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OPEN COVERS AND INFINITARY OPERATIONS IN C^∞-RINGS [1)]

by Eduardo J. DUBUC

INFINITE ADDITIONS.

In the C^∞-ring $C^\infty(\mathrm{R}^n)$ of smooth real valued functions, one can define elements (functions) *by adding certain infinite families*. A typical application of this method is the following: Suppose $M \to \mathrm{R}^n$ is a closed smooth submanifold, and let $h: M \to \mathrm{R}$ be a smooth function defined on M. By definition this means that there are open sets $U_\alpha \subset \mathrm{R}^n$, $\alpha \in \Gamma$, such that they cover M, and smooth functions $h_\alpha: U_\alpha \to \mathrm{R}$ such that

$$\forall p \in U_\alpha \cap M, \quad h_\alpha(p) = h(p).$$

Let U_0 be $\mathrm{R}^n - M$, and take any function (e.g. the zero function) $h_0: U_0 \to \mathrm{R}$. We then have an open covering U_α, $\alpha \in \Gamma + \{0\}$, of the whole space R^n, and functions $h_\alpha: U_\alpha \to \mathrm{R}$ such that

$$\forall p \in U_\alpha \cap M, \quad h_\alpha(p) = h(p).$$

(since $U_0 \cap M = \emptyset$). This family does not agree in the intersections $U_\alpha \cap U_\beta$; thus it does not define a global function $f: \mathrm{R}^n \to \mathrm{R}$ which extends h. However, we can construct an extension of h. Let W_i be a locally finite refinement of U_α, for each i take α such that $W_i \subset U_\alpha$, and let $g_i: W_i \to \mathrm{R}$ be equal to $h_\alpha \mid W_i$. Let ϕ_i be an associated partition of unity. The functions $f_i = \phi_i g_i$ are defined globally, $f_i: \mathrm{R}^n \to \mathrm{R}$, and have support contained in W_i. (This is so since $support(\phi_i) \subset W_i$ by definition.) It follows that the equality $f = \sum_i f_i$ defines a function $f: \mathrm{R}^n \to \mathrm{R}$ such that

$$\forall x \in \mathrm{R}^n, \quad f(x) = \sum_i f_i(x)$$

(which is a *finite sum* on an open neighborhood of x). If we compute f at

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a point $p \in M$, we have:

$$f(p) = \sum_i f_i(p) = \sum_{i \mid p \in W_i} f_i(p) = \sum_{i \mid p \in W_i} \phi_i(p) g_i(p)$$

$$= \sum_{i \mid p \in W_i} \phi_i(p) h(p) = h(p) \sum_i \phi_i(p) = h(p).$$

Thus f is an extension of h.

Two questions arise:

I. Which is the algebraic meaning of these infinitary additions?

II. To which extent can they be defined and performed in arbitrary C^∞-rings of finite type $A = C^\infty(\mathrm{R}^n)/I$?

In particular, can we give a meaning to expressions of the form $a = \sum_\alpha a_\alpha$, $a_\alpha \in A$? In a way such that if $a_\alpha = [f_\alpha]$, $f_\alpha \in C^\infty(\mathrm{R}^n)$, and the f_α can be added in $C^\infty(\mathrm{R}^n)$ defining a function $f = \sum_\alpha f_\alpha$ (as before), then $a = [f]$ (where brackets indicate equivalence class modulo I). *That this is not always possible is seen as follows:* Let

$$I = (h \mid h \text{ is of compact support})$$

and let ϕ_α be a partition of unity such that $\phi_\alpha \in I$ for all α. Let

$$a_\alpha = [\phi_\alpha] \quad \text{and} \quad a = \sum_\alpha a_\alpha .$$

Then $a = 1$ since $1 = \sum_\alpha \phi_\alpha$ in $C^\infty(\mathrm{R}^n)$. But also $a_\alpha = [0]$ and then $a = 0$ since $0 = \sum_\alpha 0$ in $C^\infty(\mathrm{R}^n)$. Thus $1 = 0$ in A, which is impossible since $1 \notin I$. We see that a condition is needed in the ideal that presents A. Namely, that if $l_\alpha \in I$ and the l_α can be added in $C^\infty(\mathrm{R}^n)$ defining a function $l = \sum_\alpha l_\alpha$ (as before), then $l \in I$. This leads to the notion of *ideal of local character* (cf. Definition 6, iv).

ANSWER TO QUESTION II.

