $\Lambda\tau$ is an equivalence, hence so is $\Lambda\sigma$ from which $\Lambda(E) \approx \Lambda L(E)$ follows. Thus A is connected. Next we observe that any object E split by U is split by V, by $U\times V$ and $T(V)$. Moreover from Lemma 2 it follows that the number of «leaves» is the same in each cover of the pushout diagram defining S. Thus any object split by V is split by $S(V)$, hence by $S^m(V)$ and finally by $L(V)$. To go the other way, observe there is a surjection

$$\Delta\Lambda(L(V)\times U)\times U \approx L(V)\times U \to L(V)$$

and use Lemma 2. We remark that Proposition 3 implies that

$$A\times A \xrightarrow{\tau(A)} \Delta\Lambda(A\times A)\times A$$

is an isomorphism.

PROPOSITION 4. *$\Lambda(A\times A) \approx \Gamma(A^A) = Hom(A,A)$ and is a group. In particular, every endomorphism of A is an automorphism.*

PROOF. Let $\alpha A: \Delta\Lambda(A\times A)\times A \to A$ be a map such that

$$(\alpha A, p_2) = \tau(A)^{-1}.$$

By adjunction, this corresponds to a map

$$\bar\alpha(A): \Lambda(A\times A) \to \Gamma(A^A).$$

We claim that $\bar\alpha(A)$ is an isomorphism. Let $u: 1 \to \Gamma(A^A)$ be given and $\tilde u: A \to A$ the map which corresponds. Then

$$\Lambda(\tilde u, A): \Lambda(A) \to \Lambda(A\times A).$$

PROPOSITION 5. *Let $E \twoheadrightarrow E' \rightarrowtail F$ with both E and F in $Spl(U)$. Then so is E'.*

PROOF. Let n be the image as indicated in

$$\Lambda(E\times U) \twoheadrightarrow n \rightarrowtail \Lambda(F\times U).$$

Then we have

$$\begin{tikzcd}
\Delta\Lambda(E\times U)\times U \arrow[r, two heads] & \Delta n\times U \arrow[r, tail, two heads] & \Delta\Lambda(F\times U)\times U \\
E\times U \arrow[u, "\approx"] \arrow[r, two heads] & E'\times U \arrow[u] \arrow[r, tail, two heads] & F\times U \arrow[u, "\approx"']
\end{tikzcd}$$

from which the result follows.

PROPOSITION 6. *Every endomorphism of A is an automorphism.*

PROOF. It follows from Corollary 2 and Propositions 3 and 5 that every endomorphism is epi. Now if $f: A \to A$,

$$\Lambda(f \times A): \Lambda(A \times A) \to \Lambda(A \times A)$$

is surjective in $\underline{S}$, hence has a right inverse. This implies that $f \times A$: $A \times A \to A \times A$ has a right inverse g. Then for $\delta: A \to A \times A$ the diagonal map,

$$A = p_1 \delta = p_1 . f \times A . g . \delta = f . p_1 . g . \delta .$$

But then $p_1 . g . \delta : A \to A$ is both an epi and a mono, hence an isomorphism, whence f is too.

PROPOSITION 7. *$Spl(A)$ consists of the objects with an A-presentation, i.e. all objects which are a coequalizer of a pair of maps*

$$\Delta m \times A \rightrightarrows \Delta n \times A .$$

PROOF. From Theorem 1 (v) and the fact that the projection p is an epi, it follows that every $E \in Spl(A)$ is a quotient of

$$\Delta TE = \Delta \Lambda(E \times A) \times A = \Delta n \times A .$$

The kernel pair is in $Spl(A)$ and is a subobject of

$$\Delta n \times A \times \Delta n \times A \approx \Delta n \times \Delta n \times \Delta \Lambda(A \times A) \times A \approx \Delta(n \times n \times \Lambda(A \times A)) \times A .$$

The only subobjects which belong to $Spl(A)$ are of the form $\Delta m \times A$. For the converse, it suffices to observe that the inclusion of $Spl(A) \to \underline{E}$ preserves coequalizers which is proved in the same way as Proposition 3.

PROPOSITION 8. *The inclusion $Spl(A) \to \underline{E}$ has a right adjoint R.*

PROOF. See [Barr, 1978], Section 2, especially (2.6) and (2.7).

COROLLARY 3. *$Spl(A)$ is an atomic topos with A as a generator. $Spl(A)$ is equivalent to $\underline{S}^{Aut(A)}$.*

PROOF. Let $I: Spl(A) \to \underline{E}$ be the inclusion. Then $Spl(A)$ is the category of coalgebras for the left exact cotriple arising from $I \dashv R$. It is clear that

$\Delta : \underline{S} \to \underline{E}$ factors through $Spl(A)$ from which it is evident that

$$\Delta I \dashv R\Delta \dashv \Gamma I.$$

The sequence

$$\Delta\Lambda(A\times E)\times A \approx A\times E \twoheadrightarrow E$$

shows that A is a generator. Since every object of $Spl(A)$ is a sum of indecomposables which are evidently in $Spl(A)$ and by Corollary 2 irreducible in $Spl(A)$, it follows that $Spl(A)$ is atomic. With $End(A) = Aut(A)$ the result follows from Giraud's characterization of toposes [Barr, 1971, Appendix], together with the fact that there are no non-trivial topologies on a group.