The first step is to define the *open cover topology* in the category $\mathcal{A}_{f.t}^{op}$, dual of the category of C^∞-rings of finite type. Given a C^∞-ring A and an element $a \in A$, we denote $A \to A\{a^{-1}\}$ the solution in $\mathcal{A}$ to the universal problem of making a invertible (cf. [1, 2]). The following is straightforward:

1. PROPOSITION. *Given any morphism* $\phi: A \to B$ *and element* $a \in A$, *the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\phi"'] & A\{a^{-1}\} \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

is a pushout diagram iff $C = B\{\phi(a)^{-1}\}$.

We recall the following

2. PROPOSITION. *Let* $U \subset R^n$ *be an open set, and let* f *be such that* $U = \{x \mid f(x) \neq 0\}$. *Then* $C^\infty(\mathrm{R}^n) \to C^\infty(U) = C^\infty(\mathrm{R}^n)\{f^{-1}\}$.

PROOF. The basic idea is to consider the map

$$U \to \mathrm{R}^{n+1}: p \mapsto (p, f(p)^{-1})$$

which makes U the closed sub-manifold of R^{n+1} defined by the equation $1 - x_{n+1}f(p) = 0$. Then use the fact that this equation is independent to deduce that the equalizer is preserved when taking the C^∞-rings of smooth functions (cf. [1, 2]). A different proof of the preservation of this equalizer is given in [4].

3. DEFINITION. The *open cover topology in* $\mathfrak{A}^{op}_{f.t}$ is the topology generated by the empty family covering $\{0\}$, and families of the form

$$C^\infty(\mathrm{R}^n) \to C^\infty(U_\alpha), \quad \text{for all } n \text{ and all open coverings } U_\alpha \text{ of } \mathrm{R}^n.$$

It follows from Propositions 1 and 2 that basic (generating) covers of an arbitrary C^∞-ring of finite type A are families $A \to A\{a_\alpha^{-1}\}$ which can be completed into pushout diagrams

$$\begin{tikzcd} C^\infty(\mathrm{R}^s) \arrow[r] \arrow[d, "\phi"'] & C^\infty(V_\alpha) \arrow[d] \\ A \arrow[r] & A\{a_\alpha^{-1}\} \end{tikzcd}$$

where $V_\alpha = \{x \mid g_\alpha(x) \neq 0\}$ is an open cover of R^s and $\phi(g_\alpha) = a_\alpha$. The morphism ϕ can be chosen to be a quotient map. Let $A = C^\infty(\mathrm{R}^n)/I$; then, since $C^\infty(\mathrm{R}^s)$ is free, there is a smooth function $f: \mathrm{R}^n \to \mathrm{R}^s$ making the following triangle commutative :

$$\begin{tikzcd} C^\infty(\mathrm{R}^s) \arrow[r, "f^*"] \arrow[dr, "\phi"'] & C^\infty(\mathrm{R}^n) \arrow[d] \\ & A \end{tikzcd}$$

One checks then that the following diagrams are pushouts :

$$\begin{tikzcd} C^\infty(\mathrm{R}^n) \arrow[r] \arrow[d] & C^\infty(U_\alpha) \arrow[d] \\ A \arrow[r] & A\{a_\alpha^{-1}\} \end{tikzcd}$$

where

$$U_\alpha = f^{-1}(V_\alpha) = \{ x \mid f_\alpha(x) \neq 0 \}, \quad f_\alpha = g_\alpha f,$$

is an open cover of R^n and $a_\alpha = [f_\alpha]$.

Open covers $C^\infty(\mathrm{R}^n) \to C^\infty(U_\alpha)$ are *effective epimorphic families.* This just means that given a compatible family $f_\alpha \in C^\infty(U_\alpha)$ (meaning that they agree in the intersections), there exists a unique $f \in C^\infty(\mathrm{R}^n)$ such that $f| U_\alpha = f$. (Remark that $C^\infty(U_\alpha \cap U_\beta)$ is the pushout of $C^\infty(U_\alpha)$ with $C^\infty(U_\beta)$ over $C^\infty(\mathrm{R}^n)$.) However, they are not *universal*. Open covers of an arbitrary A will not, in general, be effective epimorphic families. For example, let as before I be the ideal

$$I = (h \mid h \text{ is of compact support}),$$

and let ϕ_α be a partition of unity such that $\phi_\alpha \in I$ for all α. Then the diagrams

$$\begin{tikzcd} C^\infty(\mathrm{R}^n) \arrow[r] \arrow[d] & C^\infty(U_\alpha) \arrow[d] \\ A \arrow[r] & \{0\} = A\{a_\alpha^{-1}\} \end{tikzcd}$$

are pushout diagrams for all α, where U_α is the open covering

$$U_\alpha = \{ x \mid \phi_\alpha(x) \neq 0 \}, \quad A = C^\infty(\mathrm{R}^n)/I, \text{ and } a_\alpha = [\phi_\alpha] = 0.$$

On the way we see that the empty family covers A since it covers $\{0\}$

(use composition of coverings).

Consider now an open covering of an arbitrary C^∞-ring $A = C^\infty(\mathrm{R}^n)/I$

$$\begin{tikzcd} C^\infty(\mathrm{R}^n) \arrow[r] \arrow[d] & C^\infty(U_\alpha) \arrow[d] \\ A \arrow[r] & A\{a_\alpha^{-1}\} \end{tikzcd}$$

$$U_\alpha = \{ x \mid f_\alpha(x) \neq 0 \} \text{ and } a_\alpha = [f_\alpha].$$

For $b \in A$, we denote $b|_\alpha$ its image in $A\{a_\alpha^{-1}\}$. Suppose $b, b' \in A$ are such that $b|_\alpha = b'|_\alpha$ for all α. Let $b = [f]$ and $b' = [f']$. Then

$$[f|U_\alpha] = b|_\alpha \text{ and } [f'|U_\alpha] = b'|_\alpha.$$

Thus $[f|U_\alpha] = [f'|U_\alpha]$, which means that $(f-f')|U_\alpha \in I|U_\alpha$. If the covering is going to be effective epimorphic, it should follow that $(f-f') \in I$. This leads to the notion of *ideal of local character* (cf. Definition 6, ii).

We recall the following elementary facts in order to fix the notation.

4. PROPOSITION. *Let A be a C^∞-ring and $p: A \to \mathrm{R}$ a morphism into R, which we will also call a point of A. We denote $A \to A_p$ the solution in the category of C^∞-rings to the universal problem of making invertible all the elements $a \in A$ such that $p(a) \neq 0$. There is a factorization of p, $A \to A_p \to \mathrm{R}$, and A_p is a C^∞-local ring. If $a \in A$, we denote $a|_p \in A_p$ its image in A_p. Suppose $A = C^\infty(\mathrm{R}^n)/I$, let $U \subset \mathrm{R}^n$ open, f such that $U = \{ x \mid f(x) \neq 0 \}$, and $a = [f]$. Consider the following diagram (where the upper square is a pushout):*

$$\begin{tikzcd} C^\infty(\mathrm{R}^n) \arrow[r] \arrow[d] & C^\infty(U) \arrow[d] \\ A \arrow[r] \arrow[d] & A\{a^{-1}\} \arrow[d, dashed, "p"] \arrow[dl, dashed] \\ A_p \arrow[r] & \mathrm{R} \end{tikzcd}$$

Given a point p of A, $p: A \to \mathrm{R}$, since $C^\infty(\mathrm{R}^n)$ is free, p can be identified with a point of $Zeros(I) \subset \mathrm{R}^n$ which we will also denote p. When *there exist factorizations (necessarily unique) as shown in the dotted arrows,* p *can be identified with a point of* $A\{a^{-1}\}$, *which we will also denote by* p. *Then*

$$\text{p is a point of } A\{a^{-1}\} \iff p(a) \neq 0 \iff p \in U.$$

It follows that if $A \to A\{a_\alpha^{-1}\}$ *is an open cover and* p *is a point of* A *then there exists* α *such that* p *is a point of* $A\{a_\alpha^{-1}\}$ *(since there exist* α *such that* $p \in U_\alpha$ *).*

PROOF. This is all rather straightforward. For a proof, cf. [2, Exposé 11].

Suppose now $b, b' \in A$ are such that $b|_p = b'|_p$ for all points p of A. Let $b = [f]$ and $b' = [f']$. Then

$$b|_p = [f|_p] \quad \text{and} \quad b'|_p = [f'|_p].$$

Thus $[f|_p] = [f'|_p]$, which means that $(f-f')|_p \in I|_p$. If we want to deduce that $b = b'$, it should follows that $(f-f') \in I$. This leads to the notion of *ideal of local character* (cf. Definition 6, i).

5. DEFINITION. A family $l_i \in C^\infty(\mathbb{R}^n)$ (indexed by an arbitrary set) is *locally finite* if there is an open covering U_α such that, for every α, $l_i | U_\alpha = 0$ except for a finite number of i. Equivalently, if each point p of $\mathbb{R}^n$ has an open neighborhood U such that $l_i | U = 0$ except for a finite number of i.