THEOREM 4. *A is a ΔG-torsor in $\underline{E}$.*

PROOF. The isomorphism $A\times A \approx \Delta G\times A$ is evidently true for the object A corresponding to G in $\underline{S}^G$.

COROLLARY 4. *If $f, g: E \to A$ are two «elements» of A defined over the non-empty object E, there is a unique $h \in Aut(E)$ with $hf = g$.*

THEOREM 5. *The diagram*

$$\begin{array}{ccc} \underline{E}/A & \longrightarrow & \underline{S} \\ \downarrow & & \downarrow \\ \underline{E} & \xrightarrow{R} & \underline{S}^G \end{array}$$

is a pullback.

PROOF. $\underline{S}$ is equivalent to $\underline{S}^G/A$ (recall A is the G-set G).

The significance of this fact was pointed out by M. Tierney. The way to understand this theorem is that A is the universal covering space for those coverings that are split by U; that A, hence the topos $\underline{S}^G/A$ is the universal covering in $\underline{S}^G$ and that this is preserved under pullback.

PROPOSITION 9. *The inclusion of $Spl(A) \to \underline{E}$ preserves exponentiation. Hence R is a molecular morphism.*

PROOF. Let $E, F \in Spl(A)$. Then, for any $D \in \underline{E}$,

$$\begin{aligned}
&Hom_{\underline{E}}(D, E^F \times A) \approx Hom_{\underline{E}/A}(D\times A, E^F \times A) \approx \\
&Hom_{\underline{E}/A}(D\times A, (E\times A)^{(F\times A)}) \approx \\
&Hom_{\underline{E}/A}(D\times A, (\Delta\Lambda(E\times A)\times A)^{\Delta\Lambda(F\times A)\times A}) \approx \\
&Hom_{\underline{E}/A}(D\times A, \Delta\Lambda(E\times A)^{\Delta\Lambda(F\times A)}\times A) \approx \\
&Hom_{\underline{E}}(D, \Delta\Lambda(E\times A)^{\Delta\Lambda(F\times A)}\times A) \approx \\
&Hom_{\underline{E}}(D, \Delta(\Lambda(E\times A)^{\Lambda(F\times A)})\times A)
\end{aligned}$$

[Barr-Paré, Theorem 15] so that $E^F \in Spl(A)$ and the conclusion follows.

PROPOSITION 10. *Let $Spl(A) \subset Spl(B)$. Then the inclusion is logical and has left and right adjoints. It is induced by a surjection*

$$Aut(B) \twoheadrightarrow Aut(A).$$

PROOF. Consider the functors

$$\begin{tikzcd}
Spl(A) \arrow[rr, "I_{B,A}"] \arrow[dr, hook, "I_A"] & & Spl(B) \arrow[dl, hook', "I_B"'] \\
& \underline{E} \arrow[ul, bend left, "L_A"] \arrow[ul, bend right, "R_A"'] \arrow[ur, bend right, "L_B"'] \arrow[ur, bend left, "R_B"] &
\end{tikzcd}$$

We have, for $E \in Spl(A)$, $F \in Spl(B)$,

$$\begin{aligned}
Hom(I_{B,A}E, F) \approx Hom(I_B I_{B,A} E, I_B F) \approx Hom(I_A E, I_B F) \approx \\
Hom(E, R_A I_B F),
\end{aligned}$$

so that $I_{B,A} \dashv R_A I_B$. Similarly, $L_A I_B \dashv I_{B,A}$. For the rest, the fact that $A \in Spl(B)$ is an atom implies there is an epimorphism $f: B \to A$. For any $g: B \to B$ there is, by the corollary of Theorem 4, a unique $\phi(g): A \to A$ such that

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "f"] \\
A \arrow[r, "\phi(g)"] & A
\end{tikzcd}$$

commutes. From the uniqueness of ϕ it readily follows that ϕ is a homomorphism. The surjectivity of ϕ can be readily inferred from the fact that B is projective in $Spl(B)$.

Let $Spl(A) \subset \underline{E} \supset Spl(B)$. Then $C = A \times B$ is an object of global support and evidently $Spl(A) \subset Spl(C) \supset Spl(B)$. In other words the class of subcategories of $\underline{E}$ of the form $Spl(A)$ is filtered. It is also small. For let H be an object where subobjects generate $\underline{E}$. Given $U \twoheadrightarrow 1$, we can find an epi $\Sigma H_i \twoheadrightarrow U$ where each $H_i \subset H$. Clearly $Spl(U) \subset Spl(\Sigma H_i)$. Moreover, if any H_i is repeated, it may be omitted without changing the class of objects split. Thus every subcategory of the form $Spl(U)$ is contained in a subcategory $Spl(\Sigma H_i)$ as ΣH_i ranges over all irredundant sums of subobjects of H with global support, of which there is only a set. We let $Spl(\underline{E})$ denote the union of all the subcategories of the form $Spl(U)$. $Spl(\underline{E})$ is a disjoint union of toposes along logical morphisms and is an atomic topos. We say that $\underline{E}$ *is locally simply connected* if there is a $U \twoheadrightarrow 1$ such that $Spl(U) = Spl(\underline{E})$.