Given a locally finite family l_i, the finite sums $\sum_i l_i | U_\alpha \in C^\infty(U_\alpha)$ form a compatible family. Thus there exists a unique

$$l \in C^\infty(\mathbb{R}^n) \quad \text{such that} \quad l \mid U_\alpha = \sum_i l_i \mid U_\alpha .$$

We denote this l by the formula $l = \sum_i l_i$.

6. DEFINITION. An ideal $I \in C^\infty(\mathbb{R}^n)$ is of *local character* if it satisfies any one of the following equivalent conditions:

i) $(f|_p \in I|_p$ for all $p \in Zeros(I)) \Rightarrow f \in I$.

ii) $(f | U_\alpha \in I | U_\alpha$ for some open cover $U_\alpha) \Rightarrow f \in I$.

iii) $\phi_\alpha f \in I$ for some partition of unity $\phi_\alpha \Rightarrow f \in I$.

iv) $l_i \in I$ for all $i \Rightarrow (\sum_i l_i) \in I$ for every locally finite family l_i.

PROOF OF THE EQUIVALENCE. The antecedents of the first three implications are themselves equivalent, cf. [1], Lemma 10, and for detailed proof, [2] Théorème 1.5; thus the equivalence of the first three conditions. That iv $\Rightarrow$ iii is immediate since $\phi_a f$ is a locally finite family and $f = \Sigma \phi_a f$. Finally, it is evident that ii $\Rightarrow$ iv. Notice that the implications in the first three conditions always hold in the other sense.

It is clear that any ideal I has a «closure» of local character; namely

$$\hat{I} = \{ f \mid f_{|p} \in I_{|p} \ \ \forall p \in Zeros(I) \}.$$

$\hat{I}$ can also be seen as the closure of I under additions of locally finite families. Let $\mathfrak{B}$ be the category of C^{∞}-rings presented by an ideal of local character. Let

$$A = C^{\infty}(\mathrm{R}^n)/I \quad \text{and} \quad rA = C^{\infty}(\mathrm{R}^n)/\hat{I}.$$

Then we have a canonical (quotient) map $A \to rA$ and the passage $A \to rA$ is clearly a left adjoint for the inclusion $\mathfrak{B} \to \mathfrak{A}_{f.t}$. Thus $\mathfrak{B}$ is closed under all inverse limits and has all colimits. These colimits will not coincide in general with the respective construction in $\mathfrak{A}_{f.t}$. We remark that, since $Zeros(I) = Zeros(\hat{I})$, A and rA have the same points.

7. EXAMPLES. 1° Let $A = C_0^{\infty}(\mathrm{R})$, $B = C^{\infty}(\mathrm{R})$. Then $A, B \in \mathfrak{B}$. By construction, $A \otimes_{\infty} B = C^{\infty}(\mathrm{R}^2)/I$, where

$$I = \{ f \mid \exists \eta > 0, \ f(x,y) = 0 \ \ \forall x \mid |x| < \eta \}.$$

This ideal is not of local character:

$$\hat{I} = \{ f \mid f = 0 \text{ on an arbitrary neighborhood of the } y\text{-axis} \}.$$

The coproduct of A with B in $\mathfrak{B}$ does not coincide then with the one performed in $\mathfrak{A}_{f.t}$.

2° Let $A = C_0^{\infty}(\mathrm{R})$ and $a = x|_0$. Then

$$A\{a^{-1}\} = C_0^{\infty}(\mathrm{R})\{x|_0{}^{-1}\} = C^{\infty}(\mathrm{R}^*)/J,$$

where $\mathrm{R}^* = \mathrm{R} - \{0\}$ and

$$J = \{ f \mid \exists U \mid 0 \in U, \ f|U = 0 \} \subset C^{\infty}(\mathrm{R}^*).$$

We see this because the following is a pushout diagram:

$$\begin{tikzcd} C^\infty(\mathbb{R}) \arrow[r] \arrow[d] & C^\infty(\mathbb{R}^*) = C^\infty(\mathbb{R})\{x^{-1}\} \arrow[d] \\ C_0^\infty(\mathbb{R}) \arrow[r] & C^\infty(\mathbb{R}^*)/J = C_0^\infty(\mathbb{R})\{x|_0{}^{-1}\} \end{tikzcd}$$

(cf. Proposition 1). Now, the ring $A\{a^{-1}\}$ has the presentation

$$A\{a^{-1}\} = C^\infty(\mathbb{R}^2)/(I, 1-xy),$$

where I is the ideal in 1 above. Thus the localization $A\{a^{-1}\}$ in $\mathfrak{B}$ does not coincide with the one perfomed in $\mathfrak{A}_{f.t}$. Since $Zeros(I, 1-xy) = \emptyset$, we have that $1 \in (I, 1-xy)^{\sim}$. Thus $A\{a^{-1}\} = \{0\}$ in $\mathfrak{B}$. *This means that if $\phi : C_0^\infty(\mathbb{R}) \to B$ is any morphism, B is in $\mathfrak{B}$, and $\phi(x|_0)$ invertible, then $B = \{0\}$*. This is not so if B is not in $\mathfrak{B}$.

In what follows we will utilize the *same notation* as in $\mathfrak{A}_{f.t}$ for the constructions performed in $\mathfrak{B}$. Since they are defined by the same universal properties, we have:

8. PROPOSITION. *Propositions 1 and 4 remain valid for the category $\mathfrak{B}$. In addition, since all finitely generated ideals are of local character (cf. [1, 2]), if A is of finite presentation, then for any $a \in A$, $A\{a^{-1}\}$ constructed in $\mathfrak{A}_{f.t}$ is already in $\mathfrak{B}$. Thus Proposition 2 remains valid also for the category $\mathfrak{B}$. Furthermore, if A is in $\mathfrak{B}$, then for any elements $b, b' \in A$,*

$$b = b' \text{ iff } b|_p = b'|_p \text{ for all points } p \text{ of } A.$$

9. PROPOSITION. *The open coverings are universal effective epimorphic families in the category $\mathfrak{B}^{op}$.*

PROOF. The proof is essentially a repetition of the argument given at the beginning of this article. Let $A = C^\infty(\mathbb{R}^n)/I$, I of local character, U_α an open cover of $\mathbb{R}^n$, f_α such that $U_\alpha = \{ x \mid f_\alpha(x) \neq 0 \}$ and $a_\alpha \in A$, $a_\alpha = [f_\alpha]$. We consider the pushout diagram in $\mathfrak{B}$:

$$\begin{tikzcd} C^\infty(\mathbb{R}^n) \arrow[r] \arrow[d] & C^\infty(U_\alpha) \arrow[d] \\ A \arrow[r] & A\{a_\alpha^{-1}\} \end{tikzcd}$$

Let $b_\alpha \in A\{a_\alpha^{-1}\}$ be a compatible family. This implies that for all points of $A\{a_\alpha^{-1}, a_\beta^{-1}\}$, $b_{\alpha|p} = b_{\beta|p}$. Thus, for all p in A there is a well defined $b(p) \in A_p$, $b(p) = b_{\alpha|p}$ for any α such that p is in A_α. We shall construct an element

$$b \in A \quad \text{such that} \quad b_{|p} = b(p) \text{ for all } p \text{ in } A.$$

Let $h_\alpha \in C^\infty(U_\alpha)$ such that $b_\alpha = [h_\alpha]$, let W_i be a locally finite refinement of U_α, for each i take α such that $W_i \subset U_\alpha$, and let $g_i \in C^\infty(W_i)$ be equal to $h_\alpha | W_i$. Thus, for all p in W_i, $[g_{i|p}] = b(p)$. Let ϕ_i be an associated partition of unity. The functions $l_i = \phi_i g_i$ are defined globally, $l_i \in C^\infty(\mathbb{R}^n)$, and have support contained in W_i. Thus, given any p in A, if $p \notin W_i$, $l_{i|p} = 0$, and if $p \in W_i$,

$$[l_{i|p}] = [\phi_{i|p}]\, b(p),$$

Since the family l_i is locally finite (Definition 5), we have a function $l = \sum_i l_i$. Let $b = [l]$. Then, for any p in A,

$$b_{|p} = [\textstyle\sum_i l_{i|p}] = \sum_{i \,|\, p \in W_i} [l_{i|p}] = b(p) \sum [\phi_{i|p}] = b(p).$$

Given any α, $b_{|\alpha} = b_\alpha$ since for all p in $A\{a_\alpha^{-1}\}$, $b_{|p} = b_{\alpha|p}$ and $A\{a_\alpha^{-1}\}$ is in $\mathfrak{B}$. In the same way one checks the uniqueness of such a, b, since all points of A are in some $A\{a_\alpha^{-1}\}$. This finishes the proof.