THEOREM 6. *The following are equivalent for a molecular Grothendieck topos* $\underline{E}$:

(i) $\underline{E}$ *is locally simply connected.*

(ii) $Spl(\underline{E})$ *is cocomplete.*

(iii) $Spl(\underline{E}) \to \underline{E}$ *has a left adjoint.*

(iv) $Spl(\underline{E}) \to \underline{E}$ *has a right adjoint.*

PROOF. If (i) holds, $Spl(\underline{E}) = Spl(U)$ so that the other three hold. Each of (iii) and (iv) imply (ii). If $Spl(\underline{E})$ is cocomplete, let $\{A_i\}$ range over a set of generating atoms so that $Spl(\underline{E}) = \cup Spl(A_i)$. Let $A = \Sigma A_i$ in $Spl(\underline{E})$. Then there is a $U \in \underline{E}$ such that U splits A. Since $A_i \rightarrowtail A$, U also splits A_i. If $E \in Spl(\underline{E})$, $E \in Spl(A_i)$ so there is a presentation

$$\Delta m \times A_i \rightrightarrows \Delta n \times A_i \longrightarrow E$$

from which U also splits E.

EXAMPLES. Let X be a topological space. A continuous map $p: Y \to X$ is called a *covering* if each point $x \in X$ has a neighborhood U_x such that

$p^{-1}(U)$ is the disjoint union of a family of subsets of Y each of which is mapped homeomorphically by p onto U_x. It is clear that such a p is a sheaf on X and that if U is the disjoint union of the U_x, p is a locally constant sheaf split by U. When X is connected – which we henceforth assume – a covering is called trivial if it is split by X itself. X is said to be *simply connected* if X is locally connected and every covering is trivial. It is shown in standard texts that any contractible space is simply connected. We say that *X is locally simply connected* if it is locally connected and every point has a simply connected neighborhood. The disjoint union of such neighborhoods will split every covering so that the topos of sheaves is locally simply connected. It is not altogether clear to us that a space which has a cover $U \twoheadrightarrow X$ that splits every covering is locally connected in the sense of [Chevalley], namely that every point has a simply connected neighborhood. If there is such a V there is an A which corresponds and A is a simply connected covering which is a universal covering space. In any case, whenever the space has a universal covering space, the fundamental group as defined by Chevalley is the group of automorphisms of the universal covering.

On the other hand, the fundamental group defined as homotopy classes of closed paths might not be the same. An example is given by the long circle. To define this we describe the long line as the space

$$\Omega_1 \times [0,1] \cup \Omega_1 \quad \text{modulo the relation } (\alpha,1) \sim (\alpha+1,0) \text{ for } \alpha \in \Omega_1 .$$

This is ordered by:

$$(\alpha,t) < (\beta,u) \text{ if } \alpha = \beta \text{ and } t < u \quad \text{while } (\alpha,t) < \Omega_1 \ \forall x,\ \forall t.$$

This space equipped with the order topology is the long line. Think of it as the set of all countable ordinals plus the first uncountable one with an interval between each x and $x+1$. The long circle is the space gotten by identifying Ω_1 with $(0,0)$. Although the space has a hole in it, the hole is too big to be surrounded by a path so the path fundamental group is trivial. On the other hand the space has a covering by a space made up of countably many copies of the long line laid end to end and that space is clearly connected. It can readily be seen that it is also simply connected, from which it is trivial to see that the Chevalley fundamental group is Z.

Let X be the space consisting of infinitely many circles joined at a point with radii shrinking to 0, topologized as a subset of the plane. The space is connected and locally connected but no neighborhood of the common point is simply connected, nor does the topos of sheaves satisfy our possibly weaker form of local simple connectivity. A covering space $p: Y \to X$ must split some $U \twoheadrightarrow X$. Since U is a quotient of a sum of open subsets of X, each of which must split p, it follows that p splits over a neighborhood of the common point. This neighborhood contains all but finitely many of the circles and over this neighborhood Y must be trivial. The remaining finitely many circles may be covered by arcs and so there may be loops in Y over there. We omit the details, but for a cover U which covers all the circles but the first n, the category $Spl(U)$ is $\underline{S}^G$ where G is free on n generators. The category $Spl(Sh(X))$ is the union of these categories.

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Department of Mathematics
McGill University
805 Sherbrooke St. W.
MONTREAL, P.Q.
CANADA H3A 2K6

Department of Mathematics
Marymount College
TARRYTOWN, N.Y. 10591
U.S.A.