10. DEFINITION - PROPOSITION. Given any C^∞-ring A presented by an ideal of local character, a family $b_i \in A$ (indexed by an arbitrary set) is *locally finite* if there is an open covering $A \to A\{a_\alpha^{-1}\}$ such that for every α, $b_{i|\alpha} = 0$ except for a finite number of i. Given such a family, the finite sums $\sum_i b_{i|\alpha} \in A\{a_\alpha^{-1}\}$ form a compatible family. It follows then from the previous proposition that there exists a unique

$$b \in A \quad \text{such that} \quad b_{|\alpha} = \sum_i b_{i|\alpha}.$$

We denote this b by the formula $b = \sum_i b_i$. Given any morphism $\phi: A \to B$ in $\mathfrak{B}$, the family $\phi(b_i) \in B$ is also locally finite, and $\phi \sum_i b_i = \sum_i \phi(b_i)$.

This finishes the answer to Question II. Infinite additions of locally finite families make sense and can be perfomed in certain C^∞-rings. The

C^∞-rings which have this extra structure are precisely those presented by an ideal of local character. Before passing to Question I, we make a final remark on the open covering topology.

11. PROPOSITION. *Given any C^∞-ring $A \in \mathfrak{A}_{f.t}$ and a family $a_\alpha \in A$, $A \to A\{a_\alpha^{-1}\}$ covers in the open covering topology iff for all points p in A there exists α such that p is in $A\{a_\alpha^{-1}\}$. Thus any C^∞-ring without points is covered by the empty family.*

PROOF. One of the implications has already been seen (Proposition 4). Suppose then that for all p in A there is α such that p is in $A\{a_\alpha^{-1}\}$. Let $A = C^\infty(\mathrm{R}^n)/I$, $f_\alpha \in C^\infty(\mathrm{R}^n)$ such that

$$a_\alpha = [f_\alpha] \text{ and } U_\alpha = \{ x \mid f_\alpha(x) \neq 0 \}.$$

The hypothesis means that for all $p \in Zeros(I) \subset \mathrm{R}^n$ there is α such that $p \in U_\alpha$. If $p \notin Zeros(I)$, there is

$$l \in I \text{ such that } p \in U_l = \{ c \mid l(x) \neq 0 \}.$$

Thus the U_α together with the U_l form an open cover of R^n. Let $b_l \in A$, $b_l = [l]$. Then $A \to A\{a_\alpha^{-1}\}$ together with $A \to A\{b_l^{-1}\}$ form an open cover of A. But $A\{b_l^{-1}\} = \{0\}$; thus it is covered by the empty family. By composition of coverings, it follows that $A \to A\{a_\alpha^{-1}\}$ is an open covering of A.

ANSWER TO QUESTION I.

We consider the free C^∞-ring in λ generators, $\lambda \in \Gamma$, cf. [2]. This ring, which we denote $C^\infty(\mathrm{R}^\Gamma)$, is the ring of functions $\mathrm{R}^\Gamma \to \mathrm{R}$ which depend only on a finite number of variables, and which are smooth on these variables. Clearly, this is a ring of continuous functions (for the product topology in R^Γ). If we take a locally finite family of functions $\mathrm{R}^\Gamma \to \mathrm{R}$ in $C^\infty(\mathrm{R}^\Gamma)$, and add it up, we get a continuous function which will not be in general in $C^\infty(\mathrm{R}^\Gamma)$. However, every point will have a neighborhood in which this function will coincide with the restriction of a function in $C^\infty(\mathrm{R}^\Gamma)$. We consider all functions ϕ which locally depend on a finite number of variables. More precisely, functions for which there exists an open cover $U_\alpha \subset R^\Gamma$, finite sets J_α, smooth functions h_α and factorizations as indicated in the following diagram (where ρ is the projection):

$$\begin{tikzcd} U_\alpha \arrow[r] \arrow[d, "\rho"'] & R^\Gamma \arrow[d, "\phi"] \\ R^{J_\alpha} \arrow[r, "h_\alpha"] & R \end{tikzcd}$$

for all α. Thus we add to $C^\infty(R^\Gamma)$ all the functions needed to render the open covers of R^Γ effective epimorphic families. We will, for the lack of a better notation, denote this ring by ${}_\infty C^\infty(R^\Gamma)$. Thus a function ϕ is in ${}_\infty C^\infty(R^\Gamma)$ iff for each point $p \in R^\Gamma$, there is an open neighborhood U of p, a finite set of indices $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and a smooth function of n variables h such that for all $(x_\lambda) \in U$, $\lambda \in \Gamma$,

$$\phi(x_\lambda) = h(x_{\lambda_1}, \dots, x_{\lambda_n}).$$

We leave to the reader the proof of the following (where Γ and Λ are any sets, including finite):

12. PROPOSITION. *Given any* $\phi \in {}_\infty C^\infty(R^\Gamma)$ *and a* Γ*-tuple*

$$\phi_\lambda \in {}_\infty C^\infty(R^\Lambda), \quad (\phi_\lambda) : R^\Lambda \to R^\Gamma,$$

the composite $\phi(\phi_\lambda) \in {}_\infty C^\infty(R^\Lambda)$.

It follows then that there is an infinitary algebraic theory in the sense of Linton (cf. [3]) which has as Γ-ary operations the rings ${}_\infty C^\infty(R^\Gamma)$. We shall call this theory the *infinitary theory of C^∞-rings*. Its finitary part is the theory of C^∞-rings, since ${}_\infty C^\infty(R^n) = C^\infty(R^n)$. Thus, this ring is still free on n generators for the infinitary theory. (The action of a Γ-ary operation on $C^\infty(R^n)$ is given in Proposition 12, with $\Lambda = n$.) All ${}_\infty C^\infty$-rings of finite type are then quotients of $C^\infty(R^n)$ presented by ${}_\infty C^\infty$-ideals, that is, ideals I such that the congruence

$$f \sim g \quad \Rightarrow \quad f - g \in I$$

is an ${}_\infty C^\infty$-congruence. Recall that this congruence is always a C^∞-congruence (cf. [1, 2]).

13. PROPOSITION. *Let* $A = C^\infty(\mathrm{R}^n)/I$ *be a* C^∞*-ring presented by an ideal* I *of local character. Then,* A *is an* ${}_\infty C^\infty$*-ring. Or, equivalently,* I *is an* ${}_\infty C^\infty$*-ideal.*

PROOF. Let $\phi \in {}_\infty C^\infty(\mathrm{R}^\Gamma)$, and let

$$g_\lambda, f_\lambda \in C^\infty(\mathrm{R}^n), \quad \lambda \in \Gamma, \quad \text{such that} \quad g_\lambda - f_\lambda \in I.$$

Let p be any point of R^n and let $U \subset \mathrm{R}^\Gamma$ be an open neighborhood of the two points $(g_\lambda(p))$, $(f_\lambda(p)) \in \mathrm{R}^\Gamma$ where ϕ depends on a finite number of variables:

$$\phi(x_\lambda) = h(x_{\lambda_1}, \dots, x_{\lambda_k}) \quad \text{over } U,$$

for some $h \in C^\infty(\mathrm{R}^k)$. Let $W \subset \mathrm{R}^n$ be an open neighborgood of p such that

$$(g_\lambda)(W) \subset U \quad \text{and} \quad (f_\lambda)(W) \subset U.$$

Then

$$(\phi(g_\lambda) - \phi(f_\lambda))|W = (h(g_{\lambda_1}, \dots, g_{\lambda_k}) - h(f_{\lambda_1}, \dots, f_{\lambda_k}))|W \in I|W,$$

because

$$h(g_{\lambda_1}, \dots, g_{\lambda_k}) - h(f_{\lambda_1}, \dots, f_{\lambda_k}) \in I,$$

since all ideals are C^∞-ideals. Thus the term $(\phi(g_\lambda) - \phi(f_\lambda))_{|p}$ is in $I_{|p}$. Since I is of local character, this implies $\phi(g_\lambda) - \phi(f_\lambda) \in I$.

The converse of this proposition says, in a way, that there are enough operations in the infinitary theory of C^∞-rings to force any ${}_\infty C^\infty$-ideal to be of local character. This is actually the case, and we prove it by showing that given any locally finite family $l_\lambda \in C^\infty(\mathrm{R}^n)$, there is an infinitary operation that adds it up.

14. PROPOSITION. *Let* $l_\lambda \in C^\infty(\mathrm{R}^n)$, $\lambda \in \Gamma$, *be any locally finite family. Then, the following function:*

$$L: \mathrm{R}^n \times \mathrm{R}^\Gamma \to \mathrm{R}, \quad L(x_1, \dots, x_n, x_\lambda) = \sum_\lambda l_\lambda(x_1, \dots, x_n) x_\lambda$$

is in ${}_\infty C^\infty(\mathrm{R}^{n+\Gamma})$.

PROOF. Given any point $p \in \mathrm{R}^{n+\Gamma}$, $p = (a_1, \dots, a_n, a_\lambda)$, take an open neighborhood $U \subset \mathrm{R}^n$ of $(a_1, \dots, a_n)$, where all but a finite number of l_a

are zero. Then the open set $U \times \mathrm{R}^{\Gamma}$ is a neighborhood of p where L depends on only a finite number of variables.

Let $h_1, h_2, \ldots, h_n$ and f_λ be any $(n+\Gamma)$-tuplet of smooth functions in k variables. The family (indexed by Γ) $l_\lambda(h_1, \ldots, h_n)f_\lambda$ is a locally finite family in $C^\infty(\mathrm{R}^k)$, and it follows from Proposition 12 (with $k = \Gamma$) that the action of L for the ${}_\infty C^\infty$-ring structure of $C^\infty(\mathrm{R}^k)$ is given by the formula

$$L(h_1, \ldots, h_n, f_\lambda) = \sum_\lambda l_\lambda(h_1, \ldots, h_n)f_\lambda .$$

We have

15. PROPOSITION. *Let l_λ and L be as in Proposition 14.*

i) Given any n-tuple $h_1, \ldots, h_n \in C^\infty(\mathrm{R}^k)$, $L(h_1, \ldots, h_n, 0) = 0$.

ii) Given any ${}_\infty C^\infty$-ideal $I \subset C^\infty(\mathrm{R}^k)$ and any $n+\Gamma$-tuple $h_1, \ldots, h_n, f_\lambda \in C^\infty(\mathrm{R}^n)$, if $f_\lambda \in I$ for all λ, then $\sum_\lambda l_\lambda(h_1, \ldots, h_n)f_\lambda \in I$.

iii) Given any ${}_\infty C^\infty$-ideal $I \subset C^\infty(\mathrm{R}^n)$ and any Γ-tuple $f_\lambda \in C^\infty(\mathrm{R}^n)$ if $f_\lambda \in I$ for all λ, then $\sum_\lambda l_\lambda f_\lambda \in I$.

iv) Given any ${}_\infty C^\infty$-ideal $I \subset C^\infty(\mathrm{R}^n)$ and any locally finite family $f_\lambda \in C^\infty(\mathrm{R}^n)$, if $f_\lambda \in I$ for all λ, then $\sum_\lambda f_\lambda \in I$.

PROOF. i is clear, and ii follows clearly from i. We get iii by putting

$$k = n \quad \text{and} \quad h_i = \pi_i \text{ the projections.}$$

Finally, observe that given any open set $U \subset R^n$ and any smooth function f with $Supp(f) \subset U$, then there exists a function l with $supp(l) \subset U$ and such that $f = fl$. To prove iv, we apply this observation to each of the functions f_λ. The family l_λ so obtained is also locally finite; thus it has an associated operation L. iv follows then from iii.

16. COROLLARY. *Let A be a ${}_\infty C^\infty$-ring of finite type. Then A is a C^∞-ring presented by an ideal of local character.*

PROOF. Immediate from iv in the previous proposition and Definition 6, iv.

Thus, *the ideals of local character are exactly the congruences for* *the infinitary theory of C^∞-rings.* Remark that we have also proved that given any locally finite family f_λ in $C^\infty(\mathrm{R}^n)$, $\lambda \in \Gamma$, there is a $(n+\Gamma)$-ary operation L such that

$$L(\pi_1, \dots, \pi_n, f_\lambda) = \sum_\lambda f_\lambda ,$$

where π_i are the projections. Suppose now I is an ideal of local character, let $A = C^\infty(\mathrm{R}^n)/I$, let $e_i = [\pi_i]$ be the generators of A, and let

$$b_\lambda = f_\lambda(e_1, \dots, e_n) = [f_\lambda].$$

Then, by Definition-Proposition 10, b_λ is a locally finite family, and $[\sum_\lambda f_\lambda] = \sum_\lambda b_\lambda$. But since I is a ${}_\infty C^\infty$-ideal, we also have

$$[L(\pi_1, \dots, \pi_n, f_\lambda)] = L(e_1, \dots, e_n, b_\lambda).$$

Thus, given any locally finite family $b_\lambda = [f_\lambda]$ in a C^∞-ring A presented by an ideal of local character, $A = C^\infty(\mathrm{R}^n)/I$, if f_λ is locally finite in $C^\infty(\mathrm{R}^n)$, there is an infinitary operation L such that

$$L(e_1, \dots, e_n, b_\lambda) = \sum_\lambda b_\lambda ,$$

where $e_1, \dots, e_n$ are generators of A. With this we finish the answer to Question I.

